

## AC phase

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Resources and methods for learning about these subjects (list a few here, in preparation for your research):

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Question 1

If we were to express the series-connected DC voltages as *phasors* (arrows pointing with a particular length and a particular direction, graphically expressing magnitude and polarity of an electrical signal), how would we draw them in such a way that the total (or *resultant*) phasors accurately expressed the total voltage of each series-connected pair?

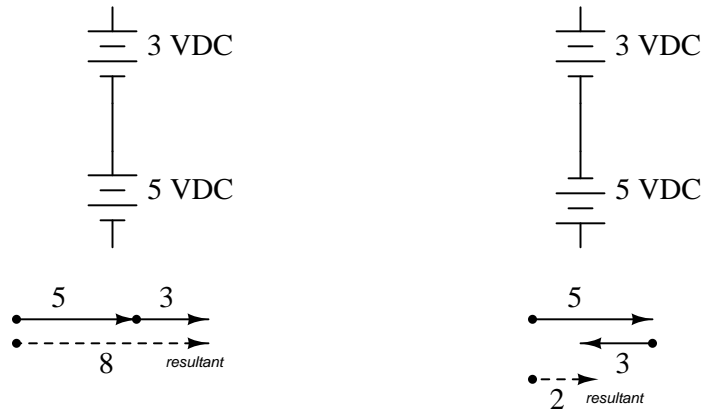


If we were to assign angle values to each of these phasors, what would you suggest?

[file 00493](#)

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Answer 1



In the right-hand circuit, where the two voltage sources are opposing, one of the phasors will have an angle of  $0^\circ$ , while the other will have an angle of  $180^\circ$ .

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Notes 1

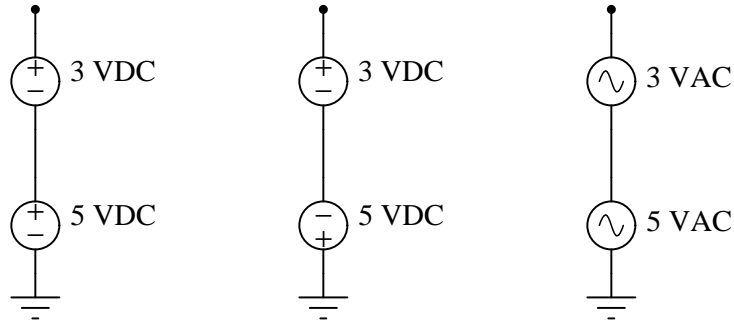
Phasors are really nothing more than an extension of the familiar "number line" most students see during their primary education years. The important difference here is that phasors are two-dimensional magnitudes, not one-dimensional, as *scalar* numbers are.

The use of degrees to measure angles should be familiar as well, even to those students without a strong mathematics background. For example, what does it mean when a skateboarder or stunt bicyclist "does a  $180^\circ$ "? It means they turn around so as to face the opposite direction ( $180^\circ$  degrees away from) their previous direction.

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Question 2

Calculate the total voltage of these series-connected voltage sources:

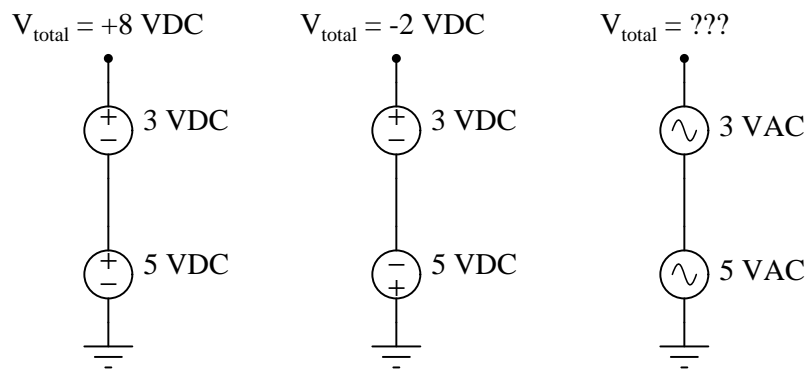


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Answer 2

This is a "trick" question, because only the total voltage of the DC sources may be predicted with certainty. There is insufficient information to calculate the total AC voltage for the two series-connected AC sources!



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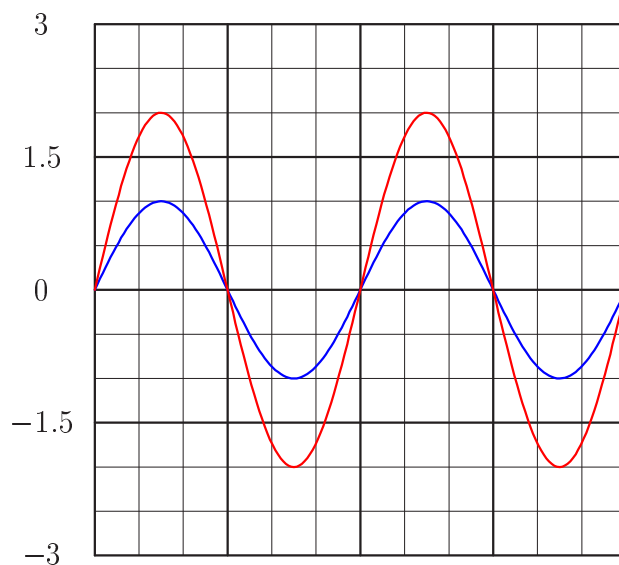
Notes 2

Discuss with your students exactly why the total voltage of the two series-connected AC sources cannot be determined, given the little information we have about them. Is it possible for their total voltage to be 8 VAC, just like the series-aiding DC sources? Is it possible for their total voltage to be 2 VAC, just like the series-opposing DC sources? Why or why not?

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Question 3

Using a computer or graphing calculator, plot the sum of these two sine waves:



What do you suppose the sum of a 1-volt (peak) sine wave and a 2-volt (peak) sine wave will be, if both waves are perfectly in-phase with each other?

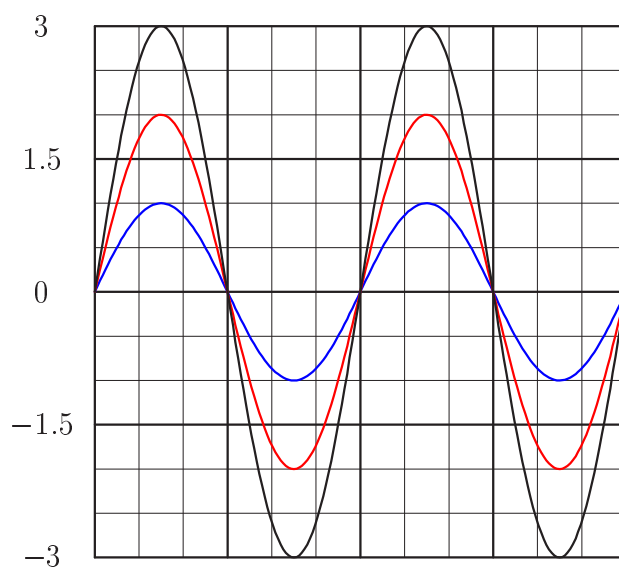
Hint: you will need to enter equations into your plotting device that look something like this:

$$y1 = \sin x$$

$$y2 = 2 * \sin x$$

$$y3 = y1 + y2$$

[file 01557](#)



