

Massachusetts Institute of Technology
Instrumentation Laboratory
Cambridge, Massachusetts

Space Guidance Analysis Memo # 10 - 66

TO: SGA Distribution
FROM: Kurth Krause - AC Electronics Resident
DATE: 28 April 1966
SUBJECT: The Minimum Fuel Return Problem

This memo presents a discussion of the problem of generating a conic trajectory which returns a spacecraft to re-entry with a specified flight-path angle from a point within the earth's sphere of influence (but outside the moon's sphere), under the constraint of minimizing the impulsive delta-velocity required to achieve the trajectory.

CONCLUSIONS

In developing a closed-form solution to the minimum-fuel return problem, it was shown that a fourth-order polynomial must be solved. Logic is required to determine the desired root, eliminating extraneous and often non-existent roots. Disregarding roots which require that the spacecraft return on a trajectory retrograde to the initial trajectory, only one root exists for most initial states. This is always true if the spacecraft is greater than 6500 miles from the earth. If the initial velocity is less than 25,000 miles per hour, the criterion that only one solution exists is that the spacecraft be greater than 5400 miles from the earth. Under some conditions, the root implying minimum fuel requires a physically impossible return trajectory. If the initial radial velocity is negative, this problem does not exist, but the root implying minimum fuel may require a trajectory which violates the maximum re-entry velocity constraint.

It was concluded that a direct iteration search, using the cotangent of the post-abort flight-path angle as an independent parameter would be a better approach to solving the problem. This approach always converges on a physically realizable solution, requires few iterations to converge, and greatly simplifies the equations. More important, for all feasible pre-abort states, the resulting ΔV required to achieve the return trajectory which satisfies all constraints is the absolute minimum.

DEVELOPMENT

Since the re-entry conditions do not constrain the trajectory plane, the minimum ΔV will result from a coplanar return, reducing the problem to two dimensions. Then the specified flight-path angle, γ_R (measured from local vertical), at the re-entry radius, RR , allows the semilatus rectum, p , to be expressed as a function of only the post-impulse flight-path angle, γ :

$$p = \frac{2 R_0 (R-1)}{R^2 (1 + \cot^2 \gamma_R) - (1 + \cot^2 \gamma)} \quad (1)$$

where R_0 = initial radial distance of the spacecraft
 $R = R_0/RR$.

Now

$$V_T^2 = \mu p / R_0^2 \quad \text{and} \quad \cot \gamma = V_R / V_T, \quad (2)$$

where

V_T = post-impulse tangential velocity component

V_R = post-impulse radial velocity component

μ = product of universal gravitational constant and mass of earth.

Combining Eqs. (1) and (2),

$$V_R^2 = [R^2 (1 + \cot^2 \gamma_R) - 1] V_T^2 - 2\mu \left(\frac{1}{RR} - \frac{1}{R_0} \right), \quad (3)$$

or, in a more illustrative form,

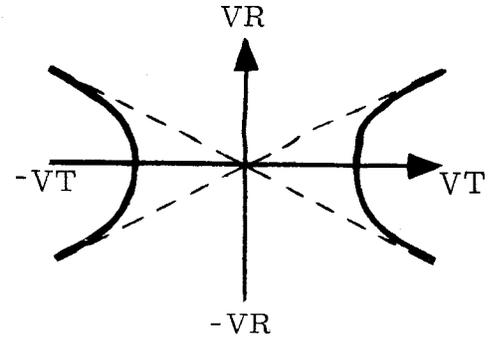
$$\frac{V_T^2}{a^2} - \frac{V_R^2}{b^2} = 1, \quad (4)$$

where

$$b^2 = 2\mu \left(\frac{1}{RR} - \frac{1}{R_0} \right)$$

$$a^2 = \frac{b^2}{R^2 (1 + \cot^2 \gamma_R) - 1}.$$

Thus, the locus of post-impulse velocity vectors which meet the re-entry constraints traces a hyperbola in the hodograph plane which is symmetric about each axis, as shown in the sketch.



