SPACE SHUTTLE GN&C EQUATION DOCUMENT

No. 3-71

Conic State Extrapolation

by

William M. Robertson

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M.I.T. Charles Stark Draper Laboratory

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FOREWORD

This document is one of a series of candidates for inclusion in the NASA Space Shuttle GN&C Equation Document. The enclosed has been prepared under NAS9-10268, Task No. 26, "GN&C Demonstration Software Support", and applies to function 1 of the Orbital Coast Navigation Module (ON2) as defined in MSC-03690, "Space Shuttle Guidance, Navigation and Control Software Functional Requirements", dated 12 October 1970.

Gerald M. Levine

Gerald M. Levine
Director,
APOLLO Space Guidance Analysis Division

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NOMENCLATURE

a	Semi-major axis of conic
c_1	First conic parameter $(\underline{r}_0 \cdot \underline{v}_0) / \mu_E$
c_2	Second conic parameter $(r_0 v_0^2 / \mu_E - 1)$
c_3	Third conic parameter $(r_0 v_0^2 / \mu_E)$
$C(\xi)$	Power series in ξ defined in text
E	Eccentric anomaly
f	True anomaly
H	Hyperbolic analog of eccentric anomaly
i	Counter
p	Semilatus rectum of conic
p_N	Normalized semilatus rectum (p / r_0)
P	Period of conic orbit
r_0	Magnitude of $\underline{r}_0$
$\underline{r}_0$	Inertial position vector corresponding to initial time t_0
r	Magnitude of $\underline{r}(t)$
$\underline{r}(t)$	Inertial position vector corresponding to time t
s	Switch used in Secant Iterator to determine whether secant method or offsetting will be performed

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r	Magnitude of $\underline{r}(t)$
$\underline{r}(t)$	Inertial position vector corresponding to time t
s	Switch used in Secant Iterator to determine whether secant method or offsetting will be performed

$S(\xi)$	Power series in ξ defined in text
t	Final time (end of time interval through which an extrapolation is made)
t_0	Initial time (beginning of time interval through which an extrapolation is to be made)
$(t - t_0)$	Specified transfer time interval
$(t - t_0)_c$	Value of the transfer time interval calculated in the Universal Kepler Equation as a function of x and the conic parameters
$(t - t_0)'_c$	Previous value of $(t - t_0)_c$
$(t - t_0)_c^{(i)}$	The "i-th" value of the transfer time interval calculated in the Universal Kepler Equation as a function of the "i-th" value x_i of x and the conic parameters
t_{ERR}	Difference between specified time interval and that calculated by Universal Kepler Equation
v_0	Magnitude of $\underline{v}_0$
$\underline{v}_0$	Inertial velocity vector corresponding to initial time t_0
$\underline{v}(t)$	Inertial velocity vector corresponding to time t
x	Universal eccentric anomaly difference (independent variable in Kepler iteration scheme)
x'	Previous value of x
x_c	Value of x to which the Kepler iteration scheme converged
x'_c	Previous value of x_c

x_i	The "i-th" value of x
x_{min}	Lower bound on x
x_{max}	Upper bound on x
α_0	Reciprocal of semi-major axis at initial point $\underline{r}_0$
α_N	Normalized semi-major axis reciprocal (αr_0)
γ_0	Angle from $\underline{r}_0$ to $\underline{v}_0$
Δt_{max}	Maximum time interval which can be used in computer due to scaling limitations
Δx	Increment in x
ϵ_t	Relative convergence tolerance factor on transfer time interval
ϵ_x	Convergence tolerance on independent variable x
θ	Transfer angle (true anomaly increment)
μ_E	Gravitational parameter of the earth
ξ	Product of α_0 and square of x
$\chi_0, \chi_1, \chi_2, \chi_3$	Coefficients of power series inversion of Universal Kepler Equation
$\underline{1}_{r_0}$	Unit vector in direction of $\underline{r}_0$
$\underline{1}_{v_0}$	Unit vector in direction of $\underline{v}_0$

1. INTRODUCTION

The Conic State Extrapolation Routine provides the capability to conically extrapolate any spacecraft inertial state vector either backwards or forwards as a function of time or as a function of transfer angle. It is merely the coded form of two versions of the analytic solution of the two-body differential equations of motion of the spacecraft center of mass. Because of its relatively fast computation speed and moderate accuracy, it serves as a preliminary navigation tool and as a method of obtaining quick solutions for targeting and guidance functions. More accurate (but slower) results are provided by the Precision State Extrapolation Routine.

2. FUNCTIONAL FLOW DIAGRAM

The Conic State Extrapolation Routine basically consists of two parts – one for extrapolating in time and one for extrapolating in transfer angle. Several portions of the formulation are, however, common to the two parts, and may be arranged as subroutines on a computer.

2.1 Conic State Extrapolation As A Function Of Time (Kepler Routine)

This routine involves a single loop iterative procedure, and hence is organized in three sections: initialization, iteration, and final computations, as shown in Fig. 1. The variable "x" is the independent variable in the iteration procedure. For a given initial state, the variable "x" measures the amount of transfer along the extrapolated trajectory. The transfer time interval and the extrapolated state vector are very conveniently expressed in terms of "x". In the iteration procedure, "x" is adjusted until the transfer time interval calculated from it agrees with the specified transfer time interval (to within a certain tolerance). Then the extrapolated state vector is calculated from this particular value of "x".

2.2 Conic State Extrapolation As A Function Of Transfer Angle (Theta Routine)

This routine makes a direct calculation (i.e. does not have an iteration scheme), as shown in Fig. 2. Again, the extrapolated state vector is calculated from the parameter "x". The value of "x" however, is obtained from a direct computation in terms of the conic parameters and the transfer angle θ. It is not necessary to iterate to determine "x", as was the case in the Kepler Routine.