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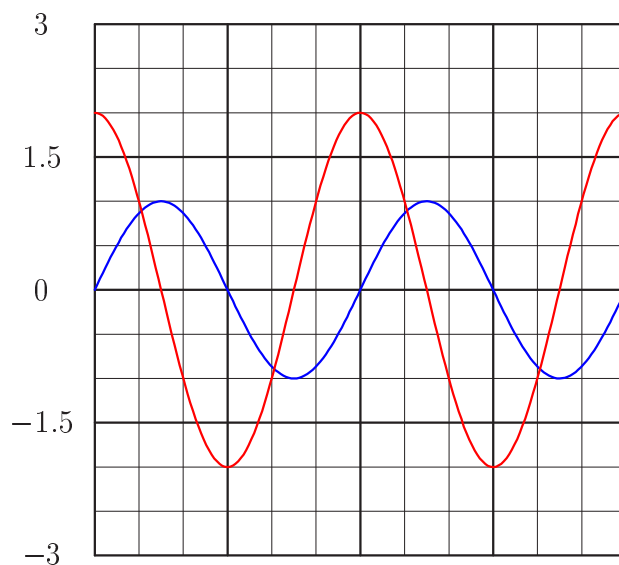
Notes 3

Graphing calculators are excellent tools to use for learning experiences such as this. In far less time than it would take to plot a third sine wave by hand, students may see the sinusoidal sum for themselves.

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Question 4

Using a computer or graphing calculator, plot the sum of these two sine waves:



Hint: you will need to enter equations into your plotting device that look something like this:

$$y1 = \sin x$$

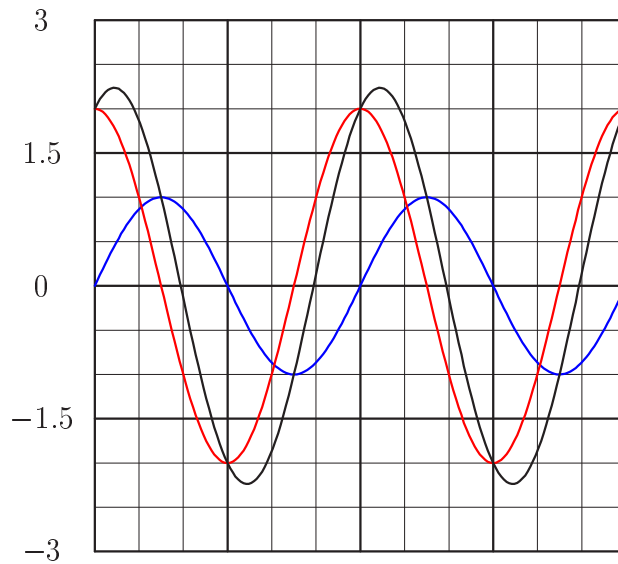
$$y2 = 2 * \sin (x + 90)$$

$$y3 = y1 + y2$$

Note: the second equation assumes your calculator has been set up to calculate trigonometric functions in angle units of *degrees* rather than *radians*. If you wish to plot these same waveforms (with the same phase shift shown) using radians as the unit of angle measurement, you must enter the second equation as follows:

$$y2 = 2 * \sin (x + 1.5708)$$

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Follow-up question: note that the sum of the 1-volt wave and the 2-volt wave does *not* equate to a 3-volt wave! Explain why.

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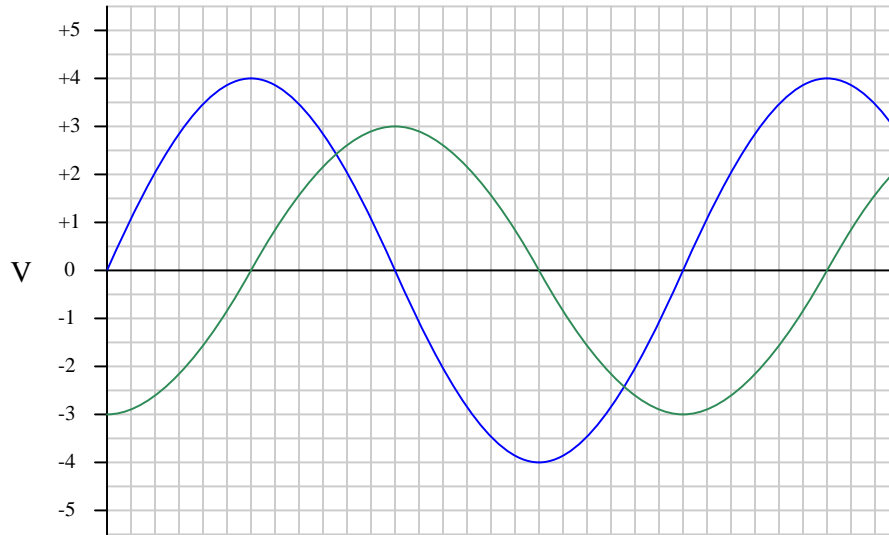
Notes 4

Graphing calculators are excellent tools to use for learning experiences such as this. In far less time than it would take to plot a third sine wave by hand, students may see the sinusoidal sum for themselves.

The point of this question is to get students thinking about how it is possible for sinusoidal voltages to not add up as one might expect. This is very important, because it indicates simple arithmetic processes like addition will not be as simple in AC circuits as it was in DC circuits, due to phase shift. Be sure to emphasize this point to your students.

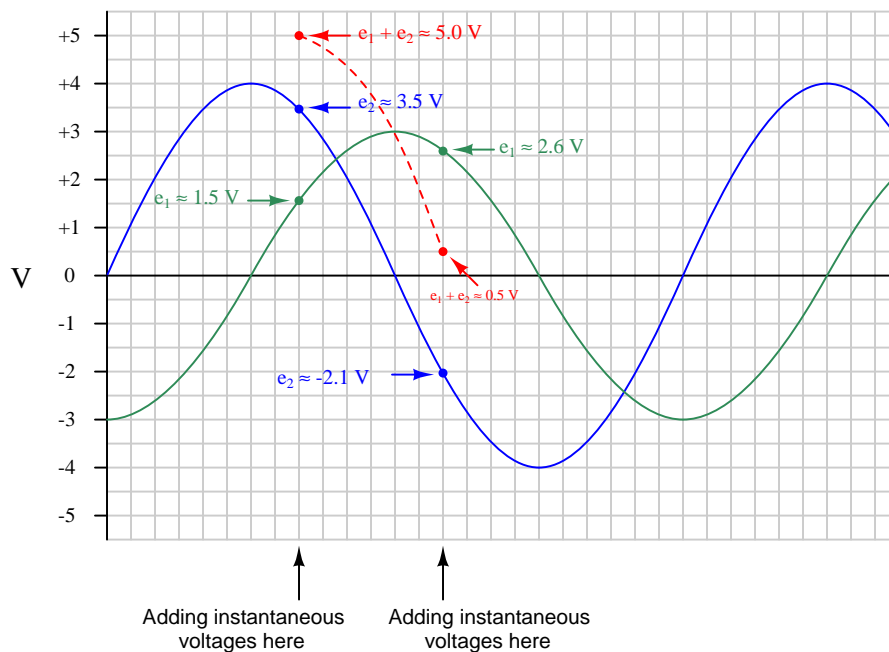
Question 5

Shown here are two sine waves of equal frequency, superimposed on the same graph:



As accurately as possible, determine the amount of phase shift between the two waves, based on the divisions shown on the graph.

Also, plot a third sine wave that is the *sum* of the two sine waves shown. Again, do this as accurately as possible, based on the divisions shown on the graph. To give an example of how you might do this, observe the following illustration:



What is the peak value of the resultant (sum) sine-wave? How does this compare with the peak values of the two original sine waves?

