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D Vankovich COUPLED TWO-DEGREE-OF-FREEDOM RESPONSE OF RESILIENTLY SUPPORTED RIGID BODY TO RANDOM EXCITATION

by

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December 1965





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COUPLED TWO-DEGREE-OF-FREEDOM RESPONSE OF RESILIENTLY SUPPORTED RIGID BODY TO RANDOM EXCITATION

ABSTRACT

Vibration isolation systems are often designed so that the rotational and the translational modes of vibration of the isolated body will be decoupled. Estimates of the system response are then based on a single-degree-of-freedom model. In practice, mode decoupling is only achieved within certain limits. In this report the effects of mode coupling are studied by computing the response of a two-degree-of-freedom model excited by a random translatory ground motion. Graphical response functions are presented from which the angular response, and the translational displacement and acceleration, of any point on the body can be calculated.

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TABLE OF CONTENTS

Section				Page
Ι	Introduction			5
II	The System and Its Equations of Motion			6
III	Mean Square Responses			10
IV.A	Graphical Results			15
IV.B	Example			21
APPENDIX A				24
APPENDIX B				27
SYMBOLS				29
BIBLIOGRAPHY			31	

I. Introduction

The coupled response of a resiliently-supported rigid body to a random vibration input is a common problem in vibration isolation applications. In practice, vibration isolation systems are often designed so that the rotational modes of vibration will be decoupled from the translational modes. Estimates of the system performance are then based on a simple single-degree-of-freedom model of the system. In actual systems, due to the ever present tolerances in design and manufacture, the modes are decoupled only within certain limits. In some critical applications it is necessary to estimate the effect of this coupling. This report considers the simplest case of a coupled system: a two-degreeof-freedom, resiliently-supported rigid body excited by a translatory ground vibration. The vibration input is assumed to be a stationary random process whose power spectral density is constant. Graphical response functions are presented from which the angular response, and the relative displacement and absolute acceleration of any point of the body, can be calculated.

II. The System and Its Equations of Motion

Consider the coupled two-degree-of-freedom system shown in Fig. 1. The support has a translational motion u(t) with respect to a fixed reference. The body is supported by an array of discrete resilient elements each of which possesses both linear velocity damping and linear elastic properties, shown as dash pots and springs, respectively.





The response to the translational input can be described by two coordinates: the absolute displacement, y_L , of a point L, taken parallel to the input direction, and a rotation, θ , about an axis perpendicular to the plane of the paper.

In order to formulate the system response in as general a way as possible, the inertial properties of the body, as well as the damping and elastic properties of the resilient elements, will be described in terms of centers of action termed the mass center, the damping center, and the elastic center, respectively. The mass center M, as is known from elementary dynamics, has the property that a unit force directed through M will cause the body to translate only with acceleration 1/m; and that the application of a unit couple will cause the body to rotate only about M with angular acceleration $1/I_{
m M}.$ The inertial properties of a body which moves in a plane are determined when the location of the mass center, the body mass, and the body moment of inertia are known. Similarly, (as shown in Appendix A) a damping center D of an array of damping elements can be defined so as to have the property that a unit force directed through D will cause the array to translate only, with a velocity 1/C, and that a unit couple will cause the array to rotate about D with angular velocity $1/C_{\theta}$. The properties of the array of damping elements are then determined when the location of the damping center, the damping constant, and the rotational damping constant, are specified. Finally, the elastic center E of an array of elastic elements is defined such that a unit force directed through E will cause the

array to translate only, with a deflection 1/K, and a unit couple will cause a rotation of the array about E, with an angular deflection $1/K_{\theta}$. As before, the elastic properties of an array of elastic elements are defined when the location of the elastic center, the elastic constant, and the rotational elastic constant are known.

The equation of motion of the body can be written in terms of the body mass m, the moment of inertia I_M , the distance of the centers E and D from the mass center M, and the resilient element constants C, C_{θ} , K, and K_{θ} . Although the response of the body is determined when the above parameters are fixed, the actual distributions of mass points and resilient elements are still somewhat arbitrary in exactly the way that fixing the mass and the moment of inertia of a body does not precisely specify its shape.

