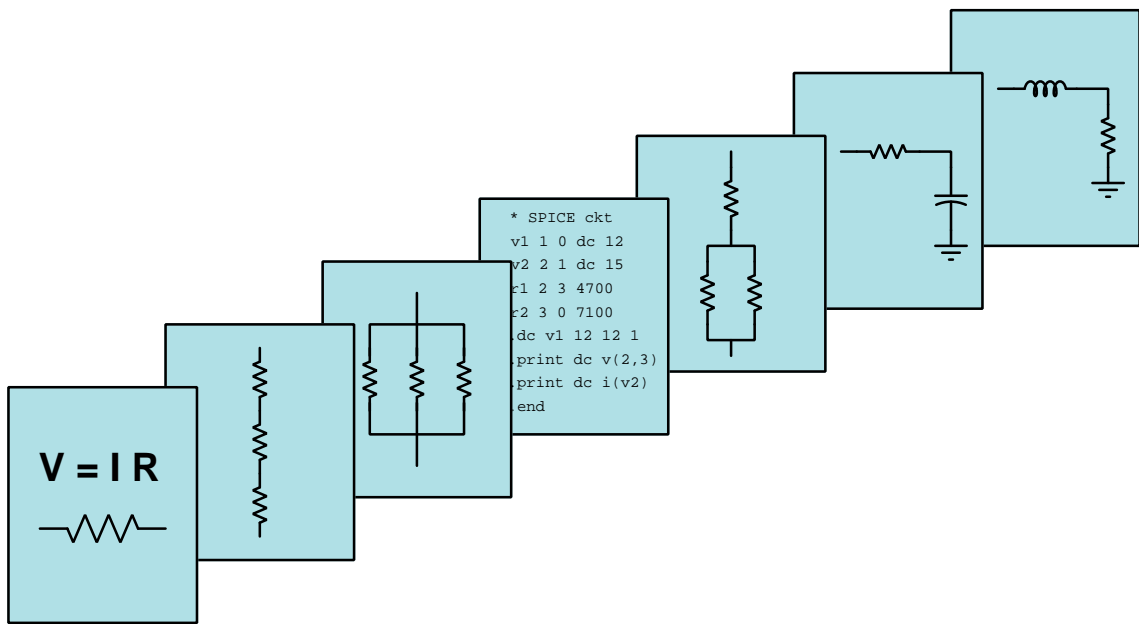


MODULAR ELECTRONICS LEARNING (MODEL) PROJECT



TRIGONOMETRY

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Chapter 1

Introduction

1.1 Recommendations for students

Trigonometry is a branch of mathematics related to triangles, and is extremely useful in a variety of technical disciplines including electric circuit analysis. Specifically, trigonometric functions describe oscillations in AC (alternating current) circuits. This module explores basic trigonometric principles.

Important concepts related to trigonometry include **angles**, **degrees**, **radians**, **minutes** and **seconds** of degrees, the **Pythagorean Theorem**, mathematical **functions**, **ratios**, **inverse functions**, **waves**, the **Law of Sines**, and the **Law of Cosines**.

Here are some good questions to ask of yourself while studying this subject:

- What are some of the different ways we have to quantify angles?
- Why is the “radian” considered a natural unit of angular measurement?
- How are the various sides of a right triangle named, in relation to its angles?
- What are some applications for the Pythagorean Theorem?
- How are the basic trigonometric functions (sine, cosine, tangent) defined?
- What is the range of the sine function (i.e. minimum to maximum value)?
- What is the range of the cosine function (i.e. minimum to maximum value)?
- What is the range of the tangent function (i.e. minimum to maximum value)?
- How do circles relate to trigonometric functions?
- Why does the voltage output by an AC generator trace the shape of a sine wave over time?
- Does the Law of Sines apply to right triangles as well as non-right triangles?
- Does the Law of Cosines apply to right triangles as well as non-right triangles?

1.2 Challenging concepts related to trigonometry

The following list cites concepts related to this module's topic that are easily misunderstood, along with suggestions for properly understanding them:

- **Radians** – degrees are so commonplace for the measurement of angles that it strikes many students as unfathomable we may have alternative units of measurement. Interestingly, since the angular unit of the *radian* is actually based on the length of a perfect circle's radius, the unit of the radian is actually considered a *natural* expression for angles unlike the degree which is completely arbitrary in its division of one rotation into 360 equal parts. Confusion between degrees and radians also expresses itself in the use of scientific calculators, where one must carefully select the proper unit for angles before using any trigonometric functions.
- **Adjacent and Opposite definitions** – while the *hypotenuse* of a right triangle is always its longest side, how we define *adjacent* versus *opposite* depends entirely on the interior angle we happen to be referencing. In other words, the hypotenuse is defined absolutely while the opposite and adjacent sides are relative to a chosen angle. One way to help students clarify which is which is to imagine the right triangle being a top-down view of a three-sided room, with the student standing in one corner. The “opposite” side is that which lies directly opposite the room from that corner, while the “adjacent” side is the non-hypotenuse side joining at that corner.
- **Mathematical functions** – sine, cosine, tangent, and other trigonometric functions take in a single argument (an angle) and return a result based on that argument. This is different from the more familiar with arithmetic functions such as addition, subtraction, multiplication, and division which take in two or more arguments and return a single result. Functional notation is different too: for example, $\sin x$ often makes students think we're multiplying something called \sin with x , but this is not the case. One helpful notational tool is to always enclose a function's argument(s) in parentheses (e.g. $\sin(x)$ rather than $\sin x$) just like computer programming languages usually do, to clearly show that the argument x is data going in to the function. Inverse functions, of course, work the same way except that for trigonometric inverse functions the argument is a ratio and the returned result is an angle. A good exploratory tool is a scientific calculator, especially when combined with a table of trigonometric functions where students can experiment with entering angles into trig functions to get ratios and vice-versa for inverse trig functions.

The *Derivations and Technical References* chapter contains a section with a table of sine, cosine, and tangent values for angles from 0 to 90° in one-degree increments. Although electronic calculators make trig tables obsolete, the tables are still an excellent learning tool for those new to trigonometry.

1.3 Recommendations for instructors

This section lists realistic student learning outcomes supported by the content of the module as well as suggested means of assessing (measuring) student learning. The outcomes state what learners should be able to do, and the assessments are specific challenges to prove students have learned.

- **Outcome** – Demonstrate effective technical reading and writing

Assessment – Students present their outlines of this module’s instructional chapters (e.g. Case Tutorial, Tutorial, Historical References, etc.) ideally as an entry to a larger Journal document chronicling their learning. These outlines should exhibit good-faith effort at summarizing major concepts explained in the text.

- **Outcome** – Apply the concept of trigonometric functions to the calculation of right-triangle dimensions

Assessment – complete the side length and angle determinations for a right triangle given some of the side lengths and angles; e.g. pose problems in the form of the “Unknown angles and side lengths” Quantitative Reasoning question.

- **Outcome** – Convert between radians and degrees

Assessment – Accurately convert from one unit of angle measurement into another; e.g. pose problems in the form of the “Radian/degree angle conversions” Quantitative Reasoning question.

- **Outcome** – Prove the concept of sine and cosine waves by experiment

Assessment – populate a computer spreadsheet with a sequential list of angle values, then use the built-in trigonometric functions to plot sine waves and cosine waves. Incorporate multiplier coefficients to alter the frequency and/or amplitude of these waves.

Assessment – do the same using on paper, plotting points of sine and/or cosine waves manually while using either a trig table or a scientific calculator to produce the ratio values for various angle values.

Chapter 2

Tutorial

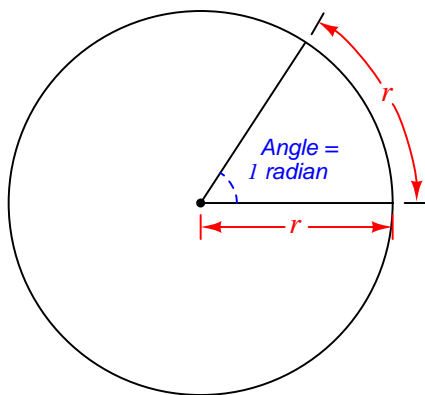
Trigonometry is the mathematical study of triangles. While this may seem at first to be overly specialized, trigonometric functions are extremely useful in the analysis of alternating-current (AC) electric circuits, and in fact for any field of study involving *waves* of any kind.

2.1 Measuring angles

An *angle* refers to a rotary displacement between two positions sharing a common center of rotation. The customary unit of measurement for angles is the *degree* there being 360 degrees in a full circle. It is also customary in mathematics that positive angles are associated with counter-clockwise displacement while negative angle represent clockwise.

When the measurement of an angle demands precision less than a degree, we may use decimal values (e.g. 34.075 degrees) or use *minutes* and *seconds* to subdivide each degree. There are 60 minutes to every degree of rotation, and 60 seconds to every minute of rotation; not unlike there being 60 minutes of time within every hour and 60 seconds within every minute. Degrees, minutes, and seconds are represented by the superscripted symbols $^{\circ}$, $'$, and $''$ respectively. For example, 34.075° could be alternatively expressed as $34^{\circ}4'30''$.

An important alternative to the use of degrees to measure angles in *radians*. The concept of a “radian” is the angle formed by marking one radius-length of a circle around its own circumference:



Since the circumference of a circle is equal to $2\pi r$, it means there are exactly 2π radians (and also 360 degrees) in one full circular rotation. This makes one radian approximately equal to 57.296° , or $57^{\circ}17'44.806''$.

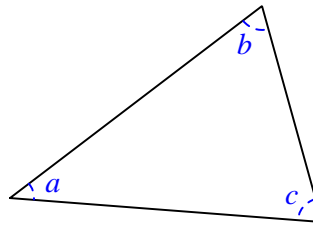
Radians are considered a “natural” unit of measurement¹ for angles, since it is based on the geometry of a perfect circle. Degrees, by comparison, are a completely arbitrary subdivision of a circle. Another unit of angular measurement – rarely used – is the *grad*, there being exactly 400 grads in a full circle.

“Scientific” electronic calculators all provide a *mode* function whereby the user gets to set the preferred unit of angular measurement. For most electrical/electronic circuit calculations the preferred unit is the *degree*, but in pure mathematical studies the preferred unit is the *radian*.

¹In fact, radians are so natural for the measurement of angles that technically the “radian” is not even a true *unit*! Unlike other units of measurement which are either based on arbitrary artifact-standards or based on constants of nature to be applied to all manner of uses, the radian is fundamental to what a circle *is*.