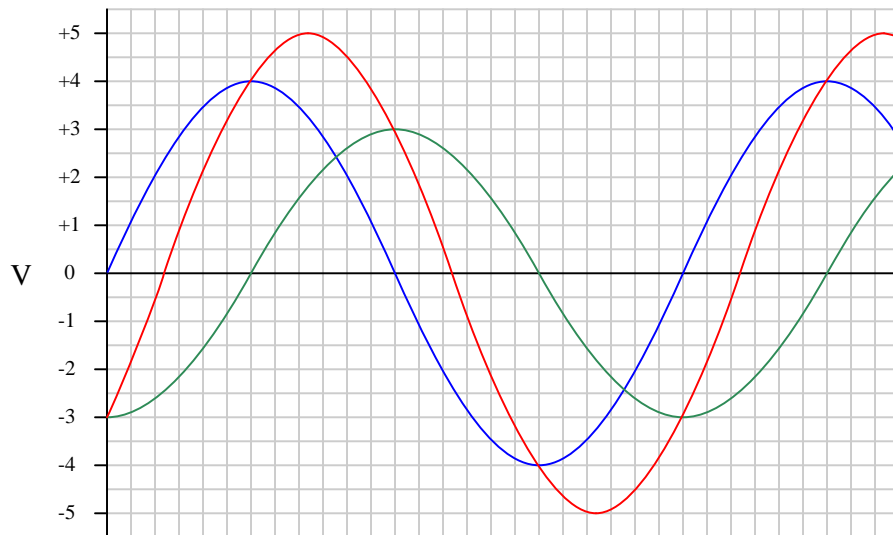
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Answer 5

The two (original) sine waves are  $90^\circ$  out of phase, one with a peak value of 4 volts, and the other with a peak value of 3 volts.



Together, they add to make a third sine wave with a peak value of 5 volts.

Follow-up question: how is it possible that the sum of 3 and 4 makes 5? Hint: you've probably seen something like this when studying trigonometry!

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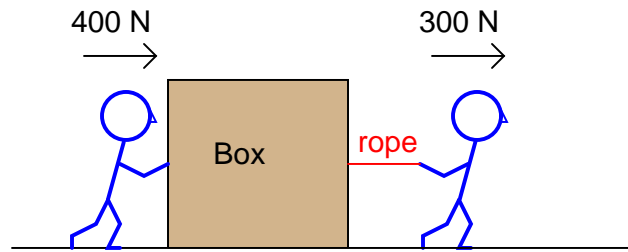
Notes 5

Though it may seem a laborious exercise at first, the point of this question is for students to realize that the sum of two equal-frequency sine waves is another sine wave. The peak values of 3 and 4, together with the phase-shift of  $90^\circ$ , was no coincidence on my part when I wrote this question. Where else in mathematics have students seen an example of the quantities 3 and 4 being combined (at 90 degrees) to make a quantity of 5?

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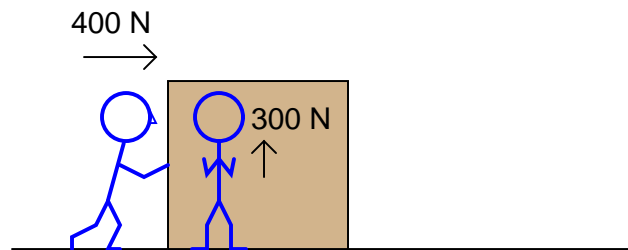
Question 6

Suppose two people work together to slide a large box across the floor, one pushing with a force of 400 newtons and the other pulling with a force of 300 newtons:



The resultant force from these two persons' efforts on the box will, quite obviously, be the sum of their forces: 700 newtons (to the right).

What if the person pulling decides to change position and push *sideways* on the box in relation to the first person, so the 400 newton force and the 300 newton force will be perpendicular to each other (the 300 newton force facing into the page, away from you)? What will the resultant force on the box be then?



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Answer 6

The resultant force on the box will be 500 newtons.

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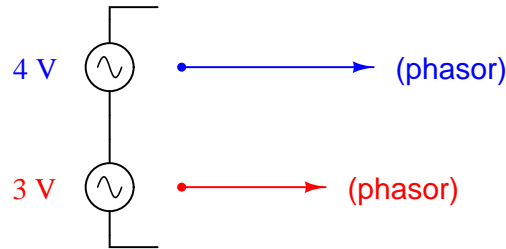
Notes 6

This is a non-electrical application of vector summation, to prepare students for the concept of using vectors to add voltages that are out-of-phase. Note how I chose to use multiples of 3, 4, and 5 for the vector magnitudes.

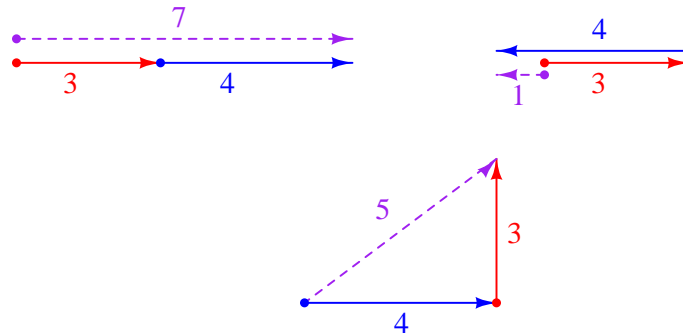
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### Question 7

Special types of vectors called *phasors* are often used to depict the magnitude and phase-shifts of sinusoidal AC voltages and currents. Suppose that the following phasors represent the series summation of two AC voltages, one with a magnitude of 3 volts and the other with a magnitude of 4 volts:



Explain what each of the following phasor diagrams represents, in electrical terms:



Also explain the significance of these sums: that we may obtain three *different* values of total voltage (7 volts, 1 volt, or 5 volts) from the same series-connected AC voltages. What does this mean for us as we prepare to analyze AC circuits using the rules we learned for DC circuits?

[file 01559](#)

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### Answer 7

Each of the phasor diagrams represents two AC voltages being added together. The dotted phasor represents the sum of the 3-volt and 4-volt signals, for different conditions of phase shift between them.

Please note that these three possibilities are not exhaustive! There are a multitude of other possible total voltages that the series-connected 3 volt and 4 volt sources may create.

Follow-up question: in DC circuits, it is permissible to connect multiple voltage sources in parallel, so long as the voltages (magnitudes) and polarities are the same. Is this also true for AC? Why or why not?

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### Notes 7

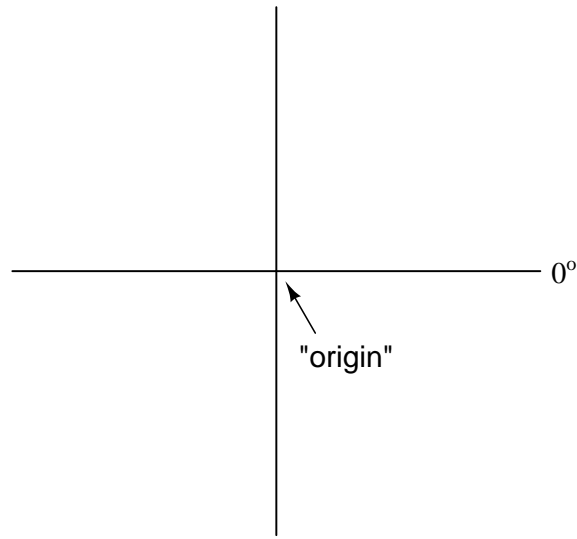
Be sure to discuss with your students that these three conditions shown are not the only conditions possible! I simply chose  $0^\circ$ ,  $180^\circ$ , and  $90^\circ$  because they all resulted in round sums for the given quantities.