When the centers M, D, and E coincide, the translational and rotational modes of vibration are decoupled, and a translational input will cause only a translational response. In general, the action centers M, D, and E will not coincide and a translational input will cause both translation and rotation in the response. In usual design practice the effect of the damping forces on coupling is either ignored or assumed to be neglible in comparison with the elastic and mass unbalances. If the elastic and damping centers are displaced equally from the mass center, the ratio of the damping coupling effect to the elastic coupling increases with frequency, and the effect of damping on coupling should not be ignored in cases of high damping and high frequency. In the present analysis the effect of damping on coupling is accounted for in a simple way by assuming that the centers E and D coincide, being displaced a distance ϵ from the mass center M. It is shown in Appendix A that E and D will coincide if the ratio

$$\frac{c_i}{k_i} = \frac{C}{K}$$

for each resilient element, a condition which is likely to be satisfied when all the elements are made of the same material and are of geometrically similar configuration.

Assumming, then, that centers E and D coincide, the system equations of motion for $\theta \ll 1$ can be written as follows:

$$Kx_{E} + m (\ddot{x}_{E} + \ddot{u} + \epsilon \ddot{\theta}) + C \dot{x}_{E} = 0$$
(1)

$$K_{\theta} \theta + C_{\theta} \dot{\theta} + I_{M} \ddot{\theta} - \epsilon (Kx_{E} + C\dot{x}_{E}) = 0$$
⁽²⁾

Introduce the new parameters ω_n , ω_{θ} , ζ , ζ_{θ} and μ (defined in glossary). Then Eqs. (1) and (2) become

$$\ddot{\mathbf{x}}_{\mathrm{E}} + 2\zeta \,\omega_{\mathrm{n}} \dot{\mathbf{x}}_{\mathrm{E}} + \omega_{\mathrm{n}}^{2} \mathbf{x}_{\mathrm{E}} + \mu \,\mathbf{R}\ddot{\theta} = - \ddot{\mathbf{u}}$$
(3)

$$-(\omega_{n}^{2} + 2\zeta \omega_{n}s)\mu x_{E} + (s^{2} + 2\zeta_{\theta} \omega_{\theta}s + \omega_{\theta}^{2})R\theta = 0$$
(4)

$$-\omega_{n}^{2}\mu x_{E} - 2\zeta \omega_{n}\mu \dot{x}_{E} + R\ddot{\theta} + 2\zeta_{\theta} \omega_{\theta}R\dot{\theta} + \omega_{\theta}^{2}R\theta = 0$$
(5)

If points ${\rm E}$ and ${\rm D}$ are to coincide, then

$$\zeta_{\theta} = \zeta \omega_{\theta} / \omega_{n} = \zeta \Omega$$
 (see Appendix A)

Then by assuming a unit acceleration input of the form $\ddot{u} = e^{st}$, the transfer functions for x_E and $R\theta$ can be obtained from Eqs. (3) and (4) as ^[1]

$$H_{x_{E}}(s) = -\left[s^{2} + \Omega^{2} (2\zeta \omega_{n} s + \omega_{n}^{2})\right] /\Delta$$

$$H_{R\theta}(s) = -\mu \left[\omega_{n}^{2} + 2\zeta \omega_{n} s\right] /\Delta$$
(6)
(7)

where

$$\Delta = s^{4} + 2\zeta \omega_{n} (1 + \mu^{2} + \Omega^{2}) s^{3} + \omega_{n}^{2} \left[1 + \mu^{2} + \Omega^{2} (1 + 4\zeta^{2}) \right] s^{2} + 4\zeta \omega_{n}^{3} \Omega^{2} s + \omega_{n}^{4} \Omega^{2}$$

and

$$s = j\omega$$
.

Similarly, using the relation that

$$\ddot{y}_{M} = \ddot{u} + \ddot{x}_{E} + \mu R \ddot{\theta}$$

transfer functions for $\ddot{\textbf{y}}_{M}$ and $\textbf{R}\ddot{\theta}$ can be obtained as

$$H_{\mathcal{Y}_{M}}(s) = \left[2\zeta \omega_{n} s^{3} + (1 + 4\zeta^{2}\Omega^{2}) \omega_{n}^{2} s^{2} + 4\zeta \Omega^{2} \omega_{n}^{3} s + \omega_{n}^{4} \Omega^{2}\right] /\Delta$$
(8)

$$H_{R\theta}(s) = -\mu \left[s^2 \left(2\zeta \omega_n s + \omega_n^2 \right) \right] /\Delta$$
(9)