2.2 Right triangles

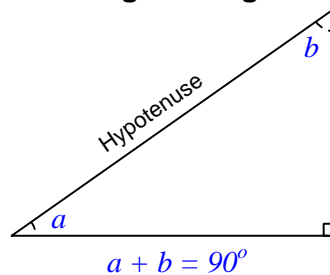
A *triangle* is a geometric object consisting of three line-segments joined together at their ends. The sum of the interior angles of any triangle, no matter its size or shape, always equals 180 degrees or π radians:



$$a + b + c = 180^\circ$$

Any triangle containing one angle at 90° ($\frac{\pi}{2}$ radians) is called a *right triangle*, with 90° being called a *right angle*. Right triangles form the basis for most of trigonometry:

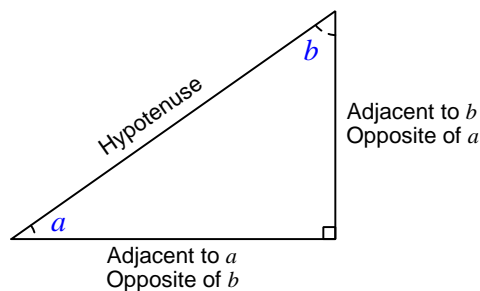
A right triangle



$$a + b = 90^\circ$$

Since the sum of all interior angles of a triangle must equal 180° , and one of those angles in a right triangle is 90° , then the sum of the other two angles (shown here as a and b) must comprise the other 90 degrees. The side of a right triangle opposite its right angle is called the *hypotenuse*.

The other two sides of a right triangle have special names as well, but their positions are relative to which of the non-right angles is referenced. The *opposite* side of a right triangle is always directly opposite the referenced angle, while the *adjacent* side of a right triangle meets at the vertex of the referenced angle.



2.3 The Pythagorean Theorem

An interesting property unique to *right* triangles is the *Pythagorean Theorem*, which states that the square of the hypotenuse's length is equal to the sum of the squares of the other two sides' lengths. Expressed as a formula:

$$A^2 + B^2 = C^2 \quad C = \sqrt{A^2 + B^2}$$

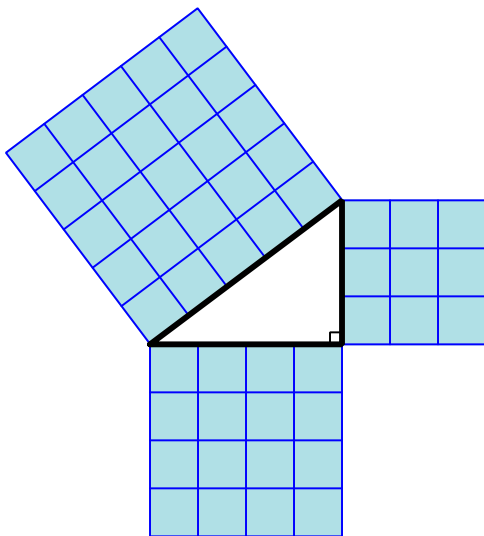
Where,

A = Length of side opposite angle a

B = Length of side opposite angle b

C = Length of hypotenuse

An interesting geometric illustration of the Pythagorean Theorem appears below, showing the square of each side length as the area of a square-shaped grid. Note how the number of internal squares held within each of the three large squares forms a sum, $9 + 16 = 25$, which is what we get for a right triangle consisting of side lengths² 3, 4, and 5:



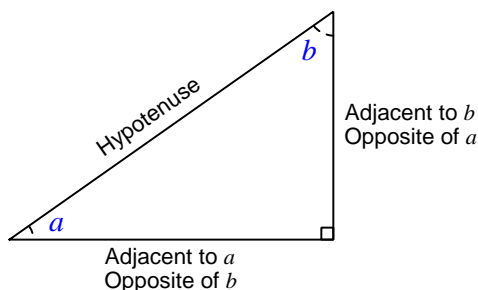
Using the Pythagorean Theorem, we may solve for the length of any side of a right triangle knowing the side-lengths of the other two.

$$C = \sqrt{A^2 + B^2} \quad A = \sqrt{C^2 - B^2} \quad B = \sqrt{C^2 - A^2}$$

²The values of 3, 4, and 5 happen to be the smallest whole numbers satisfying the Pythagorean Theorem.

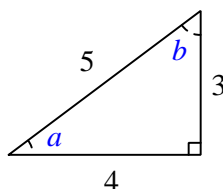
2.4 Angular functions

While the Pythagorean Theorem relates the lengths of a right triangle's sides to one another, another set of functions relate those side-lengths to their associated angles. The three basic trigonometric functions³ are *sine*, *cosine*, and *tangent*, abbreviated sin, cos, and tan:



$$\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \quad \cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \quad \tan \theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Let us return to our simple 3-4-5 right triangle to illustrate:



$$\begin{aligned} \sin \theta &= \frac{\text{Opposite}}{\text{Hypotenuse}} & \sin a &= \frac{3}{5} = 0.6 & \sin b &= \frac{4}{5} = 0.8 \\ \cos \theta &= \frac{\text{Adjacent}}{\text{Hypotenuse}} & \cos a &= \frac{4}{5} = 0.8 & \cos b &= \frac{3}{5} = 0.6 \\ \tan \theta &= \frac{\text{Opposite}}{\text{Adjacent}} & \tan a &= \frac{3}{4} = 0.75 & \tan b &= \frac{4}{3} = 1.\overline{33} \end{aligned}$$

We would use the sine function, for example, to compute the ratio between the opposite and hypotenuse lengths given a known angle value. Cosine would be useful for computing the ratio of adjacent to hypotenuse side-lengths given a known angle. With tangent it's the ratio of opposite to adjacent we could compute from a given angle.

³A popular mnemonic device for remembering these relationships is the made-up word *SOHCAHTOA*, pronounced "Soe-Kah-Toe-Ah", which reminds us that Sine is the ratio of Opposite to Hypotenuse (SOH), Cosine is the ratio of Adjacent to Hypotenuse (CAH), and that Tangent is the ratio of Opposite to Adjacent (TOA).

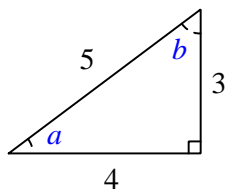
Each of these trigonometric functions has its corresponding *inverse*, or *arc-function*. These are used to compute the value of an angle given the side-lengths.

$$\sin^{-1}\left(\frac{\text{Opposite}}{\text{Hypotenuse}}\right) = \theta \quad \cos^{-1}\left(\frac{\text{Adjacent}}{\text{Hypotenuse}}\right) = \theta \quad \tan^{-1}\left(\frac{\text{Opposite}}{\text{Adjacent}}\right) = \theta$$

Please note that the $^{-1}$ superscript does *not* refer to reciprocation as it does when applied to a *variable*. When applied to a *function* the $^{-1}$ superscript represents mathematical inversion, i.e. that which un-does the original function. An alternative way to express these inverse functions is to call them *arcsine*, *arccosine*, and *arctangent*, respectively:

$$\arcsin\left(\frac{\text{Opposite}}{\text{Hypotenuse}}\right) = \theta \quad \arccos\left(\frac{\text{Adjacent}}{\text{Hypotenuse}}\right) = \theta \quad \arctan\left(\frac{\text{Opposite}}{\text{Adjacent}}\right) = \theta$$

Returning to our 3-4-5 right triangle once more, we may show multiple ways to calculate the values of the two non-right angles:



$$a = \arcsin\left(\frac{3}{5}\right) = \arccos\left(\frac{4}{5}\right) = \arctan\left(\frac{3}{4}\right) = 36.8699^\circ$$

$$b = \arcsin\left(\frac{4}{5}\right) = \arccos\left(\frac{3}{5}\right) = \arctan\left(\frac{4}{3}\right) = 53.1301^\circ$$

Of course, these functions work on *any* right triangle, not just our simple 3-4-5 example. So long as the triangle in question has one 90° angle, we know we may freely apply the Pythagorean Theorem and/or these angular functions to the calculation of angles and/or side lengths.

Modern “scientific” calculators provide sine, cosine, and tangent functions as well as their respective arc-functions. Prior to the advent of electronic calculators, though, people relied on tables of pre-computed numbers for these functions. While practically obsolete, such “trig tables” are nevertheless still useful as educational tools because they allow one to survey the range of each function at a glance for the domain of angle values encompassing a full circle. This document contains a simple set of sine, cosine, and tangent tables in the Technical References chapter, section 3.1 beginning on page 18.

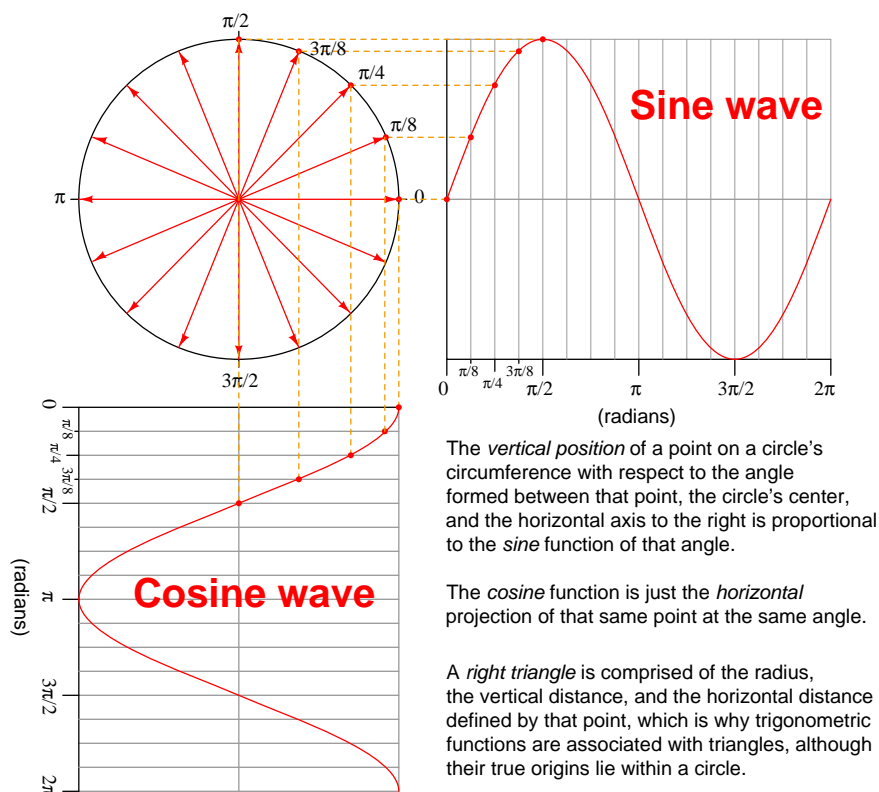
In the study of trigonometry we also encounter other angle-based functions which are nothing more than the reciprocals of sine, cosine, and tangent – these are *cosecant*, *secant*, and *cotangent*, respectively:

$$\csc \theta = \frac{1}{\sin \theta} = \frac{\text{Hypotenuse}}{\text{Opposite}} \quad \sec \theta = \frac{1}{\cos \theta} = \frac{\text{Hypotenuse}}{\text{Adjacent}} \quad \cot \theta = \frac{1}{\tan \theta} = \frac{\text{Adjacent}}{\text{Opposite}}$$

And, of course, each of these reciprocal functions has its inverse or “arc”-function as well. In most practical applications of trigonometry we rarely see these functions, but simply use sine, cosine, and tangent.