The follow-up question previews an important subject concerning AC phase: the necessary *synchronization* or paralleled AC voltage sources.

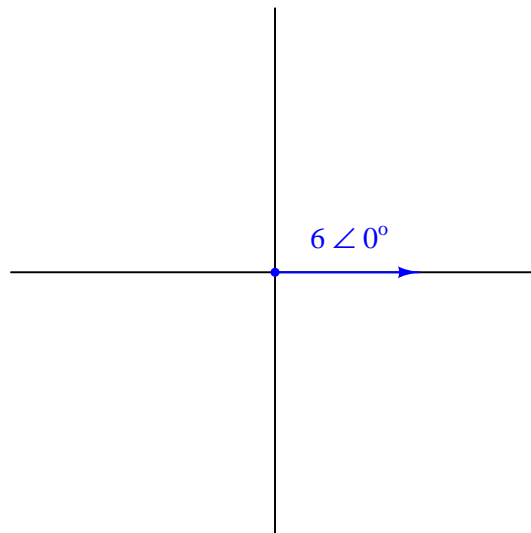
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Question 8

When drawing phasor diagrams, there is a standardized orientation for all angles used to ensure consistency between diagrams. This orientation is usually referenced to a set of perpendicular lines, like the  $x$  and  $y$  axes commonly seen when graphing algebraic functions:

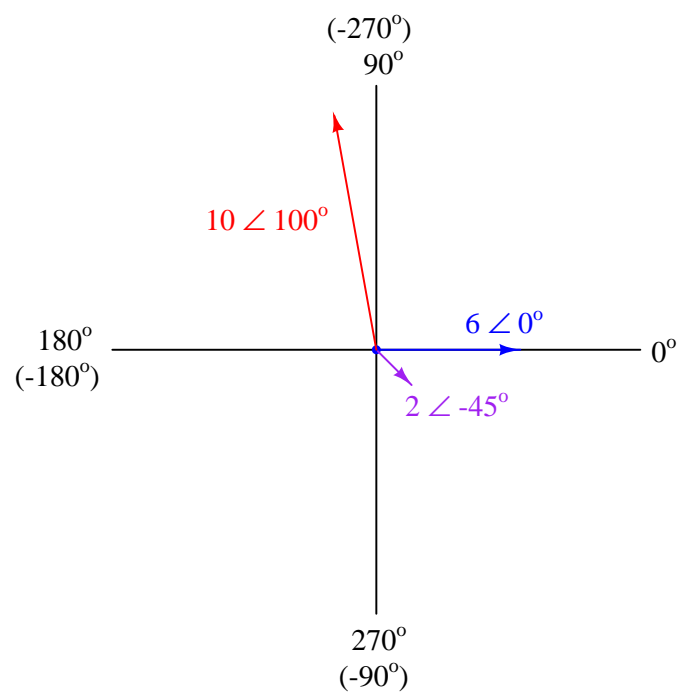


The intersection of the two axes is called the *origin*, and straight horizontal to the right is the definition of zero degrees ( $0^\circ$ ). Thus, a phasor with a magnitude of 6 and an angle of  $0^\circ$  would look like this on the diagram:



Draw a phasor with a magnitude of 10 and an angle of 100 degrees on the above diagram, as well as a phasor with a magnitude of 2 and an angle of  $-45^\circ$ . Label what directions  $90^\circ$ ,  $180^\circ$ , and  $270^\circ$  would indicate on the same diagram.

[file 02099](#)



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Notes 8

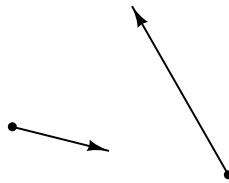
Graph paper, a ruler, and a protractor may be helpful for your students as they begin to draw and interpret phasor diagrams. Even if they have no prior knowledge of trigonometry or phasors, they should still be able to graphically represent simple phasor systems and even solve for resultant phasors.

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Question 9

What does it mean to *add* two or more phasors together, in a geometric sense? How would one draw a phasor diagram showing the following two phasors added together?

*How would you add these phasors together?*



file 02100

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Answer 9

Here are two ways of showing the same addition:



Follow-up question: how would you verbally explain the process of phasor addition? If you were to describe to someone else how to add phasors together, what would you tell them?

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Notes 9

Discuss with your students that phasors may also be subtracted, multiplied, and divided. Subtraction is not too difficult to visualize, but addition and multiplication defies geometric understanding for many.

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Question 10

Determine the sum of these two phasors, and draw a phasor diagram showing their geometric addition:

$$(4 \angle 0^\circ) + (3 \angle 90^\circ)$$

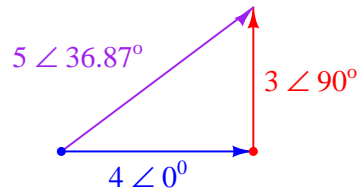
How might a phasor arithmetic problem such as this relate to an AC circuit?

[file 00495](#)

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Answer 10

$$(4 \angle 0^\circ) + (3 \angle 90^\circ) = (5 \angle 36.87^\circ)$$



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Notes 10

It is very helpful in a question such as this to graphically depict the phasors. Have one of your students draw a phasor diagram on the whiteboard for the whole class to observe and discuss.

The relation of this arithmetic problem to an AC circuit is a very important one for students to grasp. It is one thing for students to be able to mathematically manipulate and combine phasors, but quite another for them to smoothly transition between a phasor operation and comprehension of voltages and/or currents in an AC circuit. Ask your students to describe what the *magnitude* of a phasor means (in this example, the number 5), if that phasor represents an AC voltage. Ask your students to describe what the *angle* of an AC voltage phasor means, as well (in this case,  $36.87^\circ$ ), for an AC voltage.

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Question 11

Phasors may be symbolically described in two different ways: *polar notation* and *rectangular notation*. Explain what each of these notations means, and why either one may adequately describe a phasor.