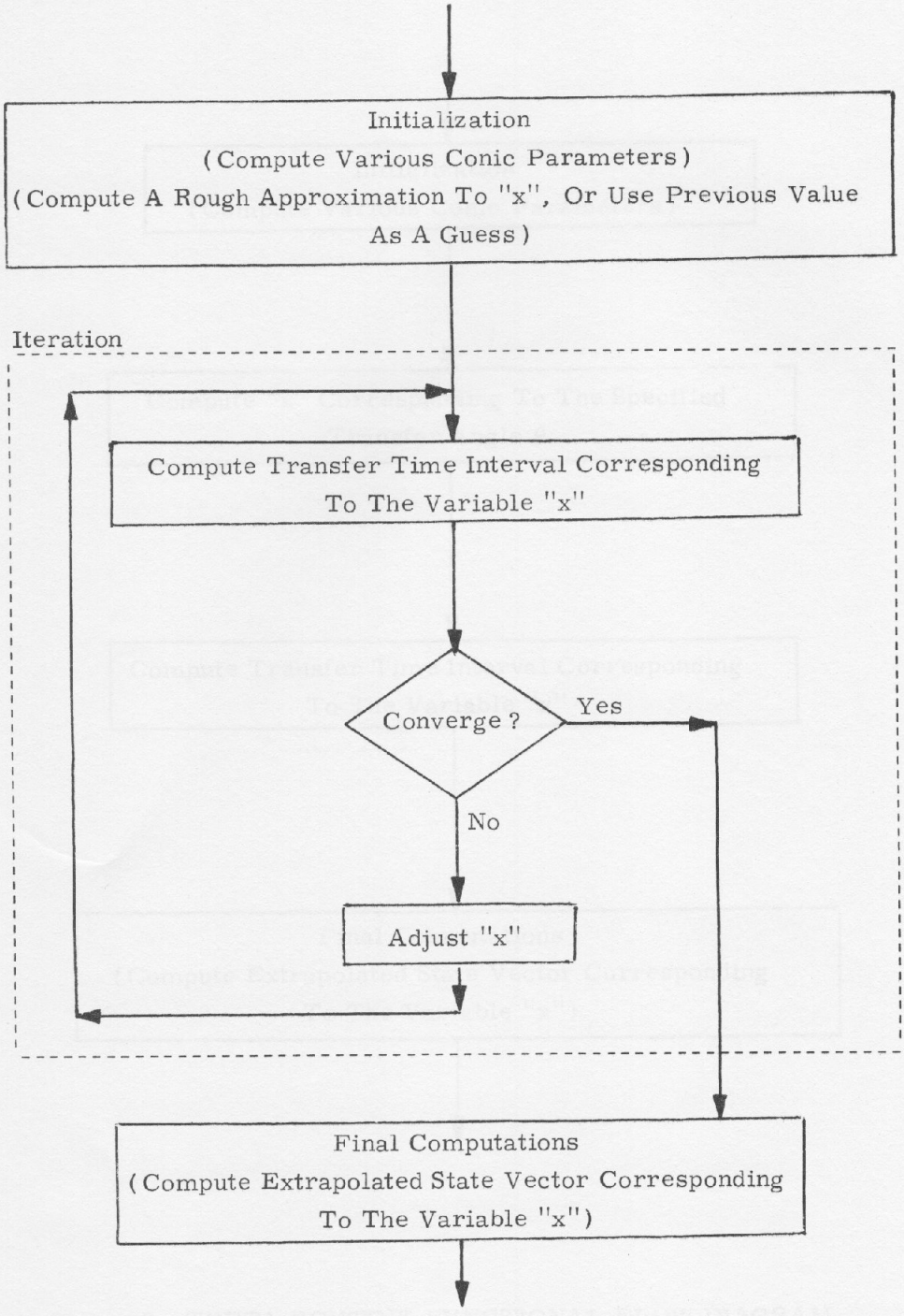

Figure 1 KEPLER ROUTINE FUNCTIONAL FLOW DIAGRAM

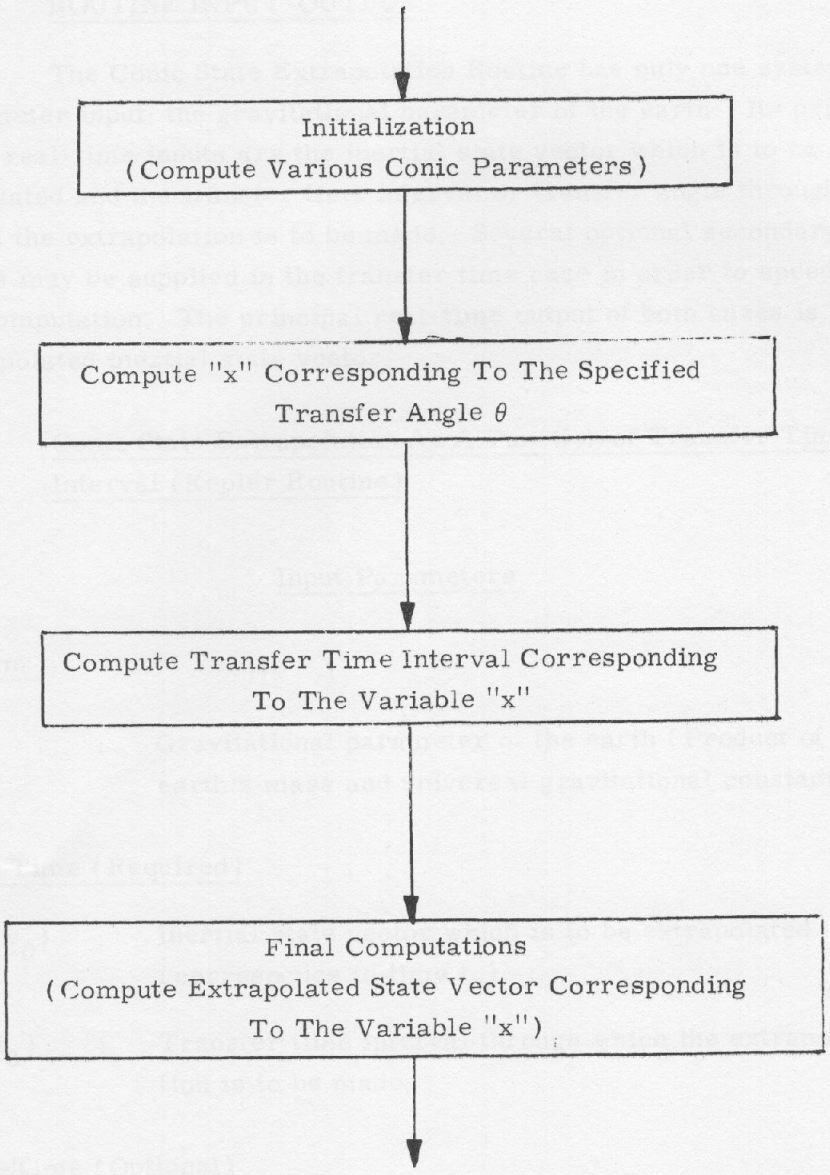

Figure 2 THETA ROUTINE FUNCTIONAL FLOW DIAGRAM

3. ROUTINE INPUT-OUTPUT

The Conic State Extrapolation Routine has only one system parameter input: the gravitational parameter of the earth. Its principal real-time inputs are the inertial state vector which is to be extrapolated and the transfer time interval or transfer angle through which the extrapolation is to be made. Several optional secondary inputs may be supplied in the transfer time case in order to speed the computation. The principal real-time output of both cases is the extrapolated inertial state vector.

3.1 Conic State Extrapolation As A Function of Transfer Time Interval (Kepler Routine)

Input Parameters

System

μ_E : Gravitational parameter of the earth (Product of earth's mass and universal gravitational constant).

Real-Time (Required)

$(\underline{r}_0, \underline{v}_0)$: Inertial state vector which is to be extrapolated (corresponds to time t_0).

$(t - t_0)$: Transfer time interval through which the extrapolation is to be made.

Real-Time (Optional)

x : Guess of independent variable corresponding to solution in Kepler iteration scheme. (Used to speed convergence).

$(t - t_0)'_c$: Value of dependent variable (the transfer time interval) in the Kepler iteration scheme, which was calculated in the last iteration of the previous call to Kepler.

x'_c : Value of the independent variable in the Kepler iteration scheme, to which the last iteration of the previous call to Kepler had converged.

Output Parameters

$(\underline{r}(t), \underline{v}(t))$: Extrapolated inertial state vector (corresponds to time t).

$(t - t_0)_c$: Value of the dependent variable (the transfer time interval) in the Kepler iteration scheme, which was calculated in the last iteration (should agree closely with $(t - t_0)$).

x_c : Value of the independent variable in the Kepler iteration scheme to which the last iteration converged.

3.2 Conic State Extrapolation As A Function Of Transfer Angle (Theta Routine)

Input Parameters

System

μ_E : Gravitational parameter of the earth (Product of earth's mass and universal gravitational constant).

Real-Time

$(\underline{r}_0, \underline{v}_0)$: Inertial state vector which is to be extrapolated.

θ : Transfer angle through which the extrapolation is to be made.

Output Parameters

$(\underline{r}, \underline{v})$: Extrapolated inertial state vector.

$(t - t_0)_c$: Transfer Time Interval corresponding to the conic extrapolation through the transfer angle θ.