Let the initial velocity be defined by $\underline{V}_0 = V_{T0} \underline{i} + V_{R0} \underline{j}$. Then ΔV is the distance between the point (V_{T0}, V_{R0}) and any point on the hyperbola. But the extrema of ΔV occur when $\Delta \underline{V}$ is perpendicular to the hyperbola at the point of intersection. Since orthogonality between two curves is guaranteed when their slopes are the negative inverses of each other at the points of intersection, the equation of the straight line along the extremum $\Delta \underline{V}$ through (V_{T0}, V_{R0}) is

$$V_R - V_{R0} = - \left(\frac{dV_R}{dV_T} \right)^{-1} (V_T - V_{T0}),$$

where
$$\frac{dV_R}{dV_T} = [R^2 (1 + \cot^2 \gamma_R) - 1] \frac{V_T}{V_R}.$$

Since this slope is itself a function of the velocity components, a second order equation results:

$$V_R - V_{R0} = - \frac{V_R}{V_T} \frac{V_T - V_{T0}}{R^2 (1 + \cot^2 \gamma_R) - 1}. \quad (5)$$

$$\text{A more familiar form is } 2(V_T - A)(V_R - B) = K^2, \quad (6)$$

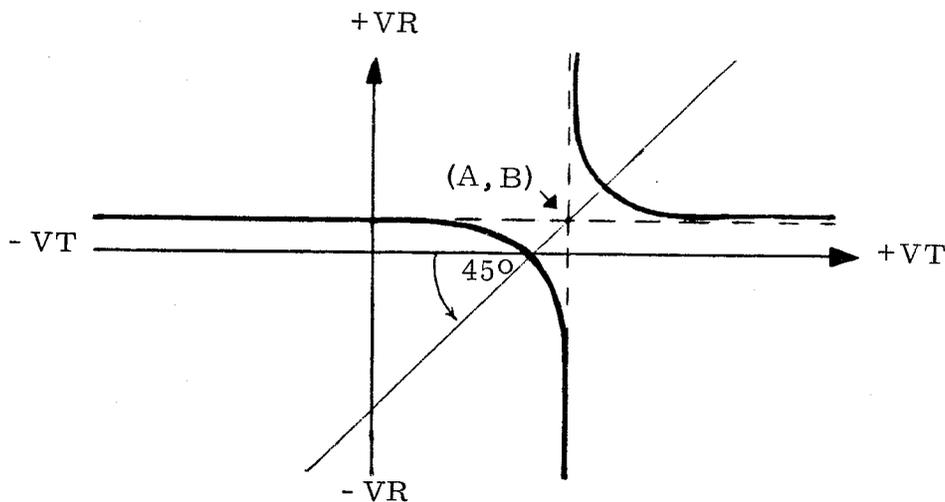
where A , B , and K^2 are constants defined by

$$A = \frac{V_{T0}}{R^2 (1 + \cot^2 \gamma_R)},$$

$$B = V_{R0} \left[1 - \frac{1}{R^2 (1 + \cot^2 \gamma_R)} \right], \text{ and}$$

$$K^2 = \frac{2 V_{T0} V_{R0}}{R^2 (1 + \cot^2 \gamma_R)} \left[1 - \frac{1}{R^2 (1 + \cot^2 \gamma_R)} \right].$$

This is the equation of a hyperbola in the hodograph plane centered at (A, B), whose axes are rotated 45° CCW, as shown in the sketch.



This sketch is valid for all positive VT0 and VR0. A corresponding negative VR0 results in the mirror image of the above sketch about the VT axis. Similarly, a corresponding negative VT0 results in the mirror image of the above sketch about the VR axis. Therefore, analysis of positive VT0 and VR0 will suffice. Comments concerning the extrema ΔV for negative VT0 and/or VR0 will be presented later in the memo.

The intersections of this hyperbola with the locus-of-solutions hyperbola determine the extrema ΔV . Note that the straight-line distance, ΔV , from the point (VT0, VR0) to any intersection of the two hyperbolas is perpendicular to the locus of return-velocity solutions, but that the hyperbolas themselves are not orthogonal.