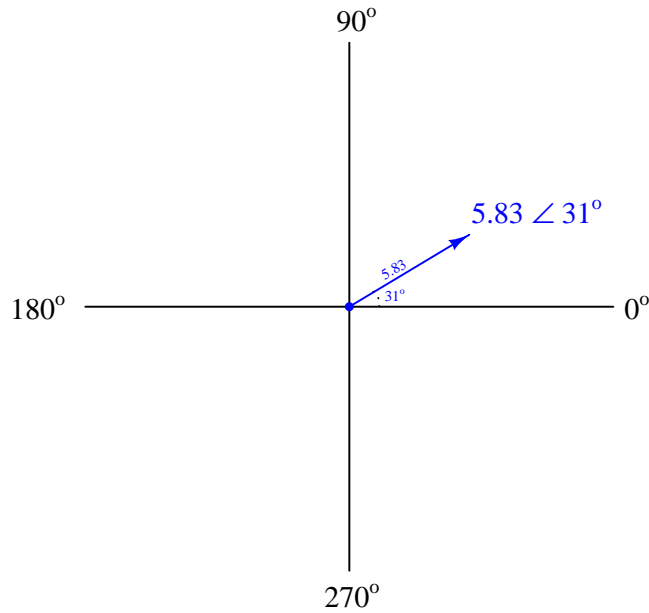
file 02101



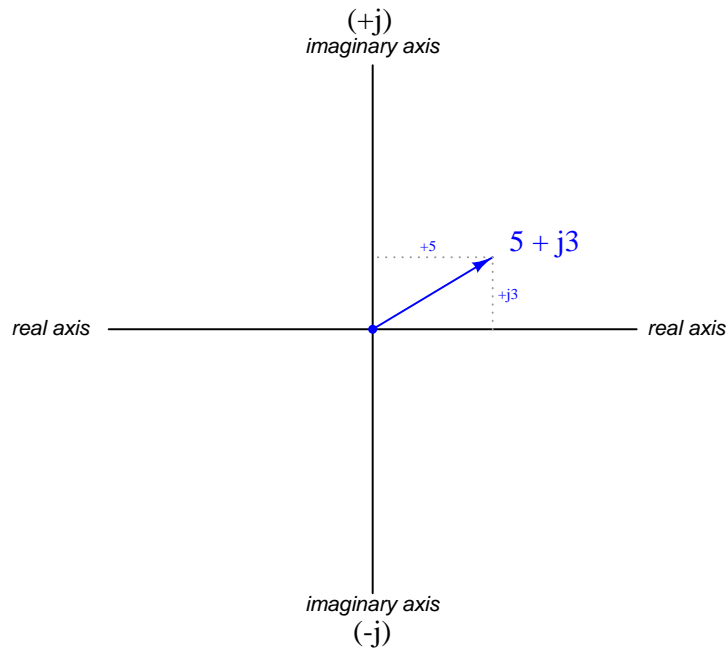
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Answer 11

Polar notation describes a phasor in terms of magnitude (length) and angle:



Rectangular notation describes a phasor in terms of horizontal and vertical displacement:



Follow-up question: why do we need the letter  $j$  in rectangular notation? What purpose does it serve, and what does it mean?

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Notes 11

When discussing the meaning of  $j$ , it might be good to explain what *imaginary numbers* are. Whether or not you choose to do this depends on the mathematical aptitude and background of your students.

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Question 12

These two phasors are written in a form known as *polar notation*. Re-write them in *rectangular notation*:

$$4 \angle 0^\circ =$$

$$3 \angle 90^\circ =$$

[file 00497](#)

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Answer 12

These two phasors, written in rectangular notation, would be  $4 + j0$  and  $0 + j3$ , respectively, although a mathematician would probably write them as  $4 + i0$  and  $0 + i3$ , respectively.

Challenge question: what does the lower-case  $j$  or  $i$  represent, in mathematical terms?

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Notes 12

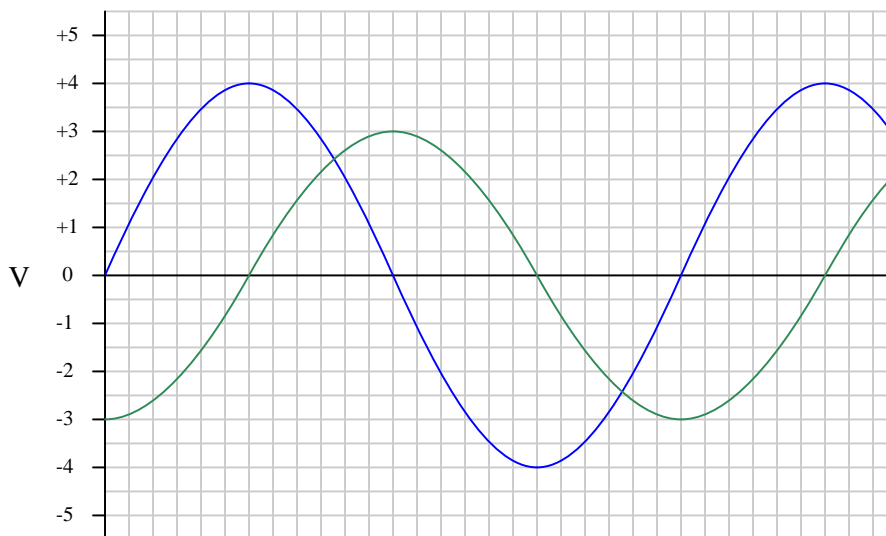
Discuss with your students the two notations commonly used with phasors: *polar* and *rectangular* form. They are merely two different ways of "saying" the same thing. A helpful "prop" for this discussion is the complex number *plane* (as opposed to a number *line* – a one-dimensional field), showing the "real" and "imaginary" axes, in addition to standard angles (right =  $0^\circ$ , left =  $180^\circ$ , up =  $90^\circ$ , down =  $270^\circ$ ). Your students should be familiar with this from their research, so have one of them draw the number plane on the whiteboard for all to view.

The challenge question regards the origin of complex numbers, beginning with the distinction of "imaginary" numbers as being a separate set of quantities from "real" numbers. Electrical engineers, of course, avoid using the lower-case letter  $i$  to denote "imaginary" because it would be so easily be confused with the standard notation for instantaneous current  $i$ .

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### Question 13

In this graph of two AC voltages, which one is *leading* and which one is *lagging*?



If the 4-volt (peak) sine wave is denoted in phasor notation as  $4 \text{ V} \angle 0^\circ$ , how should the 3-volt (peak) waveform be denoted? Express your answer in both polar and rectangular forms.

If the 4-volt (peak) sine wave is denoted in phasor notation as  $4 \text{ V} \angle 90^\circ$ , how should the 3-volt (peak) waveform be denoted? Express your answer in both polar and rectangular forms.

[file 00499](#)

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### Answer 13

The 4-volt (peak) waveform *leads* the 3-volt (peak) waveform. Conversely, the 3-volt waveform *lags* behind the 4-volt waveform.

If the 4-volt waveform is denoted as  $4 \text{ V} \angle 0^\circ$ , then the 3-volt waveform should be denoted as  $3 \text{ V} \angle -90^\circ$ , or  $0 - j3 \text{ V}$ .

If the 4-volt waveform is denoted as  $4 \text{ V} \angle 90^\circ$  ( $0 + j4 \text{ V}$  in rectangular form), then the 3-volt waveform should be denoted as  $3 \text{ V} \angle 0^\circ$ , or  $3 + j0 \text{ V}$ .

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### Notes 13

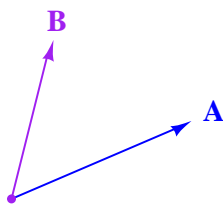
In my years of teaching, I have been surprised at how many students struggle with identifying the "leading" and "lagging" waveforms on a time-domain graph. Be sure to discuss this topic well with your students, identifying methods for correctly distinguishing "leading" waves from "lagging" waves.

This question also provides students with good practice expressing leading and lagging waves in phasor notation. One of the characteristics of phasors made evident in the answer is the relative nature of angles. Be sure to point this out to your students.

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Question 14

In this phasor diagram, determine which phasor is *leading* and which is *lagging* the other:



file 03286

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Answer 14

In this diagram, phasor **B** is leading phasor **A**.

Follow-up question: using a protractor, estimate the amount of phase shift between these two phasors.

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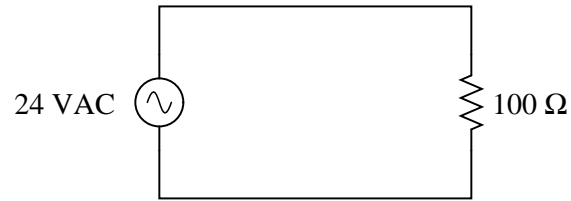
Notes 14

It may be helpful to your students to remind them of the standard orientation for phase angles in phasor diagrams (0 degrees to the right, 90 degrees up, etc.).