III. Mean Square Responses

Of all the possible statistics that could be used to describe the random response of the coupled system, the following three dimensionless mean square quantities have been chosen as being most valuable to a designer:

$$\ddot{\mathbf{Y}}^2 = \frac{\mathbf{E}[\ddot{\mathbf{y}}_L^2]}{\mathbf{E}[\ddot{\mathbf{y}}^2]_0}$$

The ratio of the mean-square absolute acceleration of a point L on the body to the absolute acceleration of a decoupled system ($\mu = 0$). The statistic can be used to estimate the structural adequacy of the body or of small attached components.

$$\Theta^2 = \frac{R^2 E[\theta^2]}{E[x^2]_0}$$

This statistic is a dimensionless measure of the mean-squared angular displacement which includes a characteristic length (the radius of gyration R) and the mean-squared translational displacement of a decoupled system. This statistic is of interest when the body is sensitive to rotation.

$$X_{L}^{2} = \frac{E[x_{L}^{2}]}{E[x^{2}]_{0}}$$

The mean-squared displacement of a point L on the body relative to the input motion, nondimensionalized by the mean-squared relative displacement of a decoupled system. This quantity provides an estimate of the sway (clearance) space requirements for the body, and an estiite of the onset and severity of nonlinearities in the resilient elements due to excessive deflection.

The above statistics are obtained by first calculating the power spectral density $S(\omega)$ of the desired response quantity, and then integrating $S(\omega)$ over all frequencies to obtain the total mean-square value. Expressions for $S(\omega)$ in terms of the input power spectrum and the system function are derived in Appendix B. Taking the easiest statistic first, for an input power spectrum S_0 which is constant with frequency,

$$S_{\boldsymbol{\Theta}}(\omega) = \frac{S_0 \left| H_{R\boldsymbol{\Theta}}(j\omega) \right|^2}{E[x^2]_0}$$
(9)

where $H_{R\theta}^{}(j\omega)$ is given by Eq. (6), and the normalizing quantity $^{[1]\ast}$

$$\mathbf{E}[\mathbf{x}^2]_0 = \frac{\pi \mathbf{S}_0}{2\zeta \omega_n^3}$$

Finally,

$$\Theta^{2} = \frac{2\xi \omega_{n}^{3}}{\pi} \int_{-\infty}^{\infty} |H_{R\Theta}(j\omega)|^{2} d\omega$$
(10)

[2] Integrals of transfer functions of the form of Eq. (10) have been tabulated, so that Eq. (10) can be written as

$$\Theta^{2} = 2\zeta \left[(b_{0}^{2}/a_{0}) (a_{2}a_{3} - a_{3}) + a_{3}b_{1}^{2} \right] / D(a)$$
(11)

where

$$a_{0} = \Omega^{2} \qquad b_{0} = \mu$$

$$a_{1} = 4\zeta \Omega^{2} \qquad b_{1} = 2\zeta \mu$$

$$a_{2} = 1 + \mu^{2} + \Omega^{2} (1 + 4\zeta^{2})$$

$$a_{3} = 2\zeta (1 + \mu^{2} + \Omega^{2})$$

$$D(a) = a_{1} (a_{2}a_{3} - a_{1}) - a_{0}a_{3}^{2}$$

This function is plotted for various ranges of ζ , μ , and Ω in Section IV.

Turning next to the calculation of X_L^2 , we follow the above procedure and first calculate the power spectral density of the displacement of a point L relative to the input motion. See Fig. 2. Now, for small θ ,

$$x_{L} = x_{E} + \ell \theta$$

where ℓ is a coordinate nominally perpendicular to the input axis, with origin at E, and whose positive direction is toward M from E. Thus x_{L} is a linear sum of two

^{*}Numbers in brackets refer to similiarly numbered references in the Bibliography.