Combining Eqs. (3) and (5) results in a closed form for finding the post-impulse velocities which yield extrema in ΔV . Unfortunately the closed form is a fourth-order polynomial:

$$X^4 + C_3 X^3 + C_2 X^2 + C_1 X + C_0 = 0$$

(X being either VR or VT), where the coefficients are non-zero functions of the initial conditions and re-entry constraints. In general, then, there are four extrema, making it necessary to determine which results in the absolute minimum ΔV .

Once the VT, VR pair is found, $\cot \gamma$ is known and ΔV can be calculated from

$$\Delta V = | \underline{V} - \underline{V0} | , \quad (7)$$

where

$$\underline{V} = \frac{1}{R0} (\mu p)^2 (\cot \gamma \text{UNIT}[\underline{R0}] + \text{UNIT}[(\underline{R0} * \underline{V0}) * \underline{R0}]), \quad (8)$$

and p is calculated by Eq. (1).

Much can be learned from the hodograph plots without having to solve the polynomial. For $VT0 > 0$, one extremum ΔV appears in the negative -VT half-plane of the hodograph, thereby requiring a retrograde impulse. This solution, though a relative minimum, always requires more ΔV than a solution in the positive -VT half-plane. Thus, this solution can be discarded, and only roots in the right half-plane need be examined.

If the vertical asymptote of the hyperbola of Eq. (6) does not intersect the locus-of-solution hyperbola, two more roots are eliminated, leaving only one ΔV extremum, the absolute minimum. Since this asymptote is located at

$$VT = \frac{VT0}{R^2(1 + \cot^2 \gamma_R)}$$

and the $VR = 0$

crossing of the first hyperbola is at $VT = +a$, the criterion that only one minimum exists is:

$$a > \frac{VT0}{R^2(1 + \cot^2 \gamma_R)},$$

or

$$\frac{2\mu \left(\frac{1}{RR} - \frac{1}{R0} \right)}{R^2(1 + \cot^2 \gamma_R) - 1} > \left[\frac{VT0}{R^2(1 + \cot^2 \gamma_R)} \right]^2 .$$

Since γ_R is near 90° , $\cot^2 \gamma_R$ can be neglected with respect to 1, reducing the criterion to

$$\frac{R^2}{R+1} > \frac{VT_0^2}{VP^2}$$

where

$$VP^2 = \frac{2\mu}{R_0}$$

Using parabolic velocity as an approximate upper bound for an acceptable re-entry velocity, the criterion reduces to $R^2 - R - 1 > 0$. Thus a sufficient, but not necessary condition that only one extremum exists is that the ratio of initial radius to re-entry radius be greater than the solution to this quadratic (which, coincidentally, is the Golden Ratio).

Therefore, under all conditions, when R_0 is greater than approximately 6500 miles, there is only one extremum.

Figure 1 is a section of the right half of a hodograph plane with the above hyperbolas plotted for a set of conditions for which all three extrema exist. VR_0 is positive. The absolute minimum ΔV is always the vector from the point (VT_0, VR_0) to the point where the upper branch of the hyperbola of Eq. (6) intersects the locus-of-solutions hyperbola. The other two extrema, when they exist, are produced by the intersections of the lower branch of the hyperbola of Eq. (6) and the $VT > 0$ branch of the locus-of-solutions hyperbola. The intersection point which results in the smaller $|VR|$ produces a local maximum ΔV , while the other produces a local minimum. If the lower branch were tangent to the locus-of-solutions hyperbola, an inflection point in a ΔV vs. $\cot \gamma$ curve would be produced. Therefore, the necessary and sufficient condition that only one extremum, the absolute minimum, exists is that this point of tangency not be reached. This point is a function of R_0 , V_0 , and VR_0 . Loci of this point with constant R_0 were plotted in Fig. 2. Only the initial velocity conditions below the line of constant R_0 have multiple ΔV extrema in the positive - VT half of the hodograph plane. Thus for FEASIBLE initial velocity, only one extrema exists above $R_0 = 5400$ miles.