4. DESCRIPTION OF EQUATIONS

4.1 Conic State Extrapolation As A Function Of Time (Kepler Routine)

The universal formulation of Stumpff-Herrick-Battin in terms of the universal eccentric anomaly difference is used. This variable, usually denoted by x, is defined by the relations:

$$x = \begin{cases} \sqrt{a}\,(E - E_0) \text{ for ellipse} \\ \sqrt{p}\,(\tan f/2 - \tan f_0/2) \text{ for parabola} \\ \sqrt{-a}\,(H - H_0) \text{ for hyperbola} \end{cases}$$

where a is the semi-major axis, E and H are the eccentric anomaly and its hyperbolic analog, p is the semi-latus rectum and f the true anomaly. The expressions for the transfer time interval $(t - t_0)$ and the extrapolated position and velocity vectors $(\underline{r}, \underline{v})$ in terms of the initial position and velocity vectors $(\underline{r}_0, \underline{v}_0)$ as functions of x are:

(Universal Kepler Equation)

$$(t - t_0) = \frac{1}{\sqrt{\mu_E}}\left[\frac{\underline{r}_0 \cdot \underline{v}_0}{\sqrt{\mu_E}} x^2 C(\alpha_0 x^2) + (1 - r_0 \alpha_0) x^3 S(\alpha_0 x^2) + r_0 x\right]$$

$$\underline{r}(t) = \left[1 - \frac{x^2}{r_0} C(\alpha_0 x^2)\right]\underline{r}_0 + \left[(t - t_0) - \frac{x^3}{\sqrt{\mu_E}} S(\alpha_0 x^2)\right]\underline{v}_0$$

$$\underline{v}(t) = \frac{\sqrt{\mu_E}}{r\, r_0}\left[\alpha_0 x^3 S(\alpha_0 x^2) - x\right]\underline{r}_0 + \left[1 - \frac{x^2}{r} C(\alpha_0 x^2)\right]\underline{v}_0$$

where

$$\alpha_0 = \frac{1}{a_0} = \frac{2}{r_0} - \frac{v_0^2}{\mu_E}$$

and

$$S(\xi) = \frac{1}{3!} - \frac{\xi}{5!} + \frac{\xi^2}{7!} - \ldots$$

$$C(\xi) = \frac{1}{2!} - \frac{\xi}{4!} + \frac{\xi^2}{6!} - \ldots$$

Since the transfer time interval $(t - t_0)$ is given, it is desired to find the x corresponding to it in the Universal Kepler Equation, and then to evaluate the extrapolated state vector $(\underline{r}, \underline{v})$ expression using that value of x. Unfortunately, the Universal Kepler Equation expresses $(t - t_0)$ as a transcendental function of x rather than conversely, and no power series inversion of the equation is known which has good convergence properties for all orbits, so it is necessary to solve the equation iteratively for the variable x.

For this purpose, the secant method (linear inverse interpolation/ extrapolation) is used. It merely finds the increment in the independent variable x which is required in order to adjust the dependent variable $(t - t_0)_c$ to the desired value $(t - t_0)$ based on a linear interpolation/extrapolation of the last two points calculated on the $(t - t_0)_c$ vs x curve. The method uses the formula

$$(x_{n+1} - x_n) = - \frac{(t - t_0)_c^{(n)} - (t - t_0)}{(t - t_0)_c^{(n)} - (t - t_0)_c^{(n-1)}} (x_n - x_{n-1})$$

where $(t - t_0)_c^{(i)}$ denotes the evaluation of the Universal Kepler Equation using the value x_i. In order to prevent the scheme from taking an increment back into regions in which it is known from past iterations that the solution does not lie, it has been found convenient to establish upper and lower bounds on the independent variable x which are continually reset during the course of the iteration as more and more values of x are found to be too large or too small. In addition, it has also been found expedient to damp by 10% any increment in the independent variable which would (if applied) take the value of the independent variable past a bound.

To start the iteration scheme, some initial guess x_0 of the independent variable is required as well as a previous point $(x_{-1}, (t - t_0)_c^{(-1)})$ on the $(t - t_0)_c$ vs x curve. If no previous point is available the point (0, 0) may be used as it lies on all $(t - t_0)_c$ vs. x curves. The closer the initial guess x_0 is to the value of x corresponding to the solution, the faster the convergence will be. One method of obtaining such a guess x_0 is to use a truncation of the infinite series obtained by direct inversion of the Kepler Equation (expressing x as a power series in $(t - t_0)$). It must be pointed out that this series diverges even for "moderate" transfer time intervals $(t - t_0)$; hence an iterative solution must be used to solve the Kepler equation for x in the general case. A third order truncation of the inversion of the Universal Kepler Equation is:

$$x = \sum_{n=0}^{3} \chi_n (t - t_0)^n$$

where

$$\chi_0 = 0, \quad \chi_1 = \sqrt{\mu_E}/r_0,$$

$$\chi_2 = -\frac{1}{2}\frac{\mu_E}{r_0^3}\left(\frac{\underline{r}_0 \cdot \underline{v}_0}{\sqrt{\mu_E}}\right),$$

$$\chi_3 = \frac{1}{6r_0}\left(\frac{\sqrt{\mu_E}}{r_0}\right)^3\left[\frac{3}{r_0}\left(\frac{\underline{r}_0 \cdot \underline{v}_0}{\sqrt{\mu_E}}\right)^2 - (1 - r_0\alpha_0)\right],$$

with $\alpha_0 = 2/r_0 - v_0^2/\mu_E$.

4.2 Conic State Extrapolation As A Function of Transfer Angle (Theta Routine)

As with the Kepler Routine, the universal formulation of Stumpff-Herrick-Battin in terms of the universal eccentric anomaly difference x is used in the Theta Routine. A completely analogous iteration scheme could have been formulated with x again as the independent variable and the transfer angle θ as the dependent variable using Marscher's universally valid equation:

$$\cot\frac{\theta}{2} = \frac{r_0\left[1 - \alpha_0 x^2 S(\alpha_0 x^2)\right]}{\sqrt{p}\, x\, C(\alpha_0 x^2)} + \cot\gamma_0$$

where

$$p = \left(\frac{r_0 v_0}{\sqrt{\mu_E}}\right)^2 \sin^2\gamma_0$$

and

$$\gamma_0 = \text{angle from } \underline{r}_0 \text{ to } \underline{v}_0.$$

However, in contrast to the Kepler equation, it is possible to invert the Marscher equation into a power series which can be made to converge as rapidly as desired, by means of which x may be calculated as a universal function of the transfer angle θ. Knowing x, we can directly calculate the transfer time interval $(t - t_0)_c$ and subsequently the extrapolated state vectors using the standard formulae.

The sequence of computations in the inversion of the Marscher Equation is asfollows:

Let

$$p_N = p/r_0,\ \alpha_N = \alpha\, r_0$$

and

$$W_1 = \sqrt{p_N}\left(\frac{\sin\theta}{1-\cos\theta} - \cot\gamma_0\right).$$

If

$$|W_1| > 1, \text{ let } V_1 = 1.$$

Let

$$W_{n+1} = +\sqrt{W_n^2 + \alpha_N} + |W_n| \qquad (|W_1| \leq 1)$$

or

$$V_{n+1} = +\sqrt{V_n^2 + \alpha_N (|1/W_1|)^2} + V_n \qquad (|W_1| > 1).$$

Let

$$\omega_n = W_n \qquad (|W_1| \leq 1)$$

or

$$1/\omega_n = (|1/W_1|)/V_n \qquad (|W_1| > 1).$$

Let

$$\Sigma = \frac{2^n}{\omega_n} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \left(\frac{\alpha_N}{\omega_n^2}\right)^j$$

where n is an integer ≥ 4. Then

$$x/\sqrt{r_0} = \begin{cases} \Sigma & (W_1 > 0) \\ 2\pi/\sqrt{\alpha_N} - \Sigma & (W_1 < 0) \end{cases}$$

The above equations have been specifically formulated to avoid certain numerical difficulties.