Fig. 2

Coupled System, E and D Coincident

variables, x_E and θ , with ℓ as a parameter. The power spectral density for such a function is derived in Appendix B. For the case of constant input spectrum, S_0 , the spectrum for X_L is given by

$$S_{X_{L}}(\omega) = \frac{S_{0}}{E[x^{2}]_{0}} \left\{ \left| H_{X_{E}}(j\omega) \right|^{2} + \left(\frac{\ell}{R}\right)^{2} \left| H_{R\theta}(j\omega) \right|^{2} + 2 \frac{\ell}{R} \mathcal{R}_{e} \left[H_{X_{E}}(j\omega) H_{R\theta}^{*}(j\omega) \right] \right\}$$

$$(12)$$

where the * denotes the complex conjugate, and \Re_e [] denotes real part of. X_L^2 is then obtained by integrating Eq. (12) term by term over all frequencies. Note that the integral of the bracketed expression Eq. (12) must equal $E[X_L^2]$. Since

$$\begin{split} \mathbf{E}[\mathbf{x}_{\mathrm{L}}^{2}] &= \mathbf{E}\left\{ \begin{bmatrix} \mathbf{x}_{\mathrm{E}} + \left(\frac{\ell}{\mathrm{R}}\right) & \mathrm{R}\boldsymbol{\theta} \end{bmatrix}^{2} \right\} \\ &= \mathbf{E}[\mathbf{x}_{\mathrm{E}}^{2}] + \left(\frac{\ell}{\mathrm{R}}\right)^{2} & \mathrm{E}\left[\left(\mathrm{R}\boldsymbol{\theta}\right)^{2}\right] + 2 \frac{\ell}{\mathrm{R}} & \mathrm{E}[\mathbf{x}_{\mathrm{E}} & \mathrm{R}\boldsymbol{\theta}] \end{split}$$

the integral of the last term in the brackets of Eq. (12) can be interpreted as $E[X_E, R\theta]$. Using the tabulations of Ref. 2, Eq. (12) can be integrated over all frequencies to give

$$X_{L}^{2} = X_{E}^{2} + \left(\frac{\ell}{R}\right)^{2} \Theta^{2} + 2 \frac{\ell}{R} X_{E} \Theta$$
(14)

(13)

where

$$X_{E}^{2} = 2\zeta \left[(b_{2}^{2}/a_{0}) (a_{2}a_{3} - a_{1}) + a_{3} (b_{3}^{2} - 2b_{2}) + a_{1} \right] / D(a)$$
$$X_{E}^{\Theta} = 2\mu\zeta \left[a_{3} (a_{2} - b_{4}) - a_{1} \right] / D(a)$$

and where a's and D(a) are as given following Eq. (11) and

 $b_2 = \Omega^2$ $b_3 = 2\zeta \Omega^2$ $b_4 = 1 - 4\zeta^2 \Omega^2$

 X_E^2 and $X_E^{}\Theta$ are plotted for various ranges of ζ , μ , and Ω in Section IV.

The calculation of Y_L^2 follows the preceding method very closely. Starting with an expression for the absolute displacement y_L of a point L on the body,

$$y_{L} = y_{M} + \ell' \theta$$
$$\ddot{y}_{L} = \ddot{y}_{M} + \ell' \ddot{\theta}$$

where $l' = l - \epsilon$. (See Fig. 2)

Following the form of Eq. (12),

$$S_{\mathbf{Y}_{L}}^{*} = \frac{S_{0}}{E[\mathbf{y}^{2}]_{0}} \left\{ \left| H_{\mathbf{y}_{M}}^{*}(j\omega) \right|^{2} + \left(\frac{\ell}{R}\right)^{2} \left| H_{R\theta}^{*}(j\omega) \right|^{2} + 2 \frac{\ell}{R} \mathcal{R}_{e} \left[H_{\mathbf{y}_{M}}^{*}(j\omega) H_{R\theta}^{*}(j\omega) \right] \right\}$$

$$(15)$$

where the normalizing quantity $E[\ddot{y}^2]_0 = \pi S_0 \omega_n (1 + 4 \zeta^2)/2\zeta$

Again using the tabulations of Ref. 2, Eq. (15) can be integrated over all frequencies to give

$$\ddot{\mathbf{Y}}_{\mathrm{L}}^{2} = \ddot{\mathbf{Y}}_{\mathrm{M}}^{2} + \left(\frac{\ell}{\mathbf{R}}\right)^{2} \quad \ddot{\boldsymbol{\Theta}}^{2} + 2\frac{\ell'}{\mathbf{R}} \quad \ddot{\mathbf{Y}}_{\mathrm{M}} \quad \ddot{\boldsymbol{\Theta}}$$
(16)

where

$$\ddot{Y}_{M}^{2} = \left(\frac{2\zeta}{1+4\zeta^{2}}\right) \left[(b_{5}^{2}/a_{0})(a_{2}a_{3}-a_{1}) + a_{3}(b_{6}^{2}-2b_{5}b_{7}) + a_{1}(b_{7}^{2}-2b_{6}b_{8}) + b_{8}^{2}(a_{1}a_{2}-a_{0}a_{3}) \right] / D(a)$$