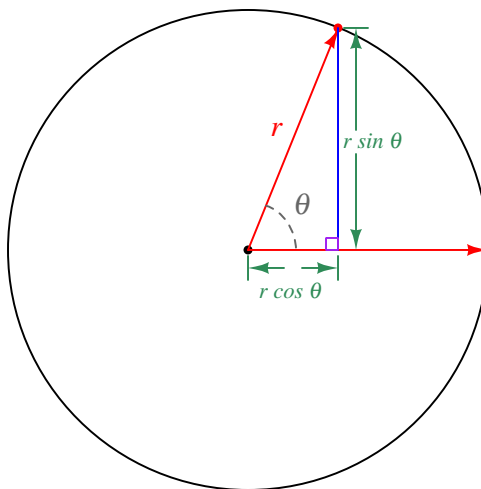
2.5 Sinusoidal waves

The angular functions of a right triangle are really related to *circles*, because the sine and cosine of any given angle really represents the projection of a point along a circle's circumference corresponding to that angle. If we plot the sine and cosine values for angles spanning the rotation of a full circle, we obtain a pair of *waves* having the same basic shape but shifted from one another by a quarter-rotation (90° , or $\frac{\pi}{2}$ radians):



For a *unit circle* where the radius has a length of one (1), the sine of a given angle represents the vertical position of the point on the circle's circumference while the cosine of that same angle represents the horizontal position of that same point.

We may see the relationship between right triangles and circles very clearly if we inscribe such a triangle inside a circle:



The length of inscribed triangle's vertical side with relation to the angle θ is the product of the circle's radius and the sine of that angle: $r \sin \theta$. Similarly, the length of the inscribed triangle's horizontal side to the angle θ involves the cosine of that angle: $r \cos \theta$.

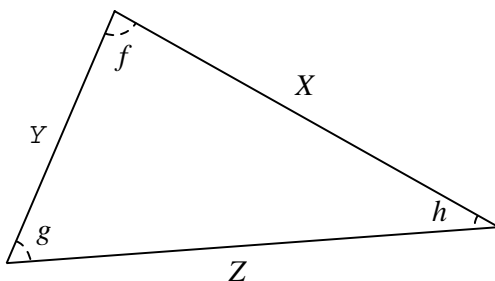
Visualizing the side-lengths of an inscribed triangle while angle θ progresses from a value of zero (0) to one full turn of the circle enables us to understand why the sine and cosine functions oscillate between -1 and $+1$ over this interval. When $\theta = 0$ the triangle is “flat” and consists merely of a horizontal line equal in length to the circle's radius; at this angle the value of the cosine function is one (1) while the sine function is zero (0). As θ increases to equal 90° or $\frac{\pi}{2}$ radians, the triangle “flattens” in the vertical direction, making the sine function have a value of one (1) and the cosine function a value of zero (0) at this angle. As θ increases to equal 180° or π radians, the triangle is once again “flat” but this time as a horizontal line to the *left* of the circle's center, making the cosine function have a value of negative one (-1) and the sine function zero (0). Moving on to 270° or $\frac{3\pi}{2}$ radians, the “flattened” triangle now forms a vertical line below the circle's center which makes the sine function have a value of negative one (-1) and the cosine function a value of zero (0).

Sinusoidal waves are ubiquitous in nature, and they appear frequently in electric circuits as well. As it so happens, when any system *resonates* at its natural frequency of motion, the pattern it traces will be sinusoidal⁴ in shape. In AC electric power systems the reason why voltage and current follow a sinusoidal trajectory over time is due to the way AC generators are built: by rotating a magnetic field past a set of fixed coils of wire (called “windings”). The rate-of-change of magnetic flux experienced by these wire coils naturally grows and decays according to the pattern of a sinusoid precisely because the magnetic field rotates in a *circular* manner past those coils, and therefore the voltage induced in these coils follows that same pattern.

⁴“Sinusoidal” simply means “like the shape of a sine wave”.

2.6 Non-right triangle properties

Although angle-based functions such as sine and cosine are defined by the side-lengths of right triangles, we also find them useful for some non-right triangle applications as well. The first and simplest application is the *Law of Sines*, applying to *any* type of triangle. This law states that the ratio of any side-length of any triangle to the sine of its opposite angle is the same as for any other side and opposite angle in that triangle:



$$\frac{X}{\sin g} = \frac{Y}{\sin h} = \frac{Z}{\sin f}$$

Of course, the Law of Sines also works for right triangles, since this is a universal law for triangles and right triangles are merely special cases of triangles. In the case of a right triangle, the ratio of the longest side (the hypotenuse) to the sine of the largest angle ($\sin 90^\circ$) is simply the length of the hypotenuse, since $\sin 90^\circ = 1$.

The *Law of Cosines* similarly applies to any and all triangles, although its form is more complex than the Law of Sines. It states that the square of any given side-length is equal to the sum of other side-lengths' squares minus two times each of the other side lengths and the cosine of its opposite angle. Illustrating on the same triangle shown above:

$$X^2 = Y^2 + Z^2 - 2YZ \cos g$$

$$Y^2 = X^2 + Z^2 - 2XZ \cos h$$

$$Z^2 = X^2 + Y^2 - 2XY \cos f$$

Chapter 3

Derivations and Technical References

This chapter is where you will find mathematical derivations too detailed to include in the tutorial, and/or tables and other technical reference material.

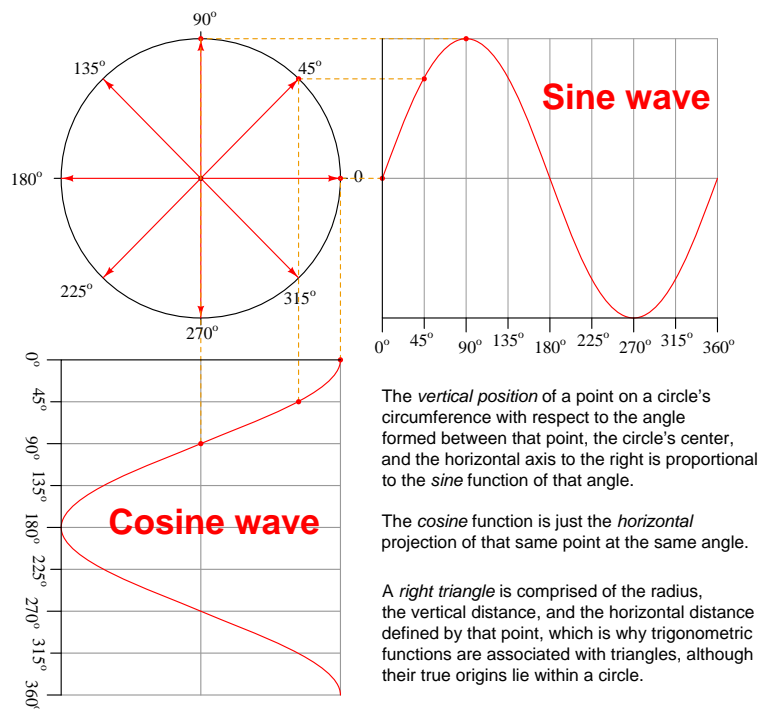
3.1 Table of sine, cosine, and tangent values

Angle	Sine	Cosine	Tangent	-----	Angle	Sine	Cosine	Tangent
0°	0.000	1.000	0.000		45°	0.707	0.707	1.000
1°	0.017	1.000	0.017		46°	0.719	0.695	1.036
2°	0.035	0.999	0.035		47°	0.731	0.682	1.072
3°	0.052	0.999	0.052		48°	0.743	0.669	1.111
4°	0.070	0.998	0.070		49°	0.755	0.656	1.150
5°	0.087	0.996	0.087		50°	0.766	0.643	1.192
6°	0.105	0.995	0.105		51°	0.777	0.629	1.235
7°	0.122	0.993	0.123		52°	0.788	0.616	1.280
8°	0.139	0.990	0.141		53°	0.799	0.602	1.327
9°	0.156	0.988	0.158		54°	0.809	0.588	1.376
10°	0.174	0.985	0.176		55°	0.819	0.574	1.428
11°	0.191	0.982	0.194		56°	0.829	0.559	1.483
12°	0.208	0.978	0.213		57°	0.839	0.545	1.540
13°	0.225	0.974	0.231		58°	0.848	0.530	1.600
14°	0.242	0.970	0.249		59°	0.857	0.515	1.664
15°	0.259	0.966	0.268		60°	0.866	0.500	1.732
16°	0.276	0.961	0.287		61°	0.875	0.485	1.804
17°	0.292	0.956	0.306		62°	0.883	0.469	1.881
18°	0.309	0.951	0.325		63°	0.891	0.454	1.963
19°	0.326	0.946	0.344		64°	0.899	0.438	2.050
20°	0.342	0.940	0.364		65°	0.906	0.423	2.145
21°	0.358	0.934	0.384		66°	0.914	0.407	2.246
22°	0.375	0.927	0.404		67°	0.921	0.391	2.356
23°	0.391	0.921	0.424		68°	0.927	0.375	2.475
24°	0.407	0.914	0.445		69°	0.934	0.358	2.605
25°	0.423	0.906	0.466		70°	0.940	0.342	2.747
26°	0.438	0.899	0.488		71°	0.946	0.326	2.904
27°	0.454	0.891	0.510		72°	0.951	0.309	3.078
28°	0.469	0.883	0.532		73°	0.956	0.292	3.271
29°	0.485	0.875	0.554		74°	0.961	0.276	3.487
30°	0.500	0.866	0.577		75°	0.966	0.259	3.732

Angle	Sine	Cosine	Tangent	----	Angle	Sine	Cosine	Tangent
31°	0.515	0.857	0.601		76°	0.970	0.242	4.011
32°	0.530	0.848	0.625		77°	0.974	0.225	4.331
33°	0.545	0.839	0.649		78°	0.978	0.208	4.705
34°	0.559	0.829	0.675		79°	0.982	0.191	5.145
35°	0.574	0.819	0.700		80°	0.985	0.174	5.671
36°	0.588	0.809	0.727		81°	0.988	0.156	6.314
37°	0.602	0.799	0.754		82°	0.990	0.139	7.115
38°	0.616	0.788	0.781		83°	0.993	0.122	8.144
39°	0.629	0.777	0.810		84°	0.995	0.105	9.514
40°	0.643	0.766	0.839		85°	0.996	0.087	11.430
41°	0.656	0.755	0.869		86°	0.998	0.070	14.301
42°	0.669	0.743	0.900		87°	0.999	0.052	19.081
43°	0.682	0.731	0.933		88°	0.999	0.035	28.636
44°	0.695	0.719	0.966		89°	1.000	0.017	57.290
					90°	1.000	0.000	∞