5. DETAILED FLOW DIAGRAMS

5.1 Conic State Extrapolation As A Function of Time (Kepler Routine)

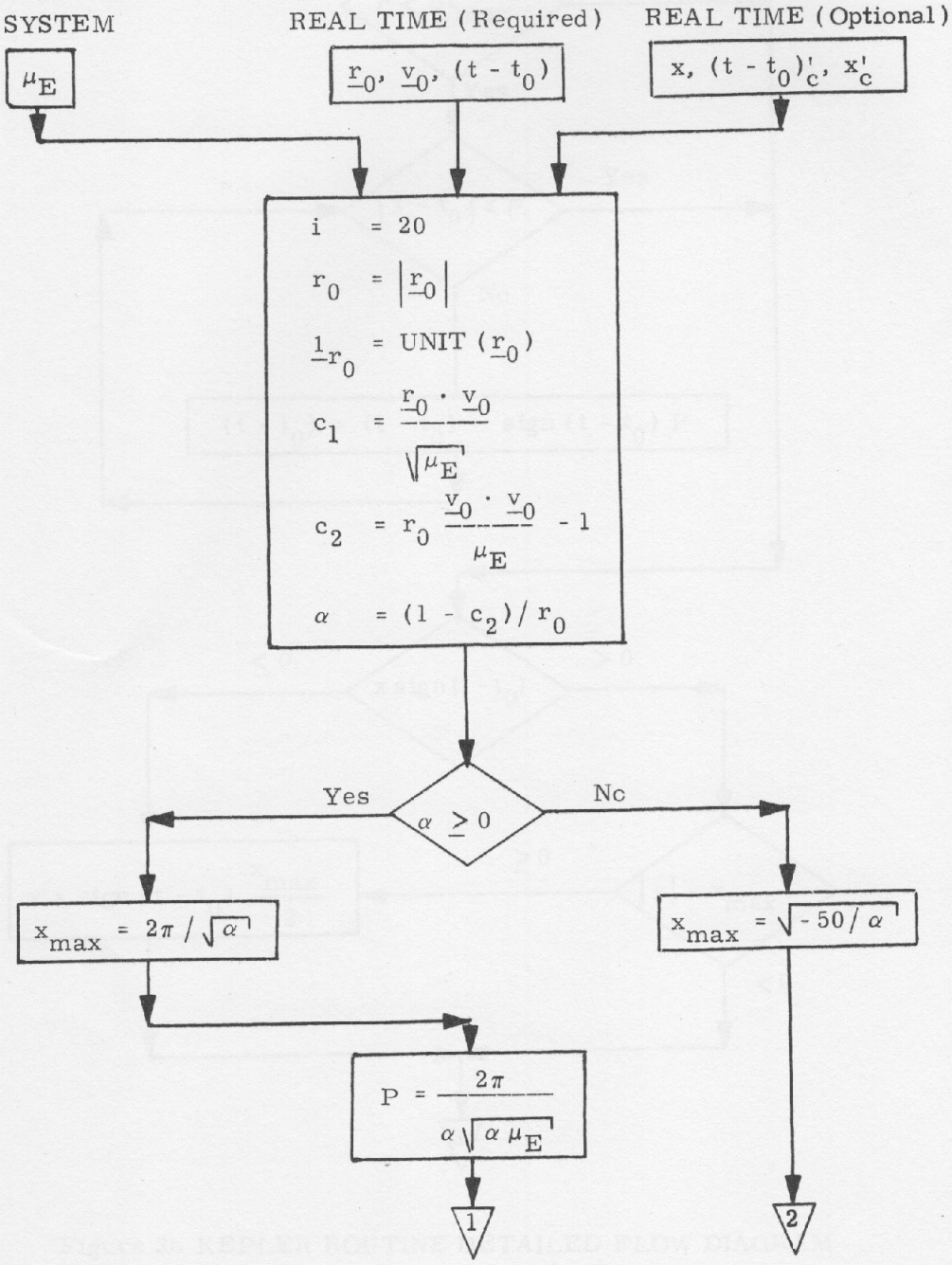

Figure 3a KEPLER ROUTINE DETAILED FLOW DIAGRAM

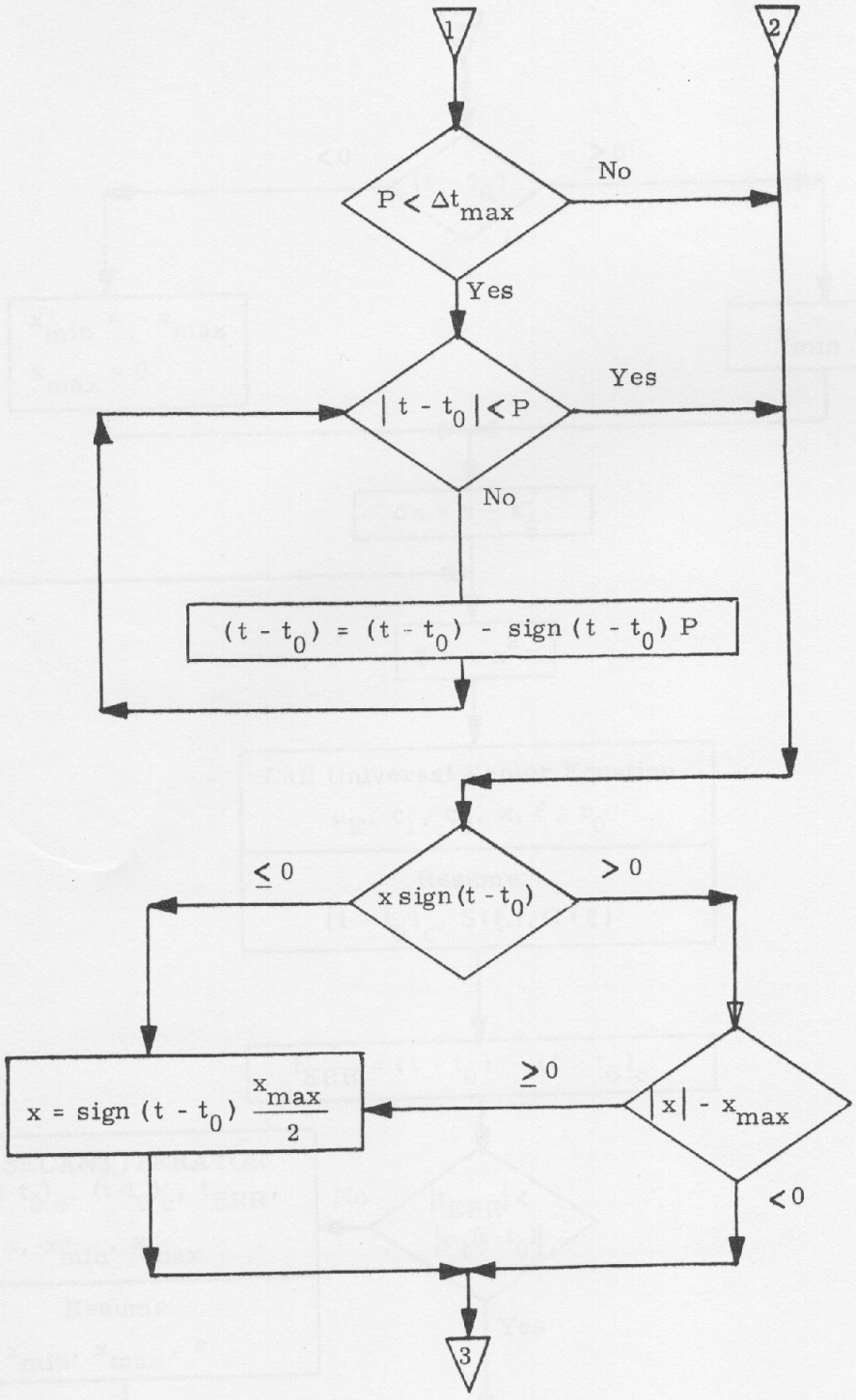

Figure 3b KEPLER ROUTINE DETAILED FLOW DIAGRAM

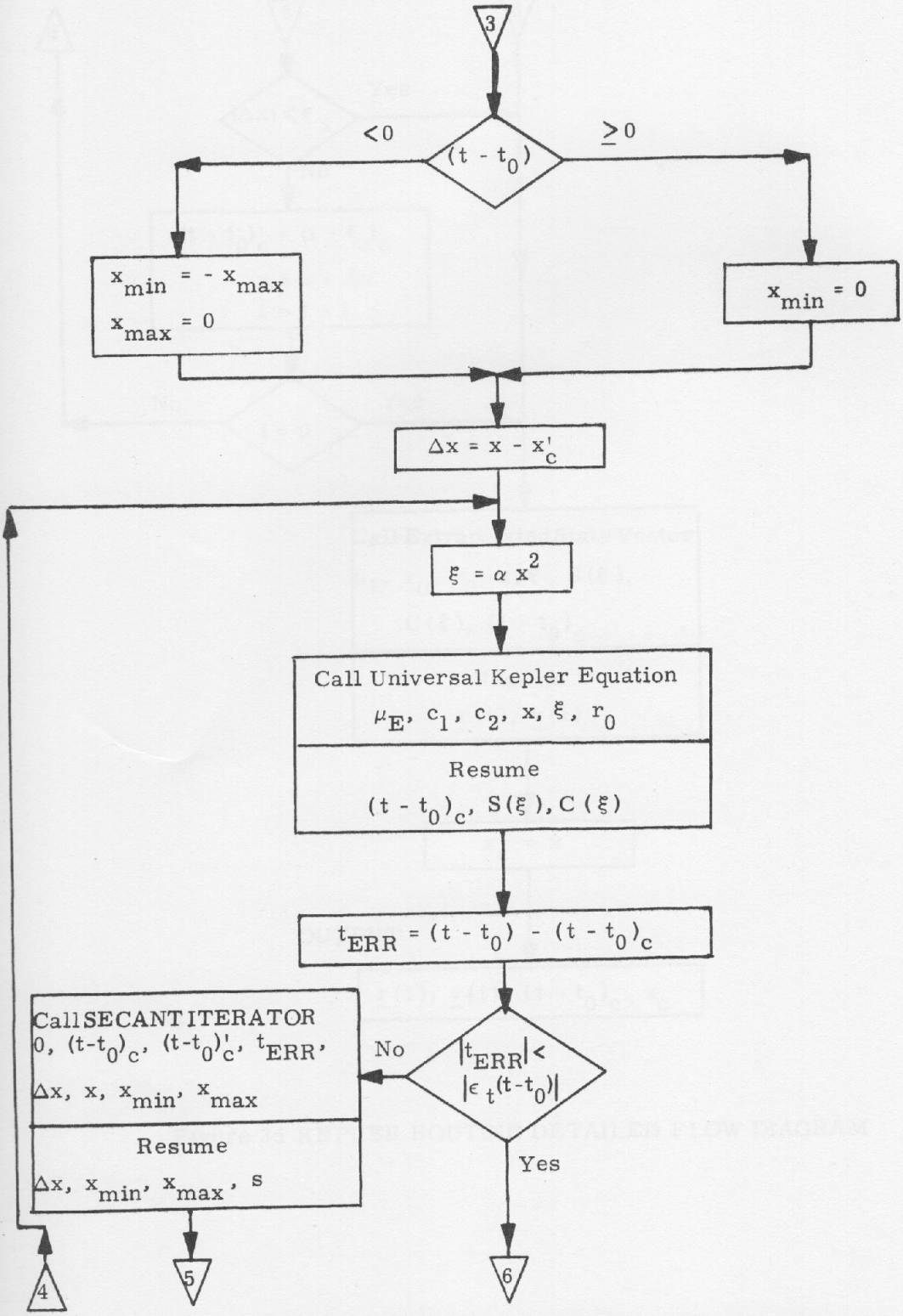

Figure 3c KEPLER ROUTINE DETAILED FLOW DIAGRAM

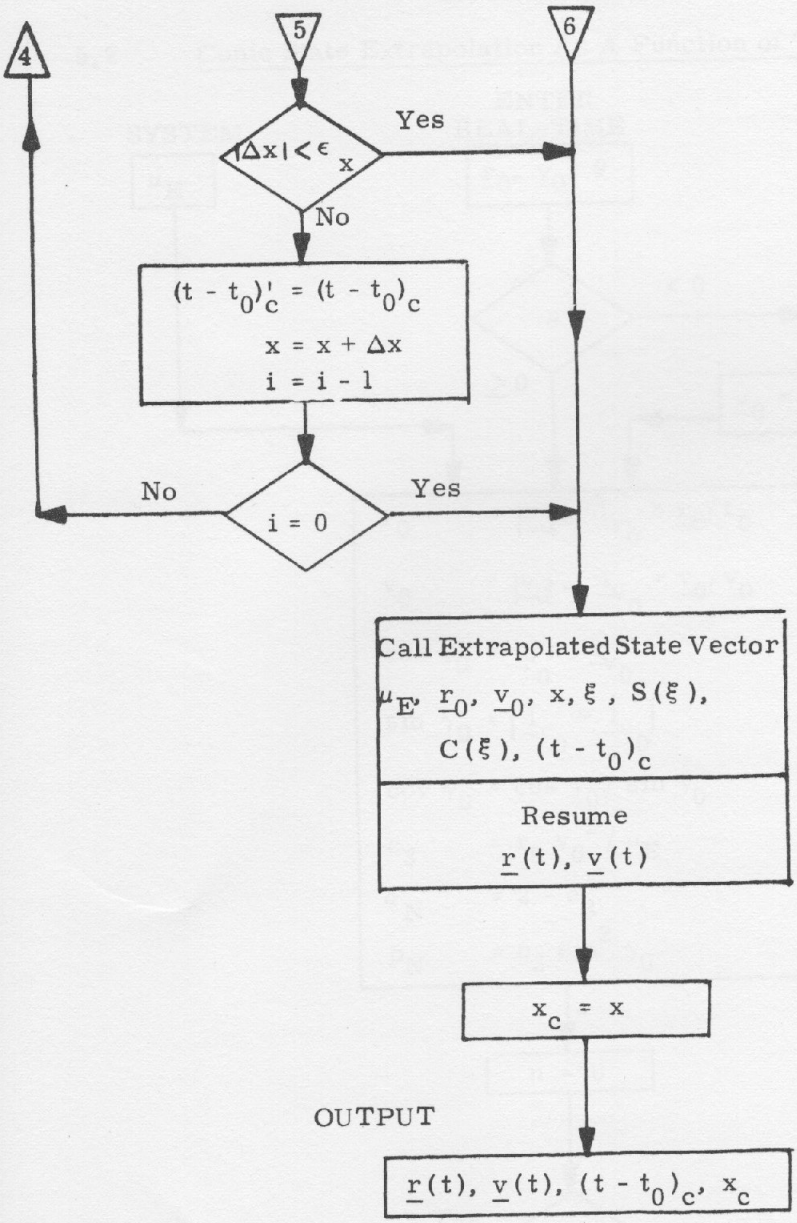

Figure 3d KEPLER ROUTINE DETAILED FLOW DIAGRAM

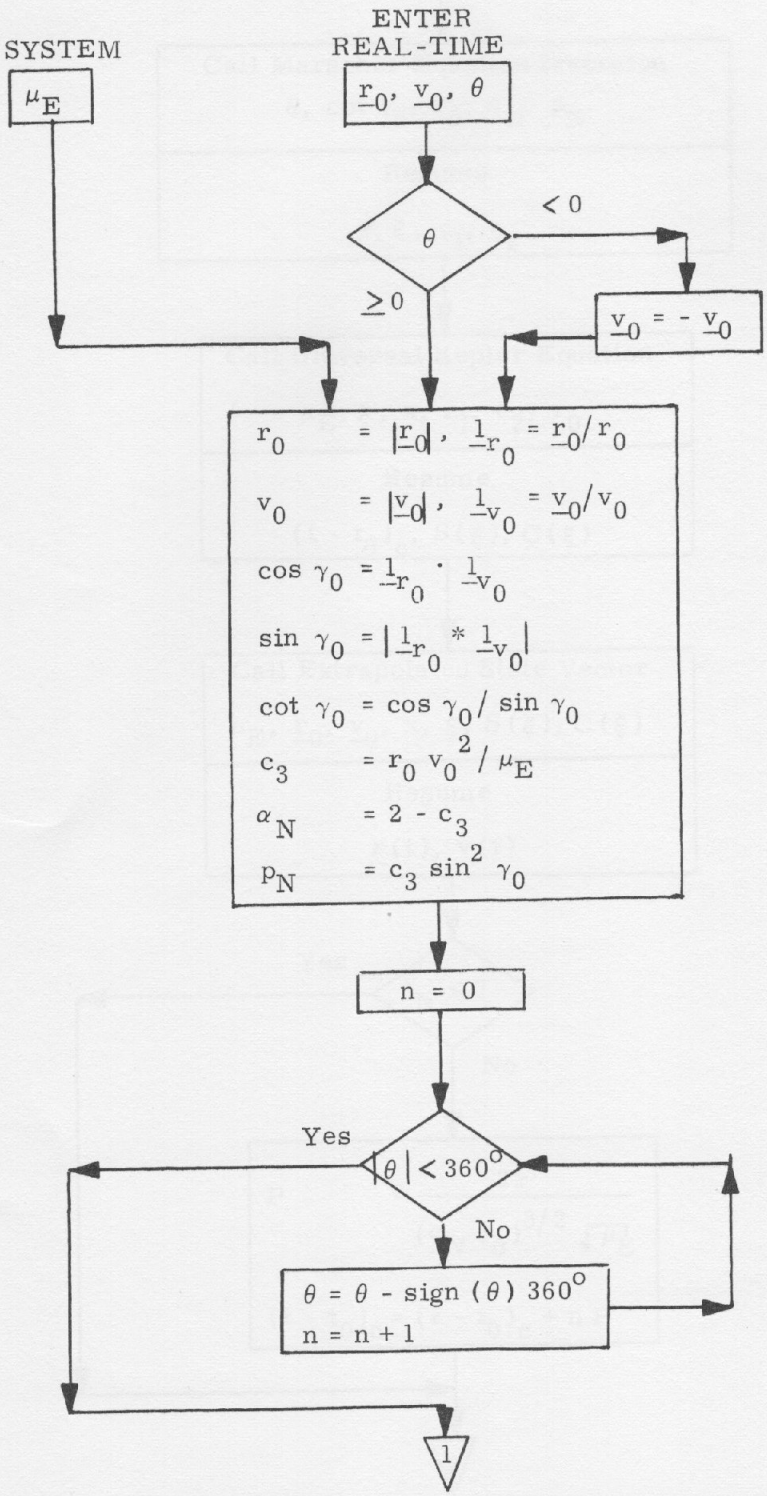

Figure 4a THETA ROUTINE DETAILED FLOW DIAGRAM