$$\ddot{\Theta}^{2} = \left(\frac{2\zeta}{1+4\zeta}2\right) \left[\mu^{2}a_{1} + b_{9}^{2}(a_{1}a_{2}-a_{0}a_{3}) \right] / D(a)$$

$$\ddot{Y} \stackrel{\sim}{\Theta} = \left(\frac{2\zeta\mu}{1+4\zeta}2\right) \left[b_{5}a_{3}-b_{10}a_{1}-b_{11}(a_{1}a_{2}-a_{0}a_{3}) \right] / D(a)$$

and where the a's and D(a) are as given after Eq. (11) and

$$b_{5} = \Omega^{2}$$

$$b_{6} = 4\zeta \Omega^{2}$$

$$b_{7} = 1 + 4\zeta^{2} \Omega^{2}$$

$$b_{8} = 2\zeta$$

$$b_{8} = 2\zeta$$

The functions \ddot{Y}_M^2 , $\dot{\Theta}^2$, and $\ddot{Y}_M^{}\dot{\Theta}$ are plotted in Section IV.

IV.A Graphical Results

The computer calculations for X_E^{Θ} , Θ^2 , Θ^2 and Y^{Θ} are plotted in Figs. 3 through 7, respectively. The plots for X_E^2 and \ddot{Y}_M^2 have been combined, since they are approximately equal, and their difference would not be detectable on these plots for the ranges of ζ , μ , and Ω presented.



Fig. 4 System Response Curves: $\mu = 0.1$



Fig 5 System Response Curves: $\mu = 0.2$



Fig. 6 System Response Curves: $\mu = 0.4$



Fig. 7 System Response Curves: $\mu = 0.6$



Fig. 8 System Response Curves: $\mu = 0.8$

IV.B Example

Consider a body with the following parameters:

weight	= 100 lb
radius of gyration R	= 6 inches
translational stiffness at elastic center K	= 2.5×10^4 lb/in.
rotational stiffness at elastic center K _A	= 1.76×10^6 in-lb/rad

It is assumed that $\frac{c_i}{k_i} = \frac{C}{K}$ for each resilient element, and that $\xi = 0.1$ for an uncoupled system. Such a system, then, has uncoupled natural frequencies

$$f_{n} = \frac{1}{2\pi} \sqrt{K/m} = \frac{1}{2\pi} \left[\frac{2.5 \times 10^{4} \times 386}{100} \right]^{1/2} = 50 \text{ cps}$$

$$f_{\theta} = \frac{1}{2\pi} \sqrt{K_{\theta}/mR^{2}} = \frac{1}{2\pi} \left[\frac{1.76 \times 10^{6} \times 386}{100 \times 36} \right]^{1/2} = 70 \text{ cps}$$

The input vibration level, expressed in the customary engineering units, is taken to be $W_0 = .10 \text{ g}^2/\text{cps}$, constant with frequency. This input level W_0 expresses the mean-square value of acceleration per unit cycle per second, assumed constant over all positive frequencies, whereas the input level S_0 used so far in this analysis is the mean-square acceleration per unit radian per second assumed constant over both positive and negative frequencies. In order that the power included in a bandwidth $f_2 - f_1$ be the same for both notations,

$$W_0 \times (f_2 - f_1) = S_0 \times 2(\omega_2 - \omega_1) = S_0 \times 4 \pi (f_2 - f_1)$$

or

$$W_0 = S_0 \times 4\pi$$

Under decoupled conditions, for any point on the body, the mean-square displacement relative to the input would be (See Eq. (9) et seq.)

$$x_{\rm RMS} = \left[\frac{\pi S_0}{2\zeta \omega_n^3}\right]^{1/2} = \left[\frac{W_0}{8\zeta \omega_n^3}\right]^{1/2} = \left[\frac{.10(386)^2}{8(.10)(50 \times 2\pi)^3}\right]^{1/2} = 2.45 \times 10^{-2} \text{ inches}$$

and the mean-square absolute acceleration would be (see Eq. (15) et seq.)