Physically, the absolute minimum ΔV solution results in a return trajectory which leaves R_0 with a positive flight-path angle, goes out to apogee, and then returns to re-entry. The local minimum ΔV solution, due to the negative path angle at R_0 , returns the spacecraft to re-entry altitude directly. There are two serious problems associated with the

absolute minimum solution in some cases. Since the locus-of-solutions equation was derived from purely geometric considerations of conics, closure of parabolas and hyperbolas through infinity were not ruled out. Therefore, it is possible for the absolute minimum solution to result in an escape trajectory which would not return to re-entry. This is the case whenever $\cot \gamma \geq [R(1 + \cot^2 \gamma_R) - 1]^{1/2}$.¹ The other problem involves a negative VR solution. If the resulting path angle at R0 is less than a prescribed negative bound, the maximum re-entry velocity constraint is violated.

ITERATIVE SEARCH APPROACH

It appears that an iterative search would be a better approach in finding the minimum ΔV . The obvious independent parameter for the iteration is $\cot \gamma$. The only equations in the iteration loop are (7), (8), and (1).

POSITIVE VT0 AND VR0 CASE

There are two advantages for using $\cot \gamma$ as the independent parameter: a) it is used directly in the equations, and b) its upper bound, $[R(1 + \cot^2 \gamma_R) - 1]^{1/2}$ serves as the starting point for the iteration. Since the absolute minimum ΔV solution comes from a $\cot \gamma$ which is very near this value, the number of iterations required for convergence is small. Also, for feasible initial velocities, if the absolute minimum solution requires that the return trajectory close through infinity, the iteration method will converge on the physically realizable solution which is closest to this analytic minimum. This solution lies just within the upper bound of $\cot \gamma$, thereby requiring only one iteration to find it.

NEGATIVE VR0 CASE

If VR0 is negative, the roots are the same as those generated by the corresponding positive VR0 except that the sign of VR is reversed. The magnitudes of the ΔV extrema are unchanged. The absolute minimum solution always results in a direct return trajectory and therefore is always

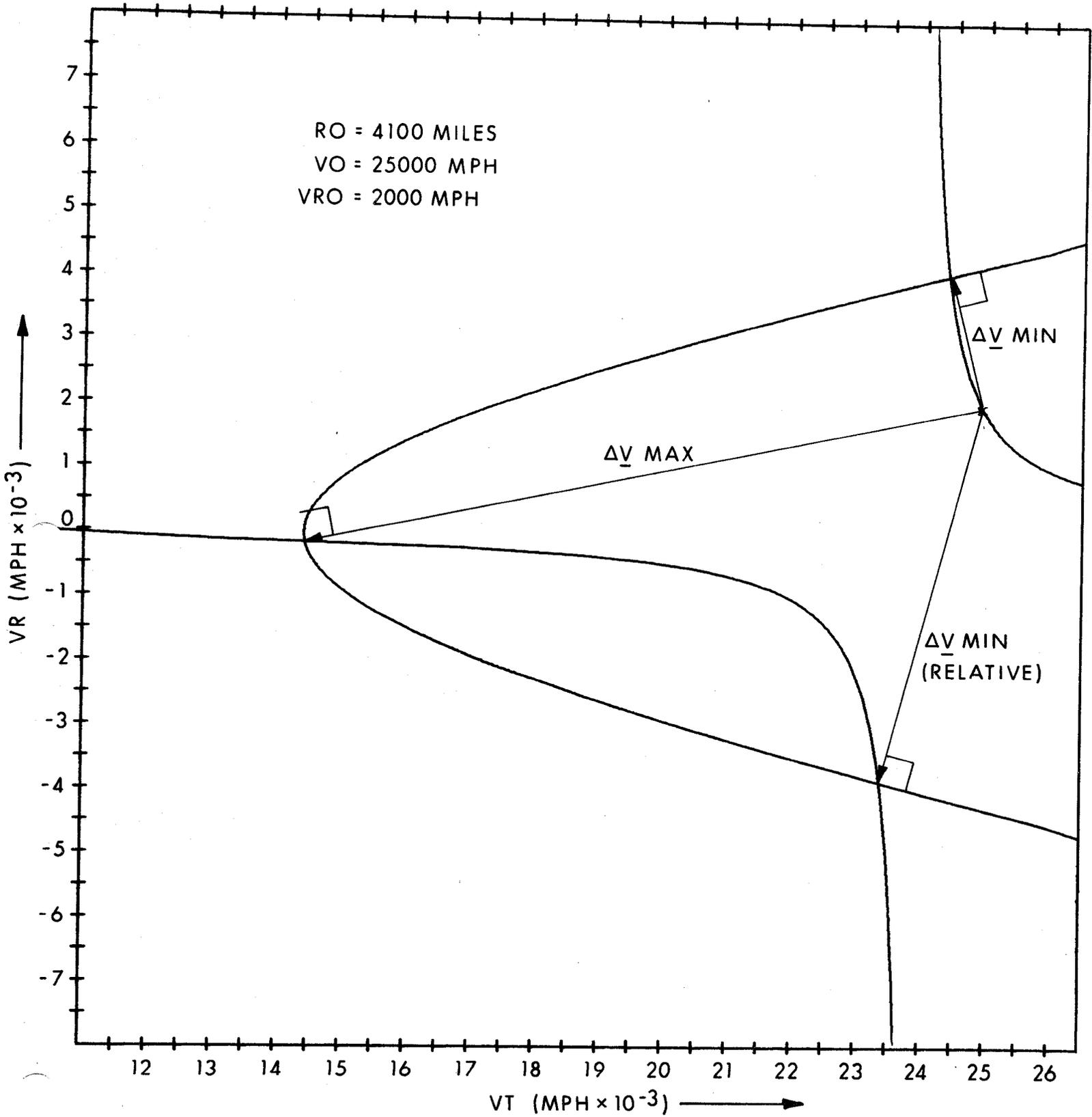
1. Marscher, W.F., "A Unified Method of Generating Conic Sections," R-479, MIT Instrumentation Laboratory, Cambridge, Mass. February 1965, pp. 16-17.