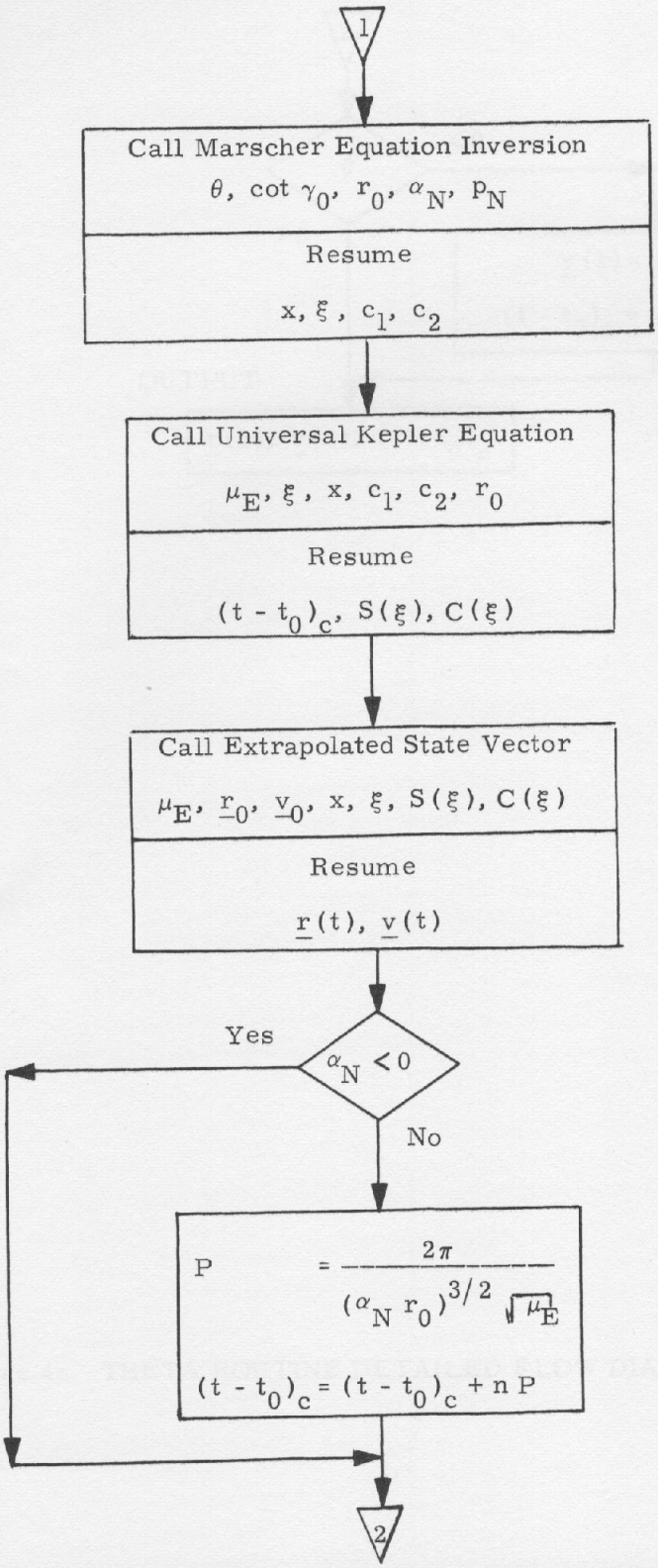

Figure 4b THETA ROUTINE DETAILED FLOW DIAGRAM

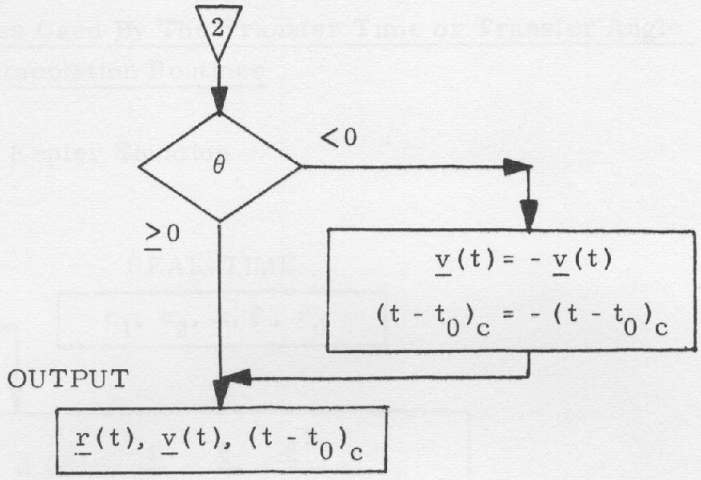

Figure 4c THETA ROUTINE DETAILED FLOW DIAGRAM

5.3 Subroutines Used By The Transfer Time or Transfer Angle Conic Extrapolation Routines

5.3.1 Universal Kepler Equation

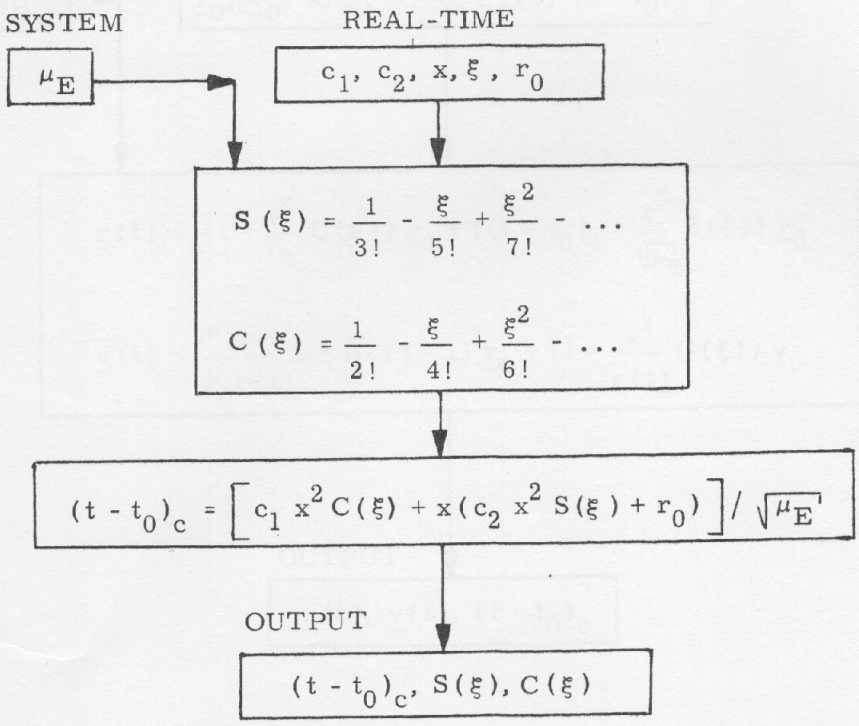

Figure 5 UNIVERSAL KEPLER EQUATION DETAILED FLOW DIAGRAM

5.3.2 Extrapolated State Vector

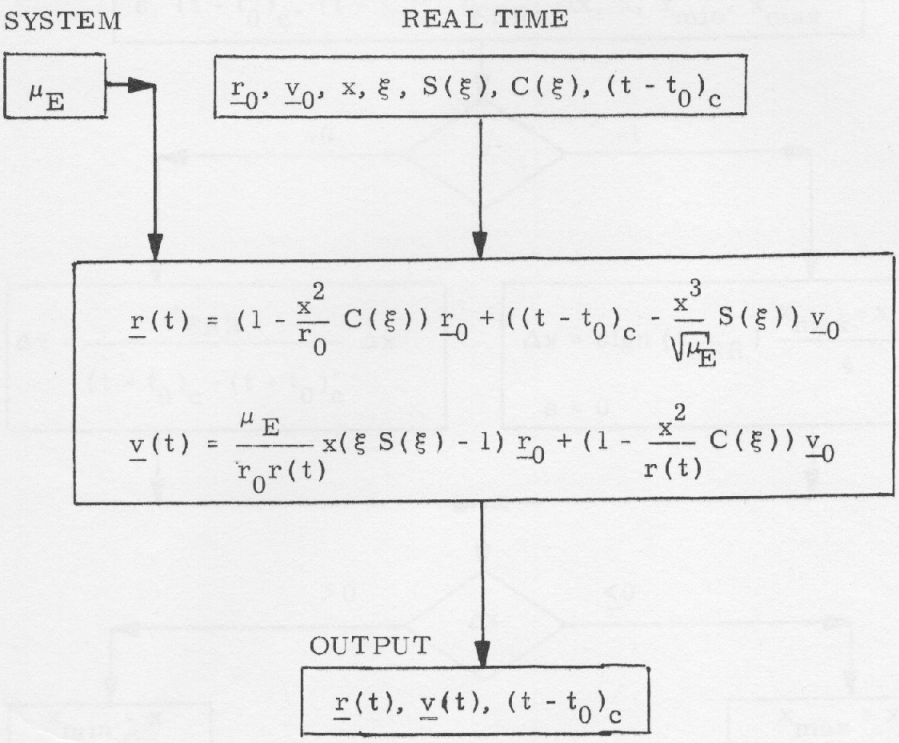

Figure 6 EXTRAPOLATED STATE VECTOR EQUATION
DETAILED FLOW DIAGRAM

5.3.3 Secant Iterator