3.2 Trigonometric identities

3.2.1 The Unit Circle



3.2.2 Pythagorean identity for sine and cosine

$$\sin^2 x + \cos^2 x = 1$$

3.2.3 Odd and even functions

Any mathematical function $f(x)$ is considered *odd* if $f(-x) = -f(x)$, as is the case with polynomial functions having only odd degree terms (e.g. $2x^7 + 8x^5 - x^3 + 9x$). Conversely, any mathematical function is considered *even* if $f(-x) = f(x)$, as with polynomial functions having only even (or zero) degree terms (e.g. $x^6 - 6x^4 + 3x^2 - 8$). The sine function is considered odd while the cosine function is even: for example, $\sin(-10) = -\sin(10)$ while $\cos(-10) = \cos(10)$.

$$\sin(-x) = -\sin(x) \quad \cos(-x) = \cos(x)$$

3.2.4 Sums and differences of angles

$$\sin(x \pm y) = (\sin x \cos y) \pm (\cos x \sin y) \quad \cos(x \pm y) = (\cos x \cos y) \mp (\sin x \sin y)$$

3.2.5 Products and sums of sine and cosine functions

$$\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2} \qquad \cos x \cos y = \frac{\cos(x-y) + \cos(x+y)}{2}$$

$$\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2} \qquad \cos x \sin y = \frac{\sin(x+y) - \sin(x-y)}{2}$$

$$\sin x + \sin y = 2 \sin \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)$$

$$\cos x + \cos y = 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)$$

$$\sin x - \sin y = 2 \cos \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right)$$

$$\cos x - \cos y = -2 \sin \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right)$$

3.2.6 Squares of sine and cosine functions

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

3.2.7 Doubled angles

$$\sin(2x) = 2 \sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

3.2.8 Halved angles

$$\sin \left(\frac{x}{2} \right) = \pm \sqrt{\frac{1 - \cos x}{2}} \qquad \cos \left(\frac{x}{2} \right) = \pm \sqrt{\frac{1 + \cos x}{2}}$$

Chapter 4

Questions

This learning module, along with all others in the ModEL collection, is designed to be used in an inverted instructional environment where students independently read¹ the tutorials and attempt to answer questions on their own *prior* to the instructor's interaction with them. In place of lecture², the instructor engages with students in Socratic-style dialogue, probing and challenging their understanding of the subject matter through inquiry.

Answers are not provided for questions within this chapter, and this is by design. Solved problems may be found in the Tutorial and Derivation chapters, instead. The goal here is *independence*, and this requires students to be challenged in ways where others cannot think for them. Remember that you always have the tools of *experimentation* and *computer simulation* (e.g. SPICE) to explore concepts!

The following lists contain ideas for Socratic-style questions and challenges. Upon inspection, one will notice a strong theme of *metacognition* within these statements: they are designed to foster a regular habit of examining one's own thoughts as a means toward clearer thinking. As such these sample questions are useful both for instructor-led discussions as well as for self-study.

¹Technical reading is an essential academic skill for any technical practitioner to possess for the simple reason that the most comprehensive, accurate, and useful information to be found for developing technical competence is in textual form. Technical careers in general are characterized by the need for continuous learning to remain current with standards and technology, and therefore any technical practitioner who cannot read well is handicapped in their professional development. An excellent resource for educators on improving students' reading prowess through intentional effort and strategy is the book *Reading For Understanding – How Reading Apprenticeship Improves Disciplinary Learning in Secondary and College Classrooms* by Ruth Schoenbach, Cynthia Greenleaf, and Lynn Murphy.

²Lecture is popular as a teaching method because it is easy to implement: any reasonably articulate subject matter expert can talk to students, even with little preparation. However, it is also quite problematic. A good lecture always makes complicated concepts seem easier than they are, which is bad for students because it instills a false sense of confidence in their own understanding; reading and re-articulation requires more cognitive effort and serves to verify comprehension. A culture of teaching-by-lecture fosters a debilitating dependence upon direct personal instruction, whereas the challenges of modern life demand independent and critical thought made possible only by gathering information and perspectives from afar. Information presented in a lecture is ephemeral, easily lost to failures of memory and dictation; text is forever, and may be referenced at any time.

GENERAL CHALLENGES FOLLOWING TUTORIAL READING

- Summarize as much of the text as you can in one paragraph of your own words. A helpful strategy is to explain ideas as you would for an intelligent child: as simple as you can without compromising too much accuracy.
- Simplify a particular section of the text, for example a paragraph or even a single sentence, so as to capture the same fundamental idea in fewer words.
- Where did the text make the most sense to you? What was it about the text's presentation that made it clear?
- Identify where it might be easy for someone to misunderstand the text, and explain why you think it could be confusing.
- Identify any new concept(s) presented in the text, and explain in your own words.
- Identify any familiar concept(s) such as physical laws or principles applied or referenced in the text.
- Devise a proof of concept experiment demonstrating an important principle, physical law, or technical innovation represented in the text.
- Devise an experiment to disprove a plausible misconception.
- Did the text reveal any misconceptions you might have harbored? If so, describe the misconception(s) and the reason(s) why you now know them to be incorrect.
- Describe any useful problem-solving strategies applied in the text.
- Devise a question of your own to challenge a reader's comprehension of the text.

GENERAL FOLLOW-UP CHALLENGES FOR ASSIGNED PROBLEMS

- Identify where any fundamental laws or principles apply to the solution of this problem, especially before applying any mathematical techniques.
- Devise a thought experiment to explore the characteristics of the problem scenario, applying known laws and principles to mentally model its behavior.
- Describe in detail your own strategy for solving this problem. How did you identify and organized the given information? Did you sketch any diagrams to help frame the problem?
- Is there more than one way to solve this problem? Which method seems best to you?
- Show the work you did in solving this problem, even if the solution is incomplete or incorrect.
- What would you say was the most challenging part of this problem, and why was it so?
- Was any important information missing from the problem which you had to research or recall?
- Was there any extraneous information presented within this problem? If so, what was it and why did it not matter?
- Examine someone else's solution to identify where they applied fundamental laws or principles.
- Simplify the problem from its given form and show how to solve this simpler version of it. Examples include eliminating certain variables or conditions, altering values to simpler (usually whole) numbers, applying a limiting case (i.e. altering a variable to some extreme or ultimate value).
- For quantitative problems, identify the real-world meaning of all intermediate calculations: their units of measurement, where they fit into the scenario at hand. Annotate any diagrams or illustrations with these calculated values.
- For quantitative problems, try approaching it qualitatively instead, thinking in terms of “increase” and “decrease” rather than definite values.
- For qualitative problems, try approaching it quantitatively instead, proposing simple numerical values for the variables.
- Were there any assumptions you made while solving this problem? Would your solution change if one of those assumptions were altered?
- Identify where it would be easy for someone to go astray in attempting to solve this problem.
- Formulate your own problem based on what you learned solving this one.

GENERAL FOLLOW-UP CHALLENGES FOR EXPERIMENTS OR PROJECTS

- In what way(s) was this experiment or project easy to complete?
- Identify some of the challenges you faced in completing this experiment or project.

- Show how thorough documentation assisted in the completion of this experiment or project.
- Which fundamental laws or principles are key to this system's function?
- Identify any way(s) in which one might obtain false or otherwise misleading measurements from test equipment in this system.
- What will happen if (component X) fails (open/shorted/etc.)?
- What would have to occur to make this system unsafe?

4.1 Conceptual reasoning

These questions are designed to stimulate your analytic and synthetic thinking³. In a Socratic discussion with your instructor, the goal is for these questions to prompt an extended dialogue where assumptions are revealed, conclusions are tested, and understanding is sharpened. Your instructor may also pose additional questions based on those assigned, in order to further probe and refine your conceptual understanding.

Questions that follow are presented to challenge and probe your understanding of various concepts presented in the tutorial. These questions are intended to serve as a guide for the Socratic dialogue between yourself and the instructor. Your instructor's task is to ensure you have a sound grasp of these concepts, and the questions contained in this document are merely a means to this end. Your instructor may, at his or her discretion, alter or substitute questions for the benefit of tailoring the discussion to each student's needs. The only absolute requirement is that each student is challenged and assessed at a level equal to or greater than that represented by the documented questions.

It is far more important that you convey your *reasoning* than it is to simply convey a correct answer. For this reason, you should refrain from researching other information sources to answer questions. What matters here is that *you* are doing the thinking. If the answer is incorrect, your instructor will work with you to correct it through proper reasoning. A correct answer without an adequate explanation of how you derived that answer is unacceptable, as it does not aid the learning or assessment process.

You will note a conspicuous lack of answers given for these conceptual questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. The best way by far is to debate the answers with fellow students and also with the instructor during the Socratic dialogue sessions intended to be used with these learning modules. Reasoning through challenging questions with other people is an excellent tool for developing strong reasoning skills.

Another means of checking your conceptual answers, where applicable, is to use circuit simulation software to explore the effects of changes made to circuits. For example, if one of these conceptual questions challenges you to predict the effects of altering some component parameter in a circuit, you may check the validity of your work by simulating that same parameter change within software and seeing if the results agree.

³*Analytical* thinking involves the “disassembly” of an idea into its constituent parts, analogous to dissection. *Synthetic* thinking involves the “assembly” of a new idea comprised of multiple concepts, analogous to construction. Both activities are high-level cognitive skills, extremely important for effective problem-solving, necessitating frequent challenge and regular practice to fully develop.

4.1.1 Reading outline and reflections

“Reading maketh a full man; conference a ready man; and writing an exact man” – Francis Bacon

Francis Bacon’s advice is a blueprint for effective education: reading provides the learner with knowledge, writing focuses the learner’s thoughts, and critical dialogue equips the learner to confidently communicate and apply their learning. Independent acquisition and application of knowledge is a powerful skill, well worth the effort to cultivate. To this end, students should read these educational resources closely, journal their own reflections on the reading, and discuss in detail their findings with classmates and instructor(s). You should be able to do all of the following after reading any instructional text:

☒ Briefly SUMMARIZE THE TEXT in the form of a journal entry documenting your learning as you progress through the course of study. Share this summary in dialogue with your classmates and instructor. Journaling is an excellent self-test of thorough reading because you cannot clearly express what you have not read or did not comprehend.

☒ Demonstrate ACTIVE READING STRATEGIES, including verbalizing your impressions as you read, simplifying long passages to convey the same ideas using fewer words, annotating text and illustrations with your own interpretations, working through mathematical examples shown in the text, cross-referencing passages with relevant illustrations and/or other passages, identifying problem-solving strategies applied by the author, etc. Technical reading is a special case of problem-solving, and so these strategies work precisely because they help solve any problem: paying attention to your own thoughts (metacognition), eliminating unnecessary complexities, identifying what makes sense, paying close attention to details, drawing connections between separated facts, and noting the successful strategies of others.