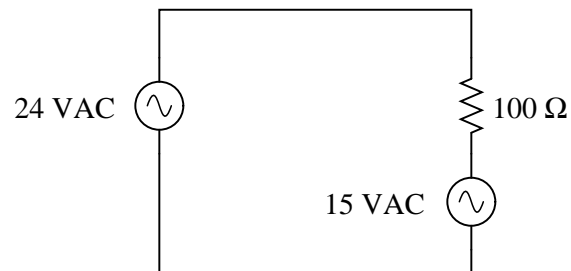
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Question 15

Is it appropriate to assign a phasor *angle* to a single AC voltage, all by itself in a circuit?



What if there is more than one AC voltage source in a circuit?



[file 00496](#)

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Answer 15

Phasor angles are *relative*, not *absolute*. They have meaning only where there is another phasor to compare against.

Angles may be associated with multiple AC voltage sources in the same circuit, but *only if those voltages are all at the same frequency*.

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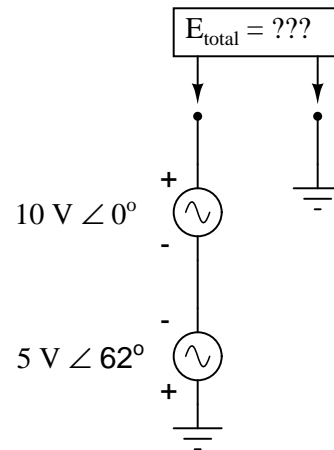
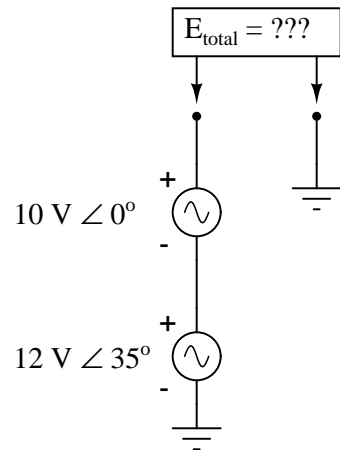
Notes 15

Discuss with your students the notion of "phase angle" in relation to AC quantities. What does it mean, exactly, if a voltage is "3 volts at an angle of 90 degrees"? You will find that such a description only makes sense where there is another voltage (i.e., "4 volts at 0 degrees") to compare to. Without a frame of reference, phasor angles are meaningless.

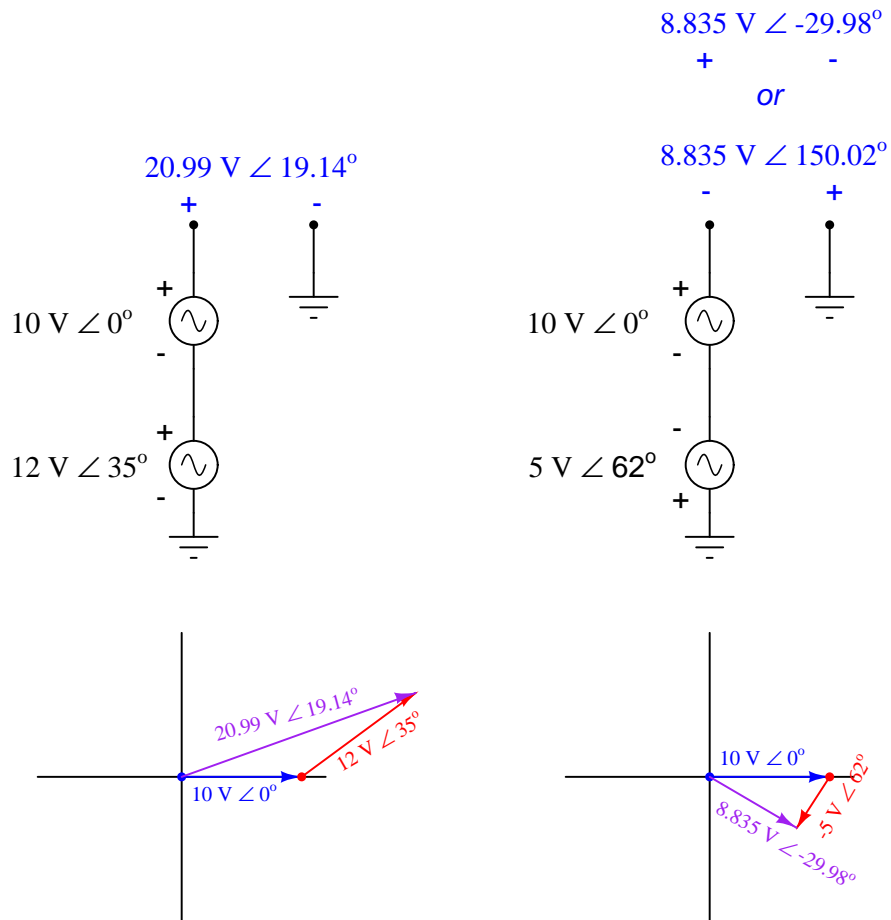
Also discuss with your students the nature of phase shifts between different AC voltage sources, if the sources are all at different frequencies. Would the phase angles be fixed, or vary over time? Why? In light of this, why do we not assign phase angles when different frequencies are involved?

Question 16

Determine the total voltage in each of these examples, drawing a phasor diagram to show how the total (resultant) voltage geometrically relates to the source voltages in each scenario:



file 00498



Notes 16

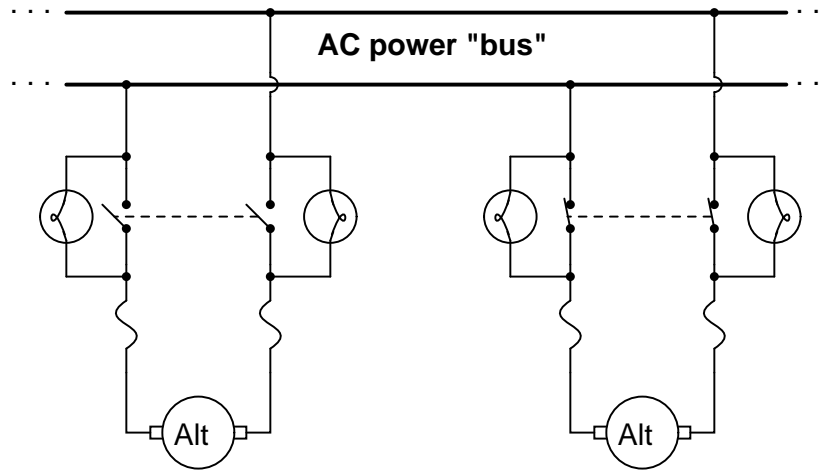
At first it may confuse students to use polarity marks (+ and -) for AC voltages. After all, doesn't the polarity of AC *alternate* back and forth, so as to be continuously changing? However, when analyzing AC circuits, polarity marks are essential for giving a frame of reference to phasor voltages, which like all voltages are measured *between two points*, and thus may be measured two different ways.

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Question 17

Before two or more operating alternators (AC generators) may be electrically coupled, they must be brought into synchronization with each other. If two alternators are out of "sync" (or out of *phase*) with each other, the result will be a large fault current when the disconnect switch is closed.