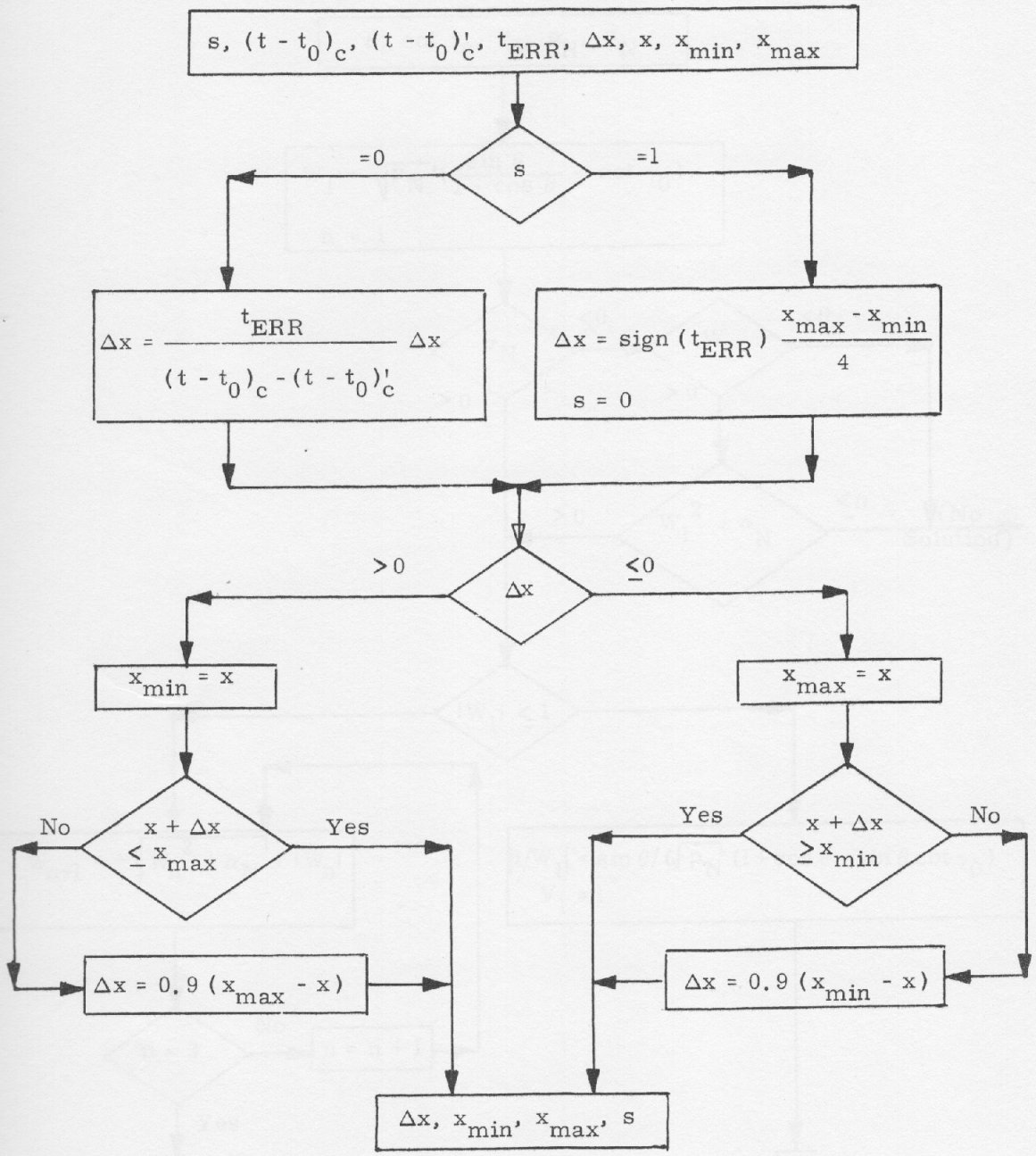

Figure 7 SECANT ITERATOR DETAILED FLOW DIAGRAM

5.3.4 Marscher Equation Inversion

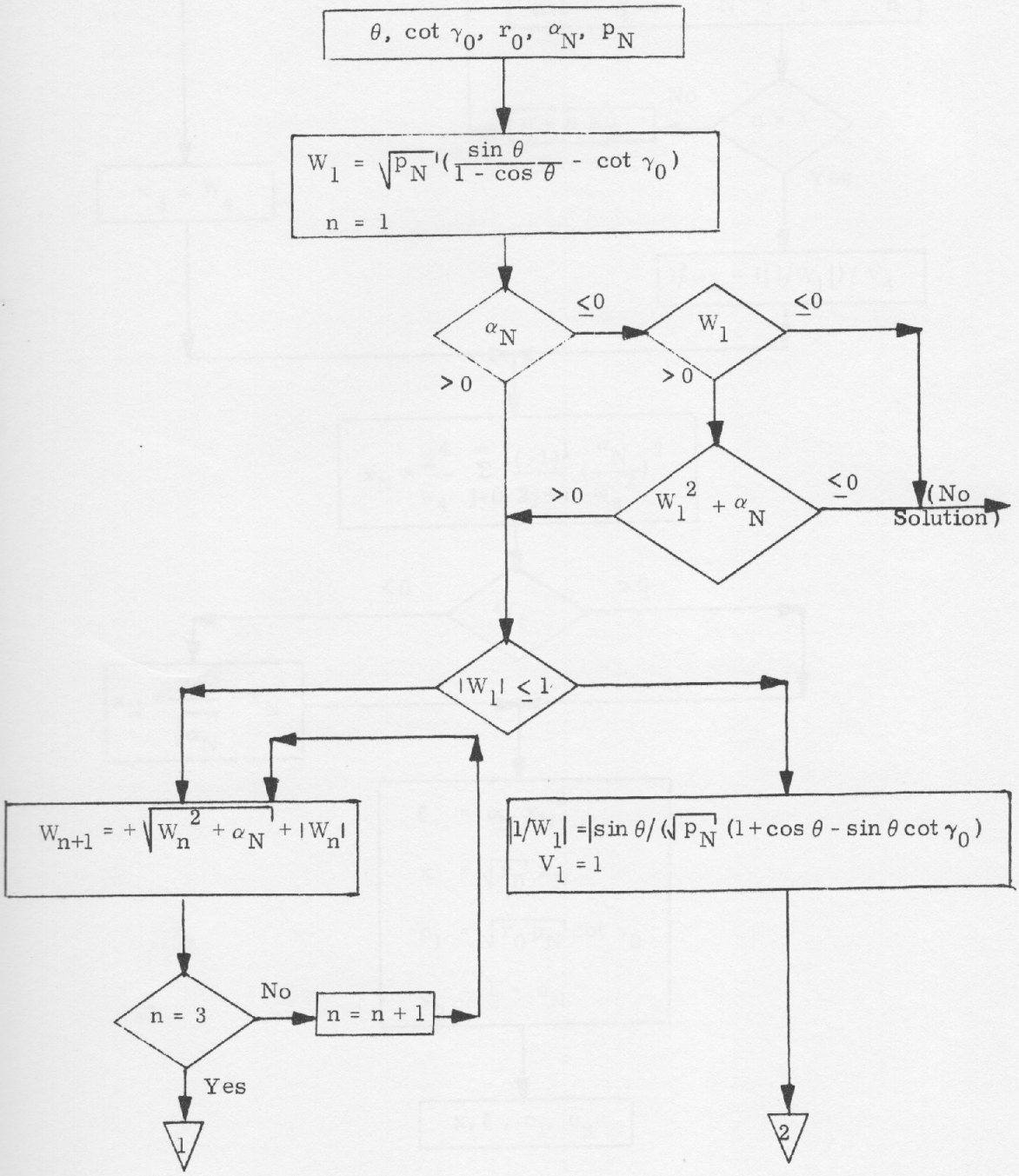

Figure 8a MARSCHER EQUATION INVERSION DETAILED FLOW DIAGRAM

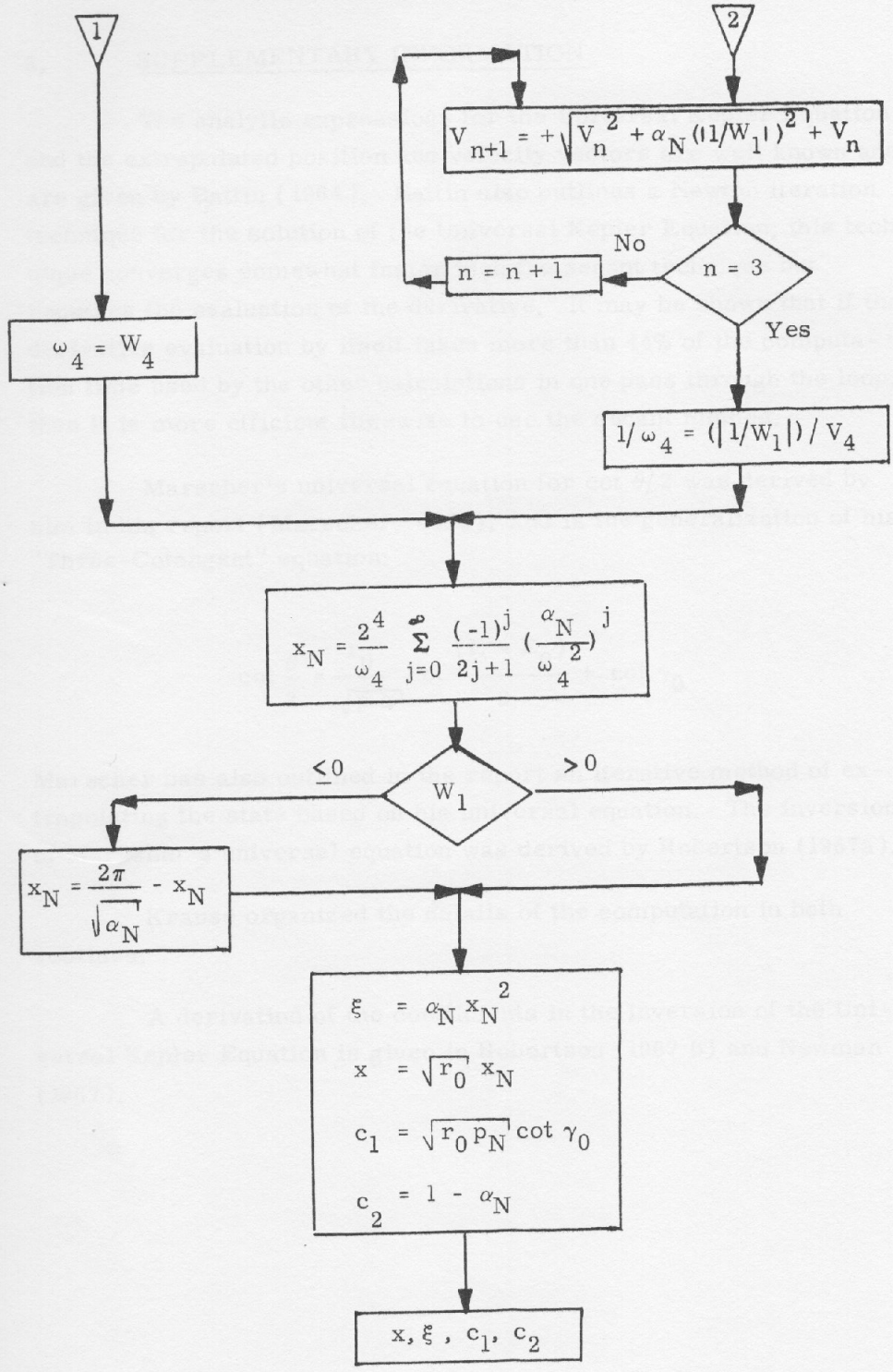

Figure 8b MARSCHER EQUATION INVERSION DETAILED FLOW DIAGRAM

6. SUPPLEMENTARY INFORMATION

The analytic expressions for the Universal Kepler Equation and the extrapolated position and velocity vectors are well known and are given by Battin (1964). Battin also outlines a Newton iteration technique for the solution of the Universal Kepler Equation; this technique converges somewhat faster than the secant technique but requires the evaluation of the derivative. It may be shown that if the derivative evaluation by itself takes more than 44% of the computation time used by the other calculations in one pass through the loop, then it is more efficient timewise to use the secant method.

Marscher's universal equation for cot $\theta/2$ was derived by him in his report (Marscher, 1965), and is the generalization of his "Three-Cotangent" equation:

$$\cot\frac{\theta}{2} = \frac{r_0}{\sqrt{p\,a}}\cot\frac{(E - E_0)}{2} + \cot\gamma_0$$

Marscher has also outlined in the report an iterative method of extrapolating the state based on his universal equation. The inversion of Marscher's universal equation was derived by Robertson (1967a).

Krause organized the details of the computation in both routines.

A derivation of the coefficients in the inversion of the Universal Kepler Equation is given in Robertson (1967 b) and Newman (1967).

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