$$\ddot{y}_{RMS} = \left[\frac{\pi S_0 \omega_n (1 + 4\zeta^2)}{2\zeta}\right]^{1/2} = \left[\frac{W_0 \pi f_0 (1 + 4\zeta^2)}{4\zeta}\right]^{1/2} = \left[\frac{.10(50\pi)(1,04)}{4(.10)}\right]^{1/2} = 6.4 \text{ g's}$$

Now assume that the center of gravity M of the body is displaced 2.4 inches from the elastic center E. Then with $\mu = \frac{\epsilon}{R} = \frac{2.4}{6} = .4$, $\Omega = \frac{f_{\Theta}}{f_{\Pi}} = \frac{70}{50} = 1.4$, and $\xi = .10$,

the following variables can be evaluated from the graphs:

$$X_{E}^{2} \approx \ddot{Y}_{M}^{2} = 0.862$$
 $\ddot{Y}_{M}^{0} = 0.184$
 $X_{E}^{0} = 0.273$ $\ddot{\Theta}^{2} = 0.138$
 $\Theta^{2} = 0.110$

Substituting into Eq. (14) and (16),

$$X_{L}^{2} = .862 + 2 \left(\frac{\ell}{R}\right) (0.273) + \left(\frac{\ell}{R}\right)^{2} (.110)$$

$$\ddot{Y}_{L}^{2} = .862 + 2 \left(\frac{\ell}{R}\right)(.184) + \left(\frac{\ell}{R}\right)^{2} (.138)$$

These equations are plotted in Fig. 3 for a range of ℓ and ℓ' .

In particular, for ℓ/R = 2.4, $X_{\rm L}^2$ = 2.81; so

$$E[x_{2,4}^2] = 1.58 E[x^2]_0$$

or

$$(x_{2,4})_{RMS} = (2.81)^{1/2} \times 2.45 \times 10^{-2} = 4.11 \times 10^{-2}$$
 inches

For the same point on the body, $\frac{\ell'}{R} = 2.0$, and $\ddot{Y}_{L}^{2} = 2.15$. So $(\ddot{y}_{2.0})_{RMS} = (2.15)^{1/2} \times 6.4 = 9.4 \text{ g}^{1}\text{s}.$

Finally, from $(\theta)^2 = .110$,

$$\theta_{\rm RMS} = \left[.110 \quad \frac{{\rm E}[{\rm x}^2]_0}{{\rm R}^2} \right]^{1/2} = \frac{.332 \times 2.45 \times 10^{-2}}{6} = 1.36 \times 10^{-3} \text{ rad}$$





APPENDIX A





Body Supported by Spring Array

As stated above in Section II, a unit force directed through the elastic center of E of an array of springs will cause a linear displacement $\frac{1}{K}$ only. Also, the application of a unit couple will cause only a rotation about point E. A demonstration of these properties of the elastic center follows.

Consider the array of springs in Fig. A-1. If the unit force causes a linear displacement only, then this displaced position of the body must be at equilibrium. For this displacement x, the ith spring force is $k_i x$.

For equilibrium,

$$\Sigma F = -1 + \Sigma k_i x = 0,$$

since

$$\Sigma$$
 k_i = K and x = $\frac{1}{K}$

Also

 $\Sigma M_{\xi=0} = \frac{1}{K} \left[k_1 \xi_1 + k_2 \xi_2 + \dots + k_n \xi_n \right] - (1) (\xi_E) = 0,$ $\xi_E = \frac{\Sigma k_i \xi_i}{K}$ (A-1)

So

which defines the position of point E.

Next, the application of a unit couple is considered. If the couple is to cause only an angular displacement about point E, then this position must be an equilibrium position. For an angular deflection θ about point E, the force of the i^{th} spring is $k_i(\xi_E - \xi_i) \theta$ and the moment of the i^{th} spring about point E is $k_i(\xi_E - \xi_i) \theta$. For equilibrium,

$$\Sigma F = \Sigma k_i (\xi_E - \xi_i) \theta = (K \xi_E - \Sigma k_i \xi_i) \theta = 0$$

which again leads to the elastic center definition given by Eq. (A-1). Also,

$$\Sigma M_{E} = \Sigma k_{i} (\xi_{E} - \xi_{i})^{2} \theta - 1 = 0$$

Introduce the definition $K_{\theta} = \sum k_i (\xi_E - \xi_i)^2$. Then

$$\theta = \frac{1}{K_{\theta}}$$

Thus an elastic center with location $\xi_E = \frac{\sum k_i \xi_i}{K}$ exists with properties as described in the first paragraph of this appendix, where

and

$$K = \Sigma_{k_{i}}$$

$$K_{\theta} = \Sigma_{k_{i}} (\xi_{E} - \xi_{i})^{2}$$
(A-2)