☒ Identify IMPORTANT THEMES, especially GENERAL LAWS and PRINCIPLES, expounded in the text and express them in the simplest of terms as though you were teaching an intelligent child. This emphasizes connections between related topics and develops your ability to communicate complex ideas to anyone.

☒ Form YOUR OWN QUESTIONS based on the reading, and then pose them to your instructor and classmates for their consideration. Anticipate both correct and incorrect answers, the incorrect answer(s) assuming one or more plausible misconceptions. This helps you view the subject from different perspectives to grasp it more fully.

☒ Devise EXPERIMENTS to test claims presented in the reading, or to disprove misconceptions. Predict possible outcomes of these experiments, and evaluate their meanings: what result(s) would confirm, and what would constitute disproof? Running mental simulations and evaluating results is essential to scientific and diagnostic reasoning.

☒ Specifically identify any points you found CONFUSING. The reason for doing this is to help diagnose misconceptions and overcome barriers to learning.

4.1.2 Foundational concepts

Correct analysis and diagnosis of electric circuits begins with a proper understanding of some basic concepts. The following is a list of some important concepts referenced in this module's full tutorial. Define each of them in your own words, and be prepared to illustrate each of these concepts with a description of a practical example and/or a live demonstration.

Variable

Superscript

Graphing as a problem-solving strategy

Inverse functions

Radius

Circumference

Angles

Degree

Radian

Right angle

Right triangle

Hypotenuse

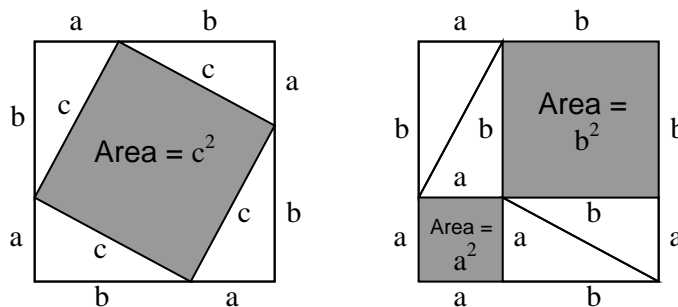
Adjacent

Opposite

Minutes and Seconds

4.1.3 Geometric proof of the Pythagorean Theorem

Geometry may be used to elegantly prove the Pythagorean Theorem. Shown here are two squares of equal area, each with four right triangles embedded within, occupying area of their own:



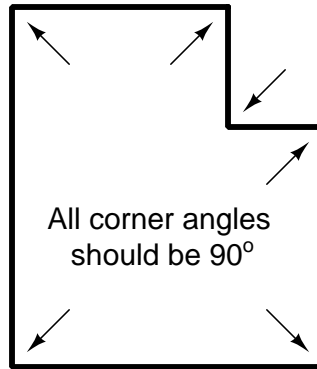
Explain how these two figures prove the Pythagorean Theorem, where $a^2 + b^2 = c^2$.

Challenges

- Write an equation for the area of each large square.

4.1.4 Squaring a house foundation

Suppose you were laying out the forms for a rectangular house foundation, and had to ensure the corners were “square” (i.e. exactly 90°):



How could you ensure this, using trigonometry? How would you ensure this if you had no calculator with you to calculate trig functions, and no protractor or framing square with you to measure angles?

Challenges

- What important assumption(s) is/are implicit in your technique?

4.2 Quantitative reasoning

These questions are designed to stimulate your computational thinking. In a Socratic discussion with your instructor, the goal is for these questions to reveal your mathematical approach(es) to problem-solving so that good technique and sound reasoning may be reinforced. Your instructor may also pose additional questions based on those assigned, in order to observe your problem-solving firsthand.

Mental arithmetic and estimations are strongly encouraged for all calculations, because without these abilities you will be unable to readily detect errors caused by calculator misuse (e.g. keystroke errors).

You will note a conspicuous lack of answers given for these quantitative questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. My advice is to use circuit simulation software such as SPICE to check the correctness of quantitative answers. Refer to those learning modules within this collection focusing on SPICE to see worked examples which you may use directly as practice problems for your own study, and/or as templates you may modify to run your own analyses and generate your own practice problems.

Completely worked example problems found in the Tutorial may also serve as “test cases⁴” for gaining proficiency in the use of circuit simulation software, and then once that proficiency is gained you will never need to rely⁵ on an answer key!

⁴In other words, set up the circuit simulation software to analyze the same circuit examples found in the Tutorial. If the simulated results match the answers shown in the Tutorial, it confirms the simulation has properly run. If the simulated results disagree with the Tutorial’s answers, something has been set up incorrectly in the simulation software. Using every Tutorial as practice in this way will quickly develop proficiency in the use of circuit simulation software.

⁵This approach is perfectly in keeping with the instructional philosophy of these learning modules: *teaching students to be self-sufficient thinkers*. Answer keys can be useful, but it is even more useful to your long-term success to have a set of tools on hand for checking your own work, because once you have left school and are on your own, there will no longer be “answer keys” available for the problems you will have to solve.

4.2.1 Miscellaneous physical constants

Note: constants shown in **bold** type are *exact*, not approximations. Values inside of parentheses show one standard deviation (σ) of uncertainty in the final digits: for example, the magnetic permeability of free space value given as $1.25663706212(19) \times 10^{-6}$ H/m represents a center value (i.e. the location parameter) of $1.25663706212 \times 10^{-6}$ Henrys per meter with one standard deviation of uncertainty equal to $0.0000000000019 \times 10^{-6}$ Henrys per meter.

Avogadro's number (N_A) = **$6.02214076 \times 10^{23}$** per mole (mol^{-1})

Boltzmann's constant (k) = **1.380649×10^{-23}** Joules per Kelvin (J/K)

Electronic charge (e) = **$1.602176634 \times 10^{-19}$** Coulomb (C)

Faraday constant (F) = **$96,485.33212...$** $\times 10^4$ Coulombs per mole (C/mol)

Magnetic permeability of free space (μ_0) = $1.25663706212(19) \times 10^{-6}$ Henrys per meter (H/m)

Electric permittivity of free space (ϵ_0) = $8.8541878128(13) \times 10^{-12}$ Farads per meter (F/m)

Characteristic impedance of free space (Z_0) = $376.730313668(57)$ Ohms (Ω)

Gravitational constant (G) = $6.67430(15) \times 10^{-11}$ cubic meters per kilogram-seconds squared ($\text{m}^3/\text{kg}\cdot\text{s}^2$)

Molar gas constant (R) = **$8.314462618...$** Joules per mole-Kelvin (J/mol-K) = $0.08205746(14)$ liters-atmospheres per mole-Kelvin

Planck constant (h) = **$6.62607015 \times 10^{-34}$** joule-seconds (J-s)

Stefan-Boltzmann constant (σ) = **$5.670374419...$** $\times 10^{-8}$ Watts per square meter-Kelvin⁴ ($\text{W}/\text{m}^2\cdot\text{K}^4$)

Speed of light in a vacuum (c) = **$299,792,458$** meters per second (m/s) = 186282.4 miles per second (mi/s)

Note: All constants taken from NIST data "Fundamental Physical Constants – Complete Listing", from <http://physics.nist.gov/constants>, National Institute of Standards and Technology (NIST), 2018 CODATA Adjustment.

4.2.2 Introduction to spreadsheets

A powerful computational tool you are encouraged to use in your work is a *spreadsheet*. Available on most personal computers (e.g. Microsoft Excel), *spreadsheet* software performs numerical calculations based on number values and formulae entered into cells of a grid. This grid is typically arranged as lettered columns and numbered rows, with each cell of the grid identified by its column/row coordinates (e.g. cell B3, cell A8). Each cell may contain a string of text, a number value, or a mathematical formula. The spreadsheet automatically updates the results of all mathematical formulae whenever the entered number values are changed. This means it is possible to set up a spreadsheet to perform a series of calculations on entered data, and those calculations will be re-done by the computer any time the data points are edited in any way.

For example, the following spreadsheet calculates average speed based on entered values of distance traveled and time elapsed:

	A	B	C	D
1	Distance traveled	46.9	Kilometers	
2	Time elapsed	1.18	Hours	
3	Average speed	= B1 / B2	km/h	
4				
5				

Text labels contained in cells A1 through A3 and cells C1 through C3 exist solely for readability and are not involved in any calculations. Cell B1 contains a sample distance value while cell B2 contains a sample time value. The formula for computing speed is contained in cell B3. Note how this formula begins with an “equals” symbol (=), references the values for distance and speed by lettered column and numbered row coordinates (B1 and B2), and uses a forward slash symbol for division (/). The coordinates B1 and B2 function as *variables*⁶ would in an algebraic formula.

When this spreadsheet is executed, the numerical value 39.74576 will appear in cell B3 rather than the formula = B1 / B2, because 39.74576 is the computed speed value given 46.9 kilometers traveled over a period of 1.18 hours. If a different numerical value for distance is entered into cell B1 or a different value for time is entered into cell B2, cell B3’s value will automatically update. All you need to do is set up the given values and any formulae into the spreadsheet, and the computer will do all the calculations for you.

Cell B3 may be referenced by other formulae in the spreadsheet if desired, since it is a variable just like the given values contained in B1 and B2. This means it is possible to set up an entire chain of calculations, one dependent on the result of another, in order to arrive at a final value. The arrangement of the given data and formulae need not follow any pattern on the grid, which means you may place them anywhere.

⁶Spreadsheets may also provide means to attach text labels to cells for use as variable names (Microsoft Excel simply calls these labels “names”), but for simple spreadsheets such as those shown here it’s usually easier just to use the standard coordinate naming for each cell.

Common⁷ arithmetic operations available for your use in a spreadsheet include the following:

- Addition (+)
- Subtraction (-)
- Multiplication (*)
- Division (/)
- Powers (^)
- Square roots (sqrt())
- Logarithms (ln() , log10())

Parentheses may be used to ensure⁸ proper order of operations within a complex formula. Consider this example of a spreadsheet implementing the *quadratic formula*, used to solve for roots of a polynomial expression in the form of $ax^2 + bx + c$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

	A	B
1	x_1	= (-B4 + sqrt((B4^2) - (4*B3*B5))) / (2*B3)
2	x_2	= (-B4 - sqrt((B4^2) - (4*B3*B5))) / (2*B3)
3	a =	9
4	b =	5
5	c =	-2

This example is configured to compute roots⁹ of the polynomial $9x^2 + 5x - 2$ because the values of 9, 5, and -2 have been inserted into cells B3, B4, and B5, respectively. Once this spreadsheet has been built, though, it may be used to calculate the roots of *any* second-degree polynomial expression simply by entering the new a , b , and c coefficients into cells B3 through B5. The numerical values appearing in cells B1 and B2 will be automatically updated by the computer immediately following any changes made to the coefficients.

⁷Modern spreadsheet software offers a bewildering array of mathematical functions you may use in your computations. I recommend you consult the documentation for your particular spreadsheet for information on operations other than those listed here.