A simple and effective means of checking for "sync" prior to closing the disconnect switch for an alternator is to have light bulbs connected in parallel with the disconnect switch contacts, like this:



What should the alternator operator look for before closing the alternator switch? Do bright lights indicate a condition of being "in-phase" with the bus, or do dim lights indicate this? What does the operator have to do in order to bring an alternator into "phase" with the bus voltage?

Also, describe what the light bulbs would do if the two alternators were spinning at slightly different speeds.

[file 00491](#)

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Answer 17

Dim lights indicate a condition of being "in-phase" with the bus. If the two alternators are spinning at slightly different speeds, there will be a *heterodyne* effect to the light bulbs' brightness: alternately growing brighter, then dimmer, then brighter again.

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Notes 17

Proper synchronization of alternators with bus voltage is a task that used to be performed exclusively by human operators, but may now be accomplished by automatic controls. It is still important, though, for students of electricity to understand the principles involved in alternator synchronization, and the simple light bulb technique of sync-indication is an excellent means of clarifying the concept.

Discuss with your students the *means* of bringing an alternator into phase with an AC bus. If the light bulbs are glowing brightly, what should the operator do to make them dim?

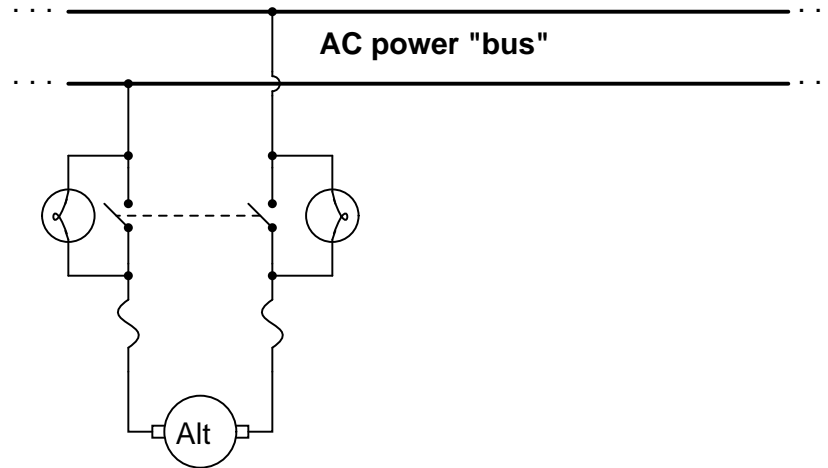
It might also be a good idea to discuss with your students what happens once two synchronized alternators become electrically coupled: the two machines become "locked" together as though they were mechanically coupled, thus maintaining synchronization from that point onward.



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Question 18

Suppose a power plant operator was about to bring this alternator on-line (connect it to the AC bus), and happened to notice that neither one of the synchronizing lights was lit at all. Thinking this to be unusual, the operator calls you to determine if something is wrong with the system. Describe what you would do to troubleshoot this system.



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Answer 18

Before you proceed with troubleshooting steps, first try to determine if there is anything wrong with this system at all. Could it be that the operator is just overly cautious, or is their caution justified?

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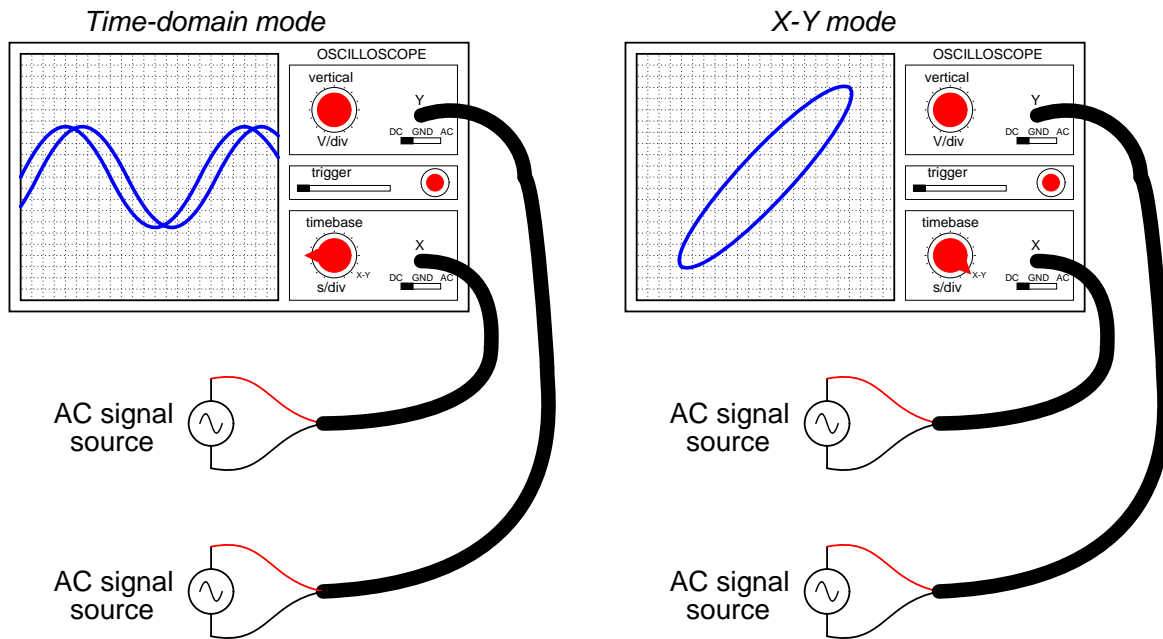
Notes 18

There may be some students who suggest there is nothing wrong at all with the system. Indeed, since dim (or dark) lights normally indicate synchronization, would not the presence of two dim lights indicate that perfect synchronization had already been achieved? Discuss the likelihood of this scenario with your students, that two independent alternators could be maintaining perfect synchronization without being coupled together.

In regard to troubleshooting, this scenario has great potential for group discussion. Despite there being a simple, single, probable condition that could cause this problem, there are several possible component failures that could have created the condition. Different students will undoubtedly have different methods of approaching the problem. Let each one share their views, and discuss together what would be the best approach.

Question 19

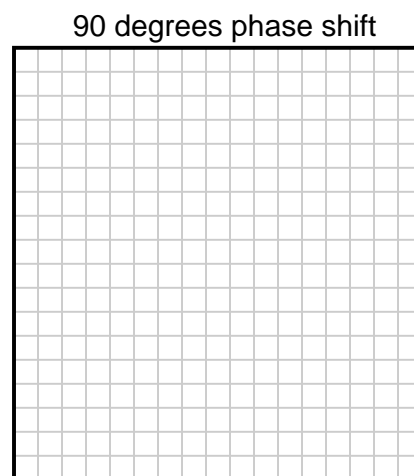
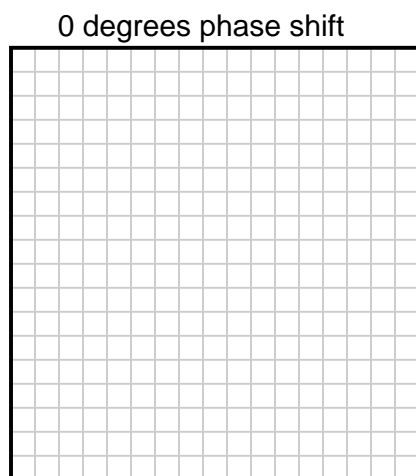
A common feature of oscilloscopes is the  $X - Y$  mode, where the vertical and horizontal plot directions are driven by external signals, rather than only the vertical direction being driven by a measured signal and the horizontal being driven by the oscilloscope's internal sweep circuitry:



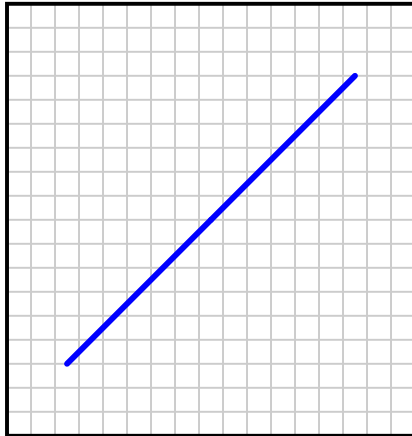
The oval pattern shown in the right-hand oscilloscope display of the above illustration is typical for two sinusoidal waveforms of the same frequency, but slightly out of phase with one another. The technical name for this type of  $X - Y$  plot is a *Lissajous figure*.