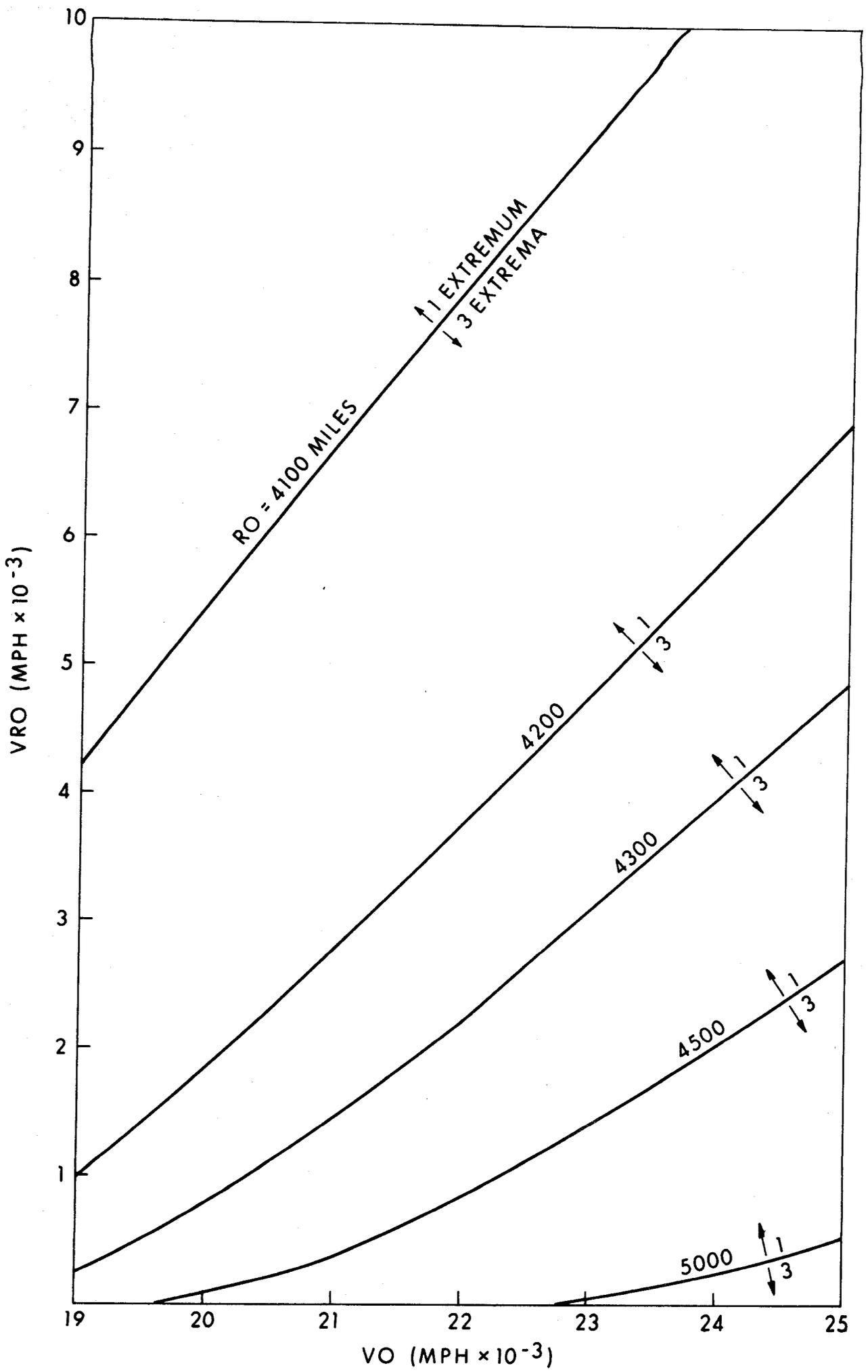
physically realizable. The local minimum solution, when it exists (again disregarding the extraneous root in the left-half plane), results in the "return-through-apogee" trajectory. The starting $\cot \gamma$ for the iteration is governed by the maximum re-entry velocity constraint. With this exception, the iteration procedure is identical to the positive VR_0 case.