⁸Spreadsheet programs, like text-based programming languages, are designed to follow standard order of operations by default. However, my personal preference is to use parentheses even where strictly unnecessary just to make it clear to any other person viewing the formula what the intended order of operations is.

⁹Reviewing some algebra here, a *root* is a value for x that yields an overall value of zero for the polynomial. For this polynomial ($9x^2 + 5x - 2$) the two roots happen to be $x = 0.269381$ and $x = -0.82494$, with these values displayed in cells B1 and B2, respectively upon execution of the spreadsheet.

Alternatively, one could break up the long quadratic formula into smaller pieces like this:

$$y = \sqrt{b^2 - 4ac} \quad z = 2a$$

$$x = \frac{-b \pm y}{z}$$

	A	B	C
1	x_1	= (-B4 + C1) / C2	= sqrt ((B4^2) - (4*B3*B5))
2	x_2	= (-B4 - C1) / C2	= 2*B3
3	a =	9	
4	b =	5	
5	c =	-2	

Note how the square-root term (y) is calculated in cell C1, and the denominator term (z) in cell C2. This makes the two final formulae (in cells B1 and B2) simpler to interpret. The positioning of all these cells on the grid is completely arbitrary¹⁰ – all that matters is that they properly reference each other in the formulae.

Spreadsheets are particularly useful for situations where the same set of calculations representing a circuit or other system must be repeated for different initial conditions. The power of a spreadsheet is that it automates what would otherwise be a tedious set of calculations. One specific application of this is to simulate the effects of various components within a circuit failing with abnormal values (e.g. a shorted resistor simulated by making its value nearly zero; an open resistor simulated by making its value extremely large). Another application is analyzing the behavior of a circuit design given new components that are out of specification, and/or aging components experiencing drift over time.

¹⁰My personal preference is to locate all the “given” data in the upper-left cells of the spreadsheet grid (each data point flanked by a sensible name in the cell to the left and units of measurement in the cell to the right as illustrated in the first distance/time spreadsheet example), sometimes coloring them in order to clearly distinguish which cells contain entered data versus which cells contain computed results from formulae. I like to place all formulae in cells below the given data, and try to arrange them in logical order so that anyone examining my spreadsheet will be able to figure out *how* I constructed a solution. This is a general principle I believe all computer programmers should follow: *document and arrange your code to make it easy for other people to learn from it.*

4.2.3 Radian/degree angle conversions

Convert between the following angle units:

- 3.1 radians = ??? degrees
- 1.5 radians = ??? degrees
- 22 radians = ??? degrees
- 199° = ??? radians
- 40° = ??? radians
- 203° = ??? radians

Challenges

- Which of the above angles represents more than one full rotation?

4.2.4 Plotting sine and cosine functions

Either plot by hand or use a computer program to plot a sine function and a cosine function given angles spanning the rotation of a full circuit (i.e. from 0 degrees to 360 degrees, or from 0 radians to 2π radians).

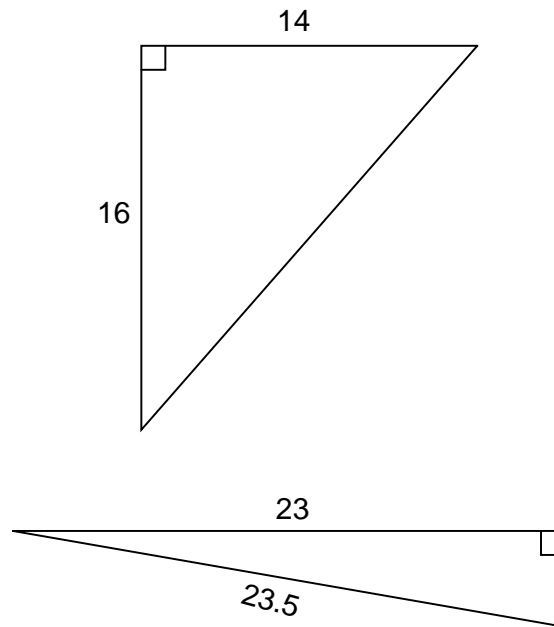
Reference a *trig table* and explain where the values for sine and cosine functions contained in that table exist on the plot you created.

Challenges

- At which angle value(s) does the sine function reach its positive peak value?
- At which angle value(s) does the sine function reach its negative peak value?
- At which angle value(s) does the cosine function reach its positive peak value?
- At which angle value(s) does the cosine function reach its negative peak value?

4.2.5 Unknown side lengths

Determine the lengths of the unmarked sides on these triangles:

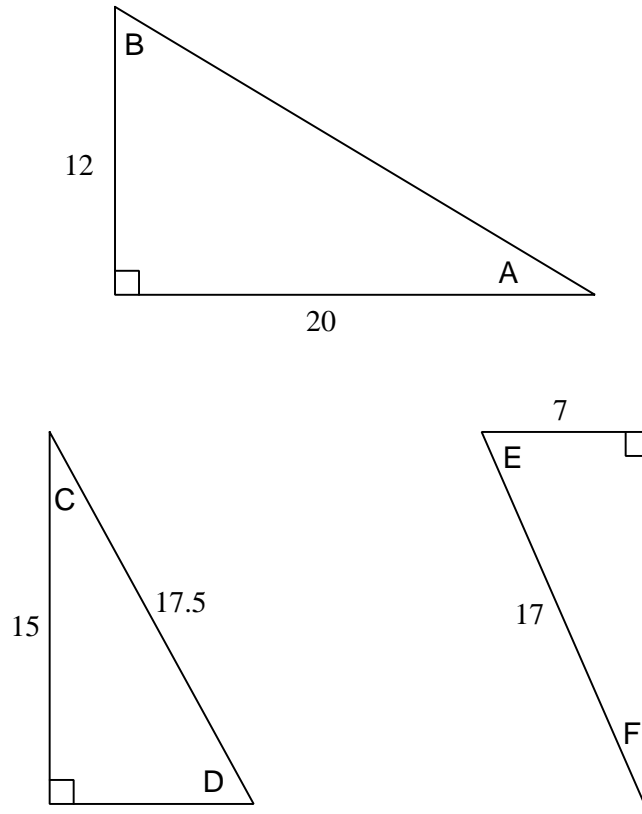


Challenges

- Why is it important for us to know that these are *right* triangles?

4.2.6 Unknown angles and side lengths

Determine the lengths of the unmarked sides and the values of the unmarked angles on these triangles:



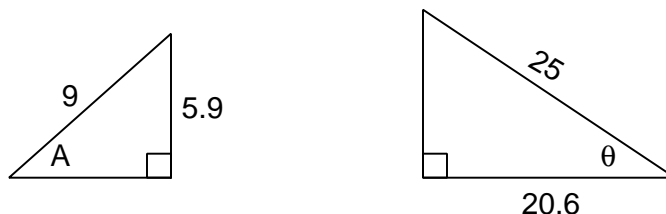
Next, show how it is possible to approximately calculate these same side lengths and angles, but without using any trigonometric function on your calculator!

Challenges

- Propose *multiple* methods for calculating each of the angles in these triangles.

4.2.7 Using trig tables

Suppose you were wishing to calculate angles A and θ for these triangles, using a trig function table rather than a *scientific* calculator:



If you are using a trig function table specifying angles to the nearest whole degree, you will find the values do not precisely match a whole-number angle, but is very close to one. What is the nearest whole-number value for angle A , and also for angle θ in these two triangles? Is the true value for angle A less than or greater than the nearest whole-number angle found in a trig table? How about the true value of angle θ ?

Challenges

- Apply linear interpolation to get a closer estimate for both angles.

4.2.8 Sine and cosine values of common angles

Express the sine and cosine values for these angles as fractions of whole numbers and square roots:

- 0°
- 30°
- 45°
- 60°
- 90°

Challenges

- What easy-to-remember pattern do see in these results?

4.3 Diagnostic reasoning

These questions are designed to stimulate your deductive and inductive thinking, where you must apply general principles to specific scenarios (deductive) and also derive conclusions about the failed circuit from specific details (inductive). In a Socratic discussion with your instructor, the goal is for these questions to reinforce your recall and use of general circuit principles and also challenge your ability to integrate multiple symptoms into a sensible explanation of what's wrong in a circuit. Your instructor may also pose additional questions based on those assigned, in order to further challenge and sharpen your diagnostic abilities.

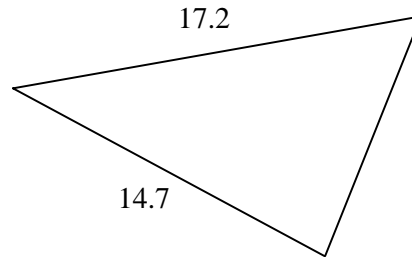
As always, your goal is to fully *explain* your analysis of each problem. Simply obtaining a correct answer is not good enough – you must also demonstrate sound reasoning in order to successfully complete the assignment. Your instructor's responsibility is to probe and challenge your understanding of the relevant principles and analytical processes in order to ensure you have a strong foundation upon which to build further understanding.

You will note a conspicuous lack of answers given for these diagnostic questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. The best way by far is to debate the answers with fellow students and also with the instructor during the Socratic dialogue sessions intended to be used with these learning modules. Reasoning through challenging questions with other people is an excellent tool for developing strong reasoning skills.

Another means of checking your diagnostic answers, where applicable, is to use circuit simulation software to explore the effects of faults placed in circuits. For example, if one of these diagnostic questions requires that you predict the effect of an open or a short in a circuit, you may check the validity of your work by simulating that same fault (substituting a very high resistance in place of that component for an open, and substituting a very low resistance for a short) within software and seeing if the results agree.

4.3.1 Find the mistake(s)

A student attempts to calculate the length of the unmarked side in this triangle, and obtains a result of 8.9303:



Explain why this side length is not necessarily correct for this triangle, and identify the unfounded assumption made by the student.

Challenges

- What other mistakes might a person make based on the same misconception/oversight?

Appendix A

Problem-Solving Strategies

The ability to solve complex problems is arguably one of the most valuable skills one can possess, and this skill is particularly important in any science-based discipline.

- Study principles, not procedures. Don't be satisfied with merely knowing how to compute solutions – learn *why* those solutions work.
- Identify what it is you need to solve, identify all relevant data, identify all units of measurement, identify any general principles or formulae linking the given information to the solution, and then identify any “missing pieces” to a solution. Annotate all diagrams with this data.
- Sketch a diagram to help visualize the problem. When building a real system, always devise a plan for that system and analyze its function *before* constructing it.
- Follow the units of measurement and meaning of every calculation. If you are ever performing mathematical calculations as part of a problem-solving procedure, and you find yourself unable to apply each and every intermediate result to some aspect of the problem, it means you don't understand what you are doing. Properly done, every mathematical result should have practical meaning for the problem, and not just be an abstract number. You should be able to identify the proper units of measurement for each and every calculated result, and show where that result fits into the problem.
- Perform “thought experiments” to explore the effects of different conditions for theoretical problems. When troubleshooting real systems, perform *diagnostic tests* rather than visually inspecting for faults, the best diagnostic test being the one giving you the most information about the nature and/or location of the fault with the fewest steps.
- Simplify the problem until the solution becomes obvious, and then use that obvious case as a model to follow in solving the more complex version of the problem.
- Check for exceptions to see if your solution is incorrect or incomplete. A good solution will work for *all* known conditions and criteria. A good example of this is the process of testing scientific hypotheses: the task of a scientist is not to find support for a new idea, but rather to *challenge* that new idea to see if it holds up under a battery of tests. The philosophical

principle of *reductio ad absurdum* (i.e. disproving a general idea by finding a specific case where it fails) is useful here.