The demonstration of the existence of a damping center D of an array of dash pots follows in a similar fashion to the elastic center calculations above. The results only are written here:

Damping center location $\xi_D = \frac{\sum c_i \xi_i}{C}$ (A-3)

Overall damping constant $C = \Sigma c_i$ (A-4)

Overall rotational damping
$$C_{\theta} = \sum c_i (\xi_D - \xi_i)^2$$
 (A-5)
constant

If centers E and D are to coincide, $\xi_{\rm E} = \xi_{\rm D}$, or

$$\frac{\sum k_i \xi_i}{K} = \frac{\sum c_i \xi_i}{C}$$
(A-6)

When there are only two resilient elements (n = 2), the relation (A-6) gives rise to only two equations which have the unique solutions

$$\frac{k_{i}}{K} = \frac{c_{i}}{C}$$
, or $\frac{k_{i}}{c_{i}} = \frac{K}{C}$ $i = 1, 2$ (A-7)

Eq. (A-7) also satisfies (A-6) when there are three or more resilient elements $(n \ge 3)$, but these solutions are not unique since the system of equations represented by (A-6) is then overdetermined.

Finally it remains to show that $\frac{\zeta}{\zeta} = \frac{\omega}{\omega_n}$; where $\zeta = \frac{C}{2\sqrt{Km}}$ and

 $\xi_{\theta} = \frac{C_{\theta}}{2\sqrt{K_{\theta}I}}$, the damping ratios for uncoupled translational and rotational vibration,

respectively. Now

$$\frac{\zeta_{\theta}}{\zeta} = \frac{C_{\theta}}{2\sqrt{K_{\theta}I}} \times \frac{2\sqrt{Km}}{C} = \frac{C_{\theta}}{C} \left[\frac{K_{\theta}}{mR^2} \frac{m}{K} \frac{K^2}{K_{\theta}^2} \right]^{1/2} = \frac{\omega_{\theta}}{\omega_n} \frac{C_{\theta}}{C} \frac{K}{K_{\theta}}$$
(A-8)

But

$$\frac{C_{\theta}}{C} \quad \frac{K}{K_{\theta}} = \frac{\Sigma \left(\xi_{D} - \xi_{i}\right)^{2} c_{i}}{\Sigma \left(\xi_{E} - \xi_{i}\right)^{2} k_{i}} \times \frac{K}{C}$$

Now if E and D coincide, $\xi_E = \xi_D$, and if Eq. (A-7) is assumed to be valid for $n \ge 3$,

then

$$\frac{C_{\theta}}{K_{\theta}} \frac{K}{C} = \frac{\Sigma \left(\xi_{E} - \xi_{i}\right)^{2} c_{i}}{\Sigma \left(\xi_{E} - \xi_{i}\right)^{2} k_{i}} \times \frac{k_{i}}{c_{i}} = 1$$
(A-9)

Combining Eqs. (A-8) and (A-9)

$$\frac{\zeta_{\theta}}{\zeta} = \frac{\omega_{\theta}}{\omega}$$

APPENDIX B



Fig. B-1

Two Linear Filters in Parallel

Consider a parallel circuit consisting of two filters, excited by a common input u(t) as shown in Fig. B-1. The autocorrelation of the sum of their outputs w(t) = $v_1(t) + v_2(t)$ is given by ^[3]

$$R_{WW}(\tau) = E \left[w(t + \tau) w^{*}(t) = E \left\{ \left[v_{1}(t + \tau) + v_{2}(t + \tau) \right] \left[v_{1}^{*}(t) + v_{2}^{*}(t) \right] \right\}_{(B-1)}$$

Multiplying out and noting that the process of taking the expectation is distributive, Eq. (B-1) becomes

$$R_{ww}(\tau) = E \left[v_1 (t + \tau) v_1^* (\tau) \right] + E \left[v_2 (t + \tau) v_2^* (\tau) \right] + E \left[v_1 (t + \tau) v_2^* (t) \right] + E \left[v_2 (t + \tau) v_1^* (t) \right] = R_{v_1 v_1}(\tau) + R_{v_2 v_2}(\tau) + R_{v_1 v_2}(\tau) + R_{v_2 v_1}(\tau)$$
(B-2)