NEGATIVE VT_0 CASE

Since positive VT_0 is the direction of the nominal tangential velocity of the spacecraft, the occurrence of negative VT_0 , although possible after lunar encounter and erroneous thrust steering, is rare. The roots are the same as those generated by the corresponding positive VT_0 except that the sign of VT is reversed. The iteration procedure is identical to that for the positive VT_0 case. The only physical difference is that the absolute minimum $\Delta \underline{V}$ results in a re-entry velocity component opposite to the direction of the earth's air mass velocity. If this is undesirable, the sign of the second unit vector in Eq. (8) must be changed. This will cause the iteration to converge on the only extremum along the positive - VT branch of the locus-of-solution hyperbola. This extremum is the minimum $\Delta \underline{V}$ which will produce a re-entry velocity component along the direction of motion of the earth's air mass.

FIGURE 1. ΔV EXTREMA





APPENDIX

A Proposed Iteration Scheme

The flow diagram of Fig. A-1 illustrates an initialization procedure necessary for implementing the iteration technique. "START" represents the initial $\cot \gamma$ to be used in the iteration loop encompassing Eqs. (1), (7), and (8). "LIMIT" is determined by the maximum re-entry velocity constraint. "MAX" is the upper bound of $\cot \gamma$. In the memo this was the parabolic limit. If it is not desirable for the trajectory to enter the moon's sphere of influence, "MAX" is determined by the distance of the sphere's edge from the earth. Eq. (8) must be modified to contain the factor "SIGN" preceding the second term of the paranthetic expression. This provides for a retrograde return velocity if negative VT at re-entry is undersirable.

If VR0 is positive at low altitudes, a option is provided to search for a solution which returns the spacecraft directly to re-entry. Although this requires more fuel than the solution which returns the spacecraft through apogee, the shorter time-of-flight may warrant the extra usage of fuel. It is possible that this direct-return, relative minimum solution may not exist, in which case the iteration will automatically converge on the absolute minimum solution.