- Work “backward” from a hypothetical solution to a new set of given conditions.
- Add quantities to problems that are qualitative in nature, because sometimes a little math helps illuminate the scenario.
- Sketch graphs illustrating how variables relate to each other. These may be quantitative (i.e. with realistic number values) or qualitative (i.e. simply showing increases and decreases).
- Treat quantitative problems as qualitative in order to discern the relative magnitudes and/or directions of change of the relevant variables. For example, try determining what happens if a certain variable were to increase or decrease before attempting to precisely calculate quantities: how will each of the dependent variables respond, by increasing, decreasing, or remaining the same as before?
- Consider limiting cases. This works especially well for qualitative problems where you need to determine which direction a variable will change. Take the given condition and magnify that condition to an extreme degree as a way of simplifying the direction of the system’s response.
- Check your work. This means regularly testing your conclusions to see if they make sense. This does *not* mean repeating the same steps originally used to obtain the conclusion(s), but rather to use some other means to check validity. Simply repeating procedures often leads to *repeating the same errors* if any were made, which is why alternative paths are better.

Appendix B

Instructional philosophy

“The unexamined circuit is not worth energizing” – Socrates (if he had taught electricity)

These learning modules, although useful for self-study, were designed to be used in a formal learning environment where a subject-matter expert challenges students to digest the content and exercise their critical thinking abilities in the answering of questions and in the construction and testing of working circuits.

The following principles inform the instructional and assessment philosophies embodied in these learning modules:

- The first goal of education is to enhance clear and independent thought, in order that every student reach their fullest potential in a highly complex and inter-dependent world. Robust reasoning is *always* more important than particulars of any subject matter, because its application is universal.
- Literacy is fundamental to independent learning and thought because text continues to be the most efficient way to communicate complex ideas over space and time. Those who cannot read with ease are limited in their ability to acquire knowledge and perspective.
- Articulate communication is fundamental to work that is complex and interdisciplinary.
- Faulty assumptions and poor reasoning are best corrected through challenge, not presentation. The rhetorical technique of *reductio ad absurdum* (disproving an assertion by exposing an absurdity) works well to discipline student’s minds, not only to correct the problem at hand but also to learn how to detect and correct future errors.
- Important principles should be repeatedly explored and widely applied throughout a course of study, not only to reinforce their importance and help ensure their mastery, but also to showcase the interconnectedness and utility of knowledge.

These learning modules were expressly designed to be used in an “inverted” teaching environment¹ where students first read the introductory and tutorial chapters on their own, then individually attempt to answer the questions and construct working circuits according to the experiment and project guidelines. The instructor never lectures, but instead meets regularly with each individual student to review their progress, answer questions, identify misconceptions, and challenge the student to new depths of understanding through further questioning. Regular meetings between instructor and student should resemble a Socratic² dialogue, where questions serve as scalpels to dissect topics and expose assumptions. The student passes each module only after consistently demonstrating their ability to logically analyze and correctly apply all major concepts in each question or project/experiment. The instructor must be vigilant in probing each student’s understanding to ensure they are truly *reasoning* and not just *memorizing*. This is why “Challenge” points appear throughout, as prompts for students to think deeper about topics and as starting points for instructor queries. Sometimes these challenge points require additional knowledge that hasn’t been covered in the series to answer in full. This is okay, as the major purpose of the Challenges is to stimulate analysis and synthesis on the part of each student.

The instructor must possess enough mastery of the subject matter and awareness of students’ reasoning to generate their own follow-up questions to practically any student response. Even completely correct answers given by the student should be challenged by the instructor for the purpose of having students practice articulating their thoughts and defending their reasoning. Conceptual errors committed by the student should be exposed and corrected not by direct instruction, but rather by reducing the errors to an absurdity³ through well-chosen questions and thought experiments posed by the instructor. Becoming proficient at this style of instruction requires time and dedication, but the positive effects on critical thinking for both student and instructor are spectacular.

An inspection of these learning modules reveals certain unique characteristics. One of these is a bias toward thorough explanations in the tutorial chapters. Without a live instructor to explain concepts and applications to students, the text itself must fulfill this role. This philosophy results in lengthier explanations than what you might typically find in a textbook, each step of the reasoning process fully explained, including footnotes addressing common questions and concerns students raise while learning these concepts. Each tutorial seeks to not only explain each major concept in sufficient detail, but also to explain the logic of each concept and how each may be developed

¹In a traditional teaching environment, students first encounter new information via *lecture* from an expert, and then independently apply that information via *homework*. In an “inverted” course of study, students first encounter new information via *homework*, and then independently apply that information under the scrutiny of an expert. The expert’s role in lecture is to simply *explain*, but the expert’s role in an inverted session is to *challenge*, *critique*, and if necessary *explain* where gaps in understanding still exist.

²Socrates is a figure in ancient Greek philosophy famous for his unflinching style of questioning. Although he authored no texts, he appears as a character in Plato’s many writings. The essence of Socratic philosophy is to leave no question unexamined and no point of view unchallenged. While purists may argue a topic such as electric circuits is too narrow for a true Socratic-style dialogue, I would argue that the essential thought processes involved with scientific reasoning on *any* topic are not far removed from the Socratic ideal, and that students of electricity and electronics would do very well to challenge assumptions, pose thought experiments, identify fallacies, and otherwise employ the arsenal of critical thinking skills modeled by Socrates.

³This rhetorical technique is known by the Latin phrase *reductio ad absurdum*. The concept is to expose errors by counter-example, since only one solid counter-example is necessary to disprove a universal claim. As an example of this, consider the common misconception among beginning students of electricity that voltage cannot exist without current. One way to apply *reductio ad absurdum* to this statement is to ask how much current passes through a fully-charged battery connected to nothing (i.e. a clear example of voltage existing without current).

from “first principles”. Again, this reflects the goal of developing clear and independent thought in students’ minds, by showing how clear and logical thought was used to forge each concept. Students benefit from witnessing a model of clear thinking in action, and these tutorials strive to be just that.

Another characteristic of these learning modules is a lack of step-by-step instructions in the Project and Experiment chapters. Unlike many modern workbooks and laboratory guides where step-by-step instructions are prescribed for each experiment, these modules take the approach that students must learn to closely read the tutorials and apply their own reasoning to identify the appropriate experimental steps. Sometimes these steps are plainly declared in the text, just not as a set of enumerated points. At other times certain steps are implied, an example being assumed competence in test equipment use where the student should not need to be told *again* how to use their multimeter because that was thoroughly explained in previous lessons. In some circumstances no steps are given at all, leaving the entire procedure up to the student.

This lack of prescription is not a flaw, but rather a feature. Close reading and clear thinking are foundational principles of this learning series, and in keeping with this philosophy all activities are designed to *require* those behaviors. Some students may find the lack of prescription frustrating, because it demands more from them than what their previous educational experiences required. This frustration should be interpreted as an unfamiliarity with autonomous thinking, a problem which must be corrected if the student is ever to become a self-directed learner and effective problem-solver. Ultimately, the need for students to read closely and think clearly is more important both in the near-term and far-term than any specific facet of the subject matter at hand. If a student takes longer than expected to complete a module because they are forced to outline, digest, and reason on their own, so be it. The future gains enjoyed by developing this mental discipline will be well worth the additional effort and delay.

Another feature of these learning modules is that they do not treat topics in isolation. Rather, important concepts are introduced early in the series, and appear repeatedly as stepping-stones toward other concepts in subsequent modules. This helps to avoid the “compartmentalization” of knowledge, demonstrating the inter-connectedness of concepts and simultaneously reinforcing them. Each module is fairly complete in itself, reserving the beginning of its tutorial to a review of foundational concepts.

This methodology of assigning text-based modules to students for digestion and then using Socratic dialogue to assess progress and hone students’ thinking was developed over a period of several years by the author with his Electronics and Instrumentation students at the two-year college level. While decidedly unconventional and sometimes even unsettling for students accustomed to a more passive lecture environment, this instructional philosophy has proven its ability to convey conceptual mastery, foster careful analysis, and enhance employability so much better than lecture that the author refuses to ever teach by lecture again.

Problems which often go undiagnosed in a lecture environment are laid bare in this “inverted” format where students must articulate and logically defend their reasoning. This, too, may be unsettling for students accustomed to lecture sessions where the instructor cannot tell for sure who comprehends and who does not, and this vulnerability necessitates sensitivity on the part of the “inverted” session instructor in order that students never feel discouraged by having their errors exposed. *Everyone* makes mistakes from time to time, and learning is a lifelong process! Part of the instructor’s job is to build a culture of learning among the students where errors are not seen as shameful, but rather as opportunities for progress.

To this end, instructors managing courses based on these modules should adhere to the following principles:

- Student questions are always welcome and demand thorough, honest answers. The only type of question an instructor should refuse to answer is one the student should be able to easily answer on their own. Remember, *the fundamental goal of education is for each student to learn to think clearly and independently*. This requires hard work on the part of the student, which no instructor should ever circumvent. Anything done to bypass the student's responsibility to do that hard work ultimately limits that student's potential and thereby does real harm.
- It is not only permissible, but encouraged, to answer a student's question by asking questions in return, these follow-up questions designed to guide the student to reach a correct answer through their own reasoning.
- All student answers demand to be challenged by the instructor and/or by other students. This includes both correct and incorrect answers – the goal is to practice the articulation and defense of one's own reasoning.
- No reading assignment is deemed complete unless and until the student demonstrates their ability to accurately summarize the major points in their own terms. Recitation of the original text is unacceptable. This is why every module contains an "Outline and reflections" question as well as a "Foundational concepts" question in the Conceptual reasoning section, to prompt reflective reading.
- No assigned question is deemed answered unless and until the student demonstrates their ability to consistently and correctly apply the concepts to *variations* of that question. This is why module questions typically contain multiple "Challenges" suggesting different applications of the concept(s) as well as variations on the same theme(s). Instructors are encouraged to devise as many of their own "Challenges" as they are able, in order to have a multitude of ways ready to probe students' understanding.
- No assigned experiment or project is deemed complete unless and until the student demonstrates the task in action. If this cannot be done "live" before the instructor, video-recordings showing the demonstration are acceptable. All relevant safety precautions must be followed, all test equipment must be used correctly, and the student must be able to properly explain all results. The student must also successfully answer all Challenges presented by the instructor for that experiment or project.