Eq. (B-2) can be written in terms of the input autocorrelation function $R_{uu}(\tau)$ and the impulse responses $h_1(\tau)$ and $h_2(\tau)$ as^[3]

$$R_{ww}(\tau) = R_{uu}(\tau) * h_{1}^{*} (-\tau) * h_{1}(\tau)$$

$$+ R_{uu}(\tau) * h_{2}^{*} (-\tau) * h_{2}(\tau)$$

$$+ R_{uu}(\tau) * h_{1}(\tau) * h_{2}^{*} (-\tau)$$

$$+ R_{uu}(\tau) * h_{2}(\tau) * h_{1}^{*} (-\tau)$$

$$(B-3)$$

where * denotes convolution.

Using the Fourier transform, Eq. (B-3) becomes, in the frequency domain,

$$S_{w}(\omega) = S_{uu}(\omega) \left[H_{1}^{*}(j_{\omega}) H_{1}(j_{\omega}) + H_{2}^{*}(j_{\omega}) H_{2}(j_{\omega}) + H_{1}(j_{\omega}) H_{2}^{*}(j_{\omega}) + H_{2}(j_{\omega}) H_{1}^{*}(j_{\omega}) \right]$$
(B-4)

The first two terms in the brackets of Eq. (B-4) are real. The last two terms will in general be complex since the imaginary parts of $H_1(j\omega)$ and $H_2(j\omega)$ will be unequal. However, when $S_w(\omega)$ is integrated over all frequencies only the real (even) parts of $H_1(j\omega) H_2^*(j\omega)$ and $H_2(j\omega) H_1^*(j\omega)$ will contribute to the integral, and the imaginary parts can be neglected. Since $H_1(j\omega) H_2^*(j\omega)$ and $H_2(j\omega) H_1^*(j\omega)$ are conjugates of each other, their real parts are equal. So Eq. B-13 can be written as

$$S_{W}(\omega) = S_{UU}(\omega) \left\{ \left| H_{1}(j\omega) \right|^{2} + \left| H_{2}(j\omega) \right|^{2} + 2 \mathcal{R}_{e} \left[H_{1}(j\omega) H_{2}^{*}(j\omega) \right] \right\}$$
(B-5)

When there is only one filter, Eq. (B-5) reduces to the well known relation

$$S_{w}(\omega) = S_{uu}(\omega) |H_{1}(j\omega)|^{2}$$
(B-6)

SYMBOLS

u	ground motion input displacement
^у г, ^у М	absolute displacement of points L, M
^x E' ^x L	displacement of points E, L relative to ground displacement
θ	angular displacement of body
L	general point on resiliently supported body
М	mass center of body; also moment (Appendix A)
D	damping center of body
Е	elastic center of body
m	mass of body
I_{M}	mass moment of inertia about point M
С	overall translational damping constant
С _ө	overall rotational damping constant
K	overall translation stiffness constant
к _ө	overall rotation stiffness constant
ω	vibration frequency, rad/sec
E	coupling parameter; See Fig. 2
°i	damping constant of i th dashpot
k.	stiffness of i th spring
ω _n	translation natural (radian) frequency of uncoupled system (ϵ = $\theta)$
^ω 0	rotational natural (radian) frequency of uncoupled system (ϵ = $ heta$)
ζ	translational damping ratio, uncoupled system; $\xi = \frac{C}{2\sqrt{Km}}$
ζθ	rotational damping ratio, uncoupled system; $\xi_{\theta} = \frac{C_{\theta}}{2\sqrt{K_{\theta}I_{M}}}$
μ	dimensionless coupling parameter; $\mu = \frac{\epsilon}{R}$

SYMBOLS (Cont'd)

- R radius of gyration of body about M
- Ω natural frequency ratio, $Ω = ω_0 / ω_n$
- Y dimensionless absolute translational response statistic defined in Section III
- (Here) dimensionless rotational response statistic defined in Section III
- X dimensionless relative translation response statistic defined in Section III
- $S(\omega)$ response power spectral density (power per rad/sec), defined for + and frequencies
- S₀ input power spectral density (power per rad/sec), assumed constant with frequency, for + and frequencies
- E mathematical expectation
- W_0 input power spectral density (power per cycle/sec), assumed constant with frequency, for + frequencies only

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