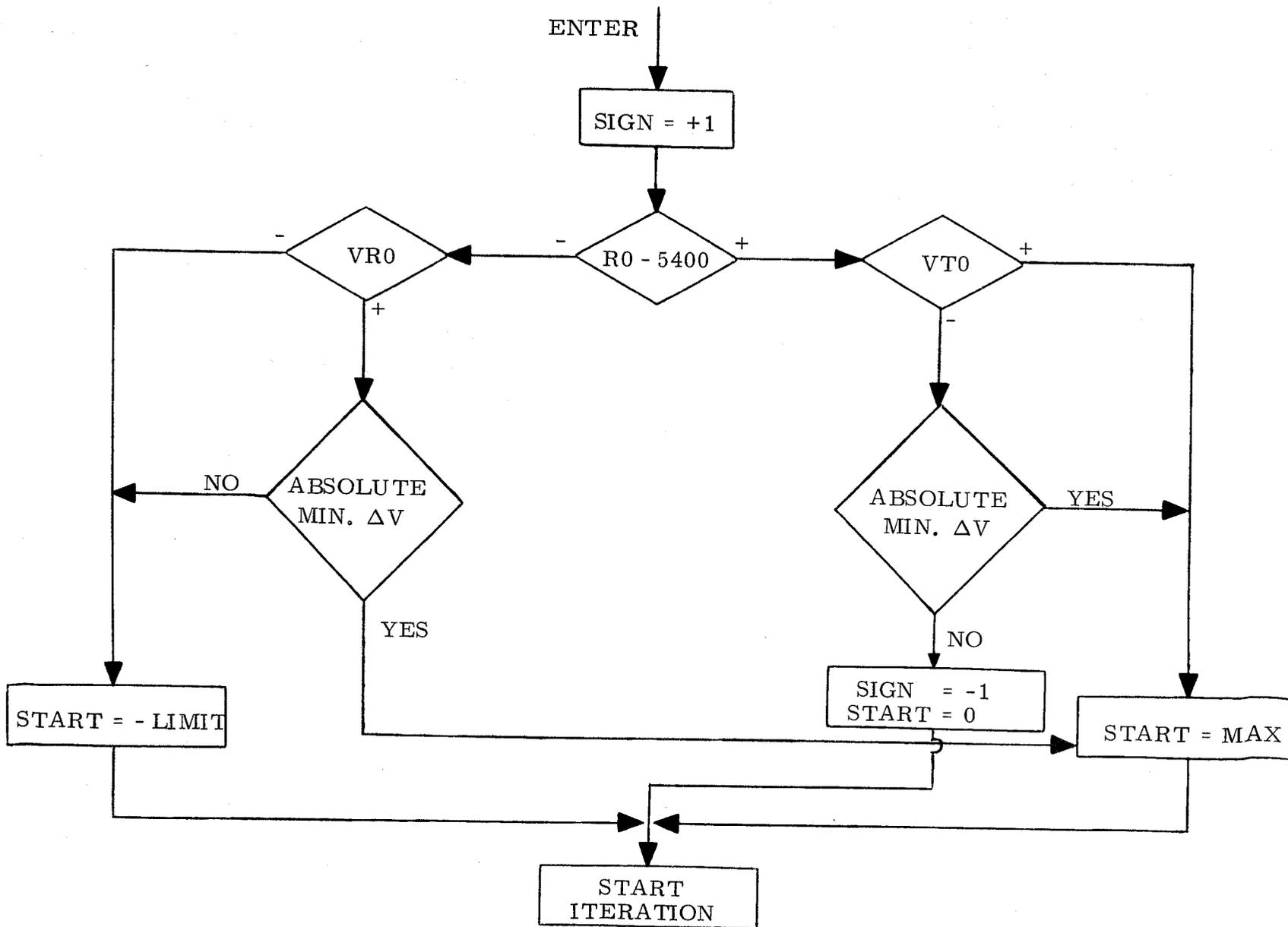


FIGURE A-1 INITIALIZATION LOGIC