Students learning from these modules would do well to abide by the following principles:

- No text should be considered fully and adequately read unless and until you can express every idea *in your own words, using your own examples*.
- You should always articulate your thoughts as you read the text, noting points of agreement, confusion, and epiphanies. Feel free to print the text on paper and then write your notes in the margins. Alternatively, keep a journal for your own reflections as you read. This is truly a helpful tool when digesting complicated concepts.
- Never take the easy path of highlighting or underlining important text. Instead, *summarize* and/or *comment* on the text using your own words. This actively engages your mind, allowing you to more clearly perceive points of confusion or misunderstanding on your own.
- A very helpful strategy when learning new concepts is to place yourself in the role of a teacher, if only as a mental exercise. Either explain what you have recently learned to someone else, or at least *imagine* yourself explaining what you have learned to someone else. The simple act of having to articulate new knowledge and skill forces you to take on a different perspective, and will help reveal weaknesses in your understanding.
- Perform each and every mathematical calculation and thought experiment shown in the text on your own, referring back to the text to see that your results agree. This may seem trivial and unnecessary, but it is critically important to ensuring you actually understand what is presented, especially when the concepts at hand are complicated and easy to misunderstand. Apply this same strategy to become proficient in the use of *circuit simulation software*, checking to see if your simulated results agree with the results shown in the text.
- Above all, recognize that learning is hard work, and that a certain level of frustration is unavoidable. There are times when you will struggle to grasp some of these concepts, and that struggle is a natural thing. Take heart that it will yield with persistent and varied⁴ effort, and never give up!

Students interested in using these modules for self-study will also find them beneficial, although the onus of responsibility for thoroughly reading and answering questions will of course lie with that individual alone. If a qualified instructor is not available to challenge students, a workable alternative is for students to form study groups where they challenge⁵ one another.

To high standards of education,

Tony R. Kuphaldt

⁴As the old saying goes, “Insanity is trying the same thing over and over again, expecting different results.” If you find yourself stumped by something in the text, you should attempt a different approach. Alter the thought experiment, change the mathematical parameters, do whatever you can to see the problem in a slightly different light, and then the solution will often present itself more readily.

⁵Avoid the temptation to simply share answers with study partners, as this is really counter-productive to learning. Always bear in mind that the answer to any question is far less important in the long run than the method(s) used to obtain that answer. The goal of education is to empower one’s life through the improvement of clear and independent thought, literacy, expression, and various practical skills.

Appendix C

Tools used

I am indebted to the developers of many open-source software applications in the creation of these learning modules. The following is a list of these applications with some commentary on each.

You will notice a theme common to many of these applications: a bias toward *code*. Although I am by no means an expert programmer in any computer language, I understand and appreciate the flexibility offered by code-based applications where the user (you) enters commands into a plain ASCII text file, which the software then reads and processes to create the final output. Code-based computer applications are by their very nature *extensible*, while WYSIWYG (What You See Is What You Get) applications are generally limited to whatever user interface the developer makes for you.

The GNU/Linux computer operating system

There is so much to be said about Linus Torvalds' **Linux** and Richard Stallman's **GNU** project. First, to credit just these two individuals is to fail to do justice to the *mob* of passionate volunteers who contributed to make this amazing software a reality. I first learned of **Linux** back in 1996, and have been using this operating system on my personal computers almost exclusively since then. It is *free*, it is completely *configurable*, and it permits the continued use of highly efficient **Unix** applications and scripting languages (e.g. shell scripts, Makefiles, **sed**, **awk**) developed over many decades. **Linux** not only provided me with a powerful computing platform, but its open design served to inspire my life's work of creating open-source educational resources.

Bram Moolenaar's **Vim** text editor

Writing code for any code-based computer application requires a *text editor*, which may be thought of as a word processor strictly limited to outputting plain-ASCII text files. Many good text editors exist, and one's choice of text editor seems to be a deeply personal matter within the programming world. I prefer **Vim** because it operates very similarly to **vi** which is ubiquitous on **Unix/Linux** operating systems, and because it may be entirely operated via keyboard (i.e. no mouse required) which makes it fast to use.

Donald Knuth's \TeX typesetting system

Developed in the late 1970's and early 1980's by computer scientist extraordinaire Donald Knuth to typeset his multi-volume magnum opus *The Art of Computer Programming*, this software allows the production of formatted text for screen-viewing or paper printing, all by writing plain-text code to describe how the formatted text is supposed to appear. \TeX is not just a markup language for documents, but it is also a Turing-complete programming language in and of itself, allowing useful algorithms to be created to control the production of documents. Simply put, *\TeX is a programmer's approach to word processing*. Since \TeX is controlled by code written in a plain-text file, this means anyone may read that plain-text file to see exactly how the document was created. This openness afforded by the code-based nature of \TeX makes it relatively easy to learn how other people have created their own \TeX documents. By contrast, examining a beautiful document created in a conventional WYSIWYG word processor such as Microsoft **Word** suggests nothing to the reader about *how* that document was created, or what the user might do to create something similar. As Mr. Knuth himself once quipped, conventional word processing applications should be called WYSIAYG (What You See Is *All* You Get).

Leslie Lamport's \LaTeX extensions to \TeX

Like all true programming languages, \TeX is inherently extensible. So, years after the release of \TeX to the public, Leslie Lamport decided to create a massive extension allowing easier compilation of book-length documents. The result was \LaTeX , which is the markup language used to create all ModEL module documents. You could say that \TeX is to \LaTeX as **C** is to **C++**. This means it is permissible to use any and all \TeX commands within \LaTeX source code, and it all still works. Some of the features offered by \LaTeX that would be challenging to implement in \TeX include automatic index and table-of-content creation.

Tim Edwards' **Xcircuit** drafting program

This wonderful program is what I use to create all the schematic diagrams and illustrations (but not photographic images or mathematical plots) throughout the ModEL project. It natively outputs PostScript format which is a true vector graphic format (this is why the images do not pixellate when you zoom in for a closer view), and it is so simple to use that I have never had to read the manual! Object libraries are easy to create for **Xcircuit**, being plain-text files using PostScript programming conventions. Over the years I have collected a large set of object libraries useful for drawing electrical and electronic schematics, pictorial diagrams, and other technical illustrations.

Gimp graphic image manipulation program

Essentially an open-source clone of Adobe's **PhotoShop**, I use **Gimp** to resize, crop, and convert file formats for all of the photographic images appearing in the **Model** modules. Although **Gimp** does offer its own scripting language (called **Script-Fu**), I have never had occasion to use it. Thus, my utilization of **Gimp** to merely crop, resize, and convert graphic images is akin to using a sword to slice bread.

SPICE circuit simulation program

SPICE is to circuit analysis as **T_EX** is to document creation: it is a form of markup language designed to describe a certain object to be processed in plain-ASCII text. When the plain-text "source file" is compiled by the software, it outputs the final result. More modern circuit analysis tools certainly exist, but I prefer **SPICE** for the following reasons: it is *free*, it is *fast*, it is *reliable*, and it is a fantastic tool for *teaching* students of electricity and electronics how to write simple code. I happen to use rather old versions of **SPICE**, version 2g6 being my "go to" application when I only require text-based output. **NGSPICE** (version 26), which is based on Berkeley **SPICE** version 3f5, is used when I require graphical output for such things as time-domain waveforms and Bode plots. In all **SPICE** example netlists I strive to use coding conventions compatible with all **SPICE** versions.

Andrew D. Hwang's ePiX mathematical visualization programming library

This amazing project is a **C++** library you may link to any **C/C++** code for the purpose of generating PostScript graphic images of mathematical functions. As a completely free and open-source project, it does all the plotting I would otherwise use a Computer Algebra System (CAS) such as **Mathematica** or **Maple** to do. It should be said that **ePiX** is *not* a Computer Algebra System like **Mathematica** or **Maple**, but merely a mathematical *visualization* tool. In other words, it won't determine integrals for you (you'll have to implement that in your own **C/C++** code!), but it can graph the results, and it does so beautifully. What I really admire about **ePiX** is that it is a **C++** programming library, which means it builds on the existing power and toolset available with that programming language. Mr. Hwang could have probably developed his own stand-alone application for mathematical plotting, but by creating a **C++** library to do the same thing he accomplished something much greater.

`gnuplot` mathematical visualization software

Another open-source tool for mathematical visualization is `gnuplot`. Interestingly, this tool is *not* part of Richard Stallman’s GNU project, its name being a coincidence. For this reason the authors prefer “gnu” *not* be capitalized at all to avoid confusion. This is a much “lighter-weight” alternative to a spreadsheet for plotting tabular data, and the fact that it easily outputs directly to an X11 console or a file in a number of different graphical formats (including PostScript) is very helpful. I typically set my `gnuplot` output format to default (X11 on my Linux PC) for quick viewing while I’m developing a visualization, then switch to PostScript file export once the visual is ready to include in the document(s) I’m writing. As with my use of `Gimp` to do rudimentary image editing, my use of `gnuplot` only scratches the surface of its capabilities, but the important points are that it’s *free* and that it *works well*.

Python programming language

Both Python and C++ find extensive use in these modules as instructional aids and exercises, but I’m listing Python here as a *tool* for myself because I use it almost daily as a *calculator*. If you open a Python interpreter console and type `from math import *` you can type mathematical expressions and have it return results just as you would on a hand calculator. Complex-number (i.e. *phasor*) arithmetic is similarly supported if you include the complex-math library (`from cmath import *`). Examples of this are shown in the Programming References chapter (if included) in each module. Of course, being a fully-featured programming language, Python also supports conditionals, loops, and other structures useful for calculation of quantities. Also, running in a console environment where all entries and returned values show as text in a chronologically-ordered list makes it easy to copy-and-paste those calculations to document exactly how they were performed.

Appendix D

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Appendix E

References

Appendix F

Version history

This is a list showing all significant additions, corrections, and other edits made to this learning module. Each entry is referenced by calendar date in reverse chronological order (newest version first), which appears on the front cover of every learning module for easy reference. Any contributors to this open-source document are listed here as well.

25 August 2024 – divided the Introduction chapter into sections, one with recommendations for students, one with a listing of challenging concepts, and one with recommendations for instructors.

12 March 2023 – added a new Quantitative Reasoning question on plotting sine and cosine functions. Also corrected a mis-spelled word.

13 June 2022 – corrected an error in the equation for cotangent, courtesy of Caleb Colby.

18 May 2022 – minor edits made to the “Find the mistake(s)” Diagnostic Reasoning question.

8 May 2021 – commented out or deleted empty chapters.

20 January 2021 – significantly edited the Introduction chapter to make it more suitable as a pre-study guide and to provide cues useful to instructors leading “inverted” teaching sessions.

2-3 January 2021 – added content to Tutorial as well as Questions chapters.

1 January 2021 – document first created.

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