What should the Lissajous figure look like for two sinusoidal waveforms that are at exactly the same frequency, and exactly the same phase (0 degrees phase shift between the two)? What should the Lissajous figure look like for two sinusoidal waveforms that are exactly 90 degrees out of phase?

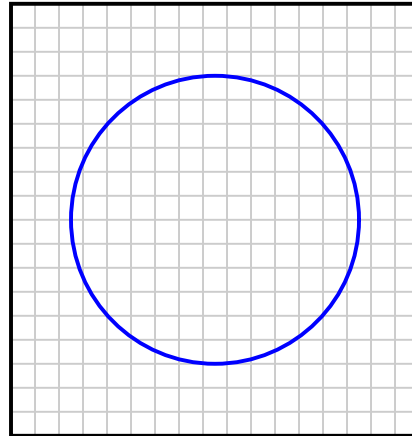
A good way to answer each of these questions is to plot the specified waveforms over time on graph paper, then determine their instantaneous amplitudes at equal time intervals, and then determine where that would place the "dot" on the oscilloscope screen at those points in time, in  $X - Y$  mode. To help you, I'll provide two blank oscilloscope displays for you to draw the Lissajous figures on:



0 degrees phase shift



90 degrees phase shift



Challenge question: what kind of Lissajous figures would be plotted by the oscilloscope if the signals were non-sinusoidal? Perhaps the simplest example of this would be two square waves instead of two sine waves.

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#### Notes 19

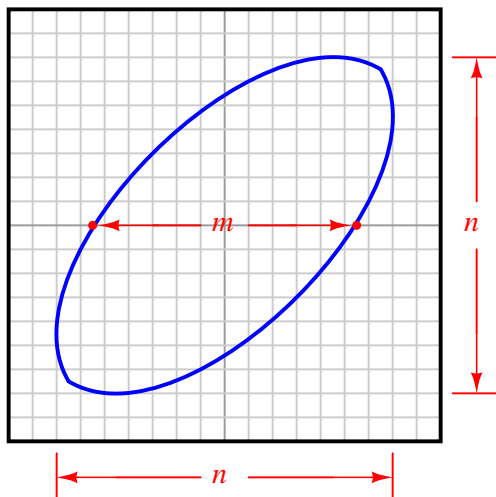
Many students seem to have trouble grasping how Lissajous figures are formed. One of the demonstrations I use to overcome this conceptual barrier is an analog oscilloscope and two signal generators set to very low frequencies, so students can see the "dot" being swept across the screen by both waveforms in slow-motion. Then, I speed up the signals and let them see how the Lissajous pattern becomes more "solid" with persistence of vision and the inherent phosphor delay of the screen.

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Question 20

*Lissajous figures*, drawn by an oscilloscope, are a powerful tool for visualizing the phase relationship between two waveforms. In fact, there is a mathematical formula for calculating the amount of phase shift between two sinusoidal signals, given a couple of dimensional measurements of the figure on the oscilloscope screen.

The procedure begins with adjusting the vertical and horizontal amplitude controls so that the Lissajous figure is proportional: just as tall as it is wide on the screen ( $n$ ). Then, we make sure the figure is centered on the screen and we take a measurement of the distance between the x-axis intercept points ( $m$ ), as such:



Determine what the formula is for calculating the phase shift angle for this circuit, given these dimensions. Hint: the formula is trigonometric! If you don't know where to begin, recall what the respective Lissajous figures look like for a  $0^\circ$  phase shift and for a  $90^\circ$  phase shift, and work from there.

[file 01481](#)

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Answer 20

$$\Theta = \sin^{-1} \left( \frac{m}{n} \right)$$

Challenge question: what kind of Lissajous figure would be drawn by two sinusoidal waveforms at slightly different frequencies?

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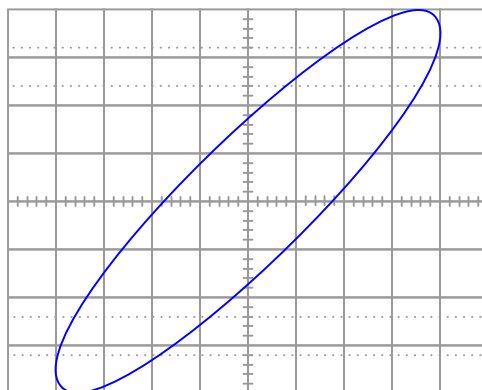
Notes 20

This is a great exercise in teaching students how to derive an equation from physical measurements when the fundamental nature of that equation (trigonometric) is already known. They should already know what the Lissajous figures for both  $0^\circ$  and  $90^\circ$  look like, and should have no trouble figuring out what  $a$  and  $b$  values these two scenarios would yield if measured similarly on the oscilloscope display. The rest is just fitting the pieces together so that the trigonometric function yields the correct angle(s).

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Question 21

Calculate the amount of phase shift indicated by this Lissajous figure:



file 03578

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Answer 21

$$\Theta \approx 25.9^\circ$$

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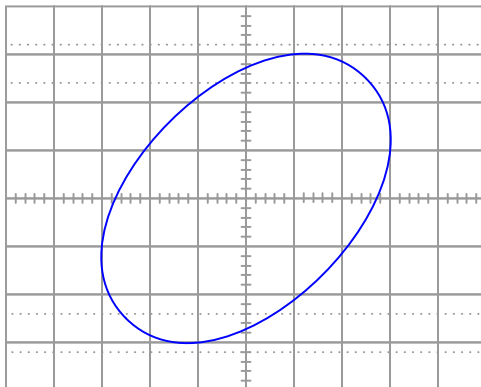
Notes 21

This question is nothing more than an exercise in Lissajous figure interpretation.

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Question 22

Calculate the amount of phase shift indicated by this Lissajous figure:



file 03575

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Answer 22

$$\Theta \approx 64.2^\circ$$

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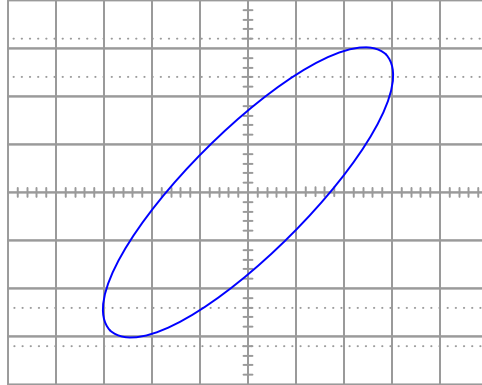
Notes 22

This question is nothing more than an exercise in Lissajous figure interpretation.

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Question 23

Calculate the amount of phase shift indicated by this Lissajous figure:



file 03577

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Answer 23

$$\Theta \approx 34.5^\circ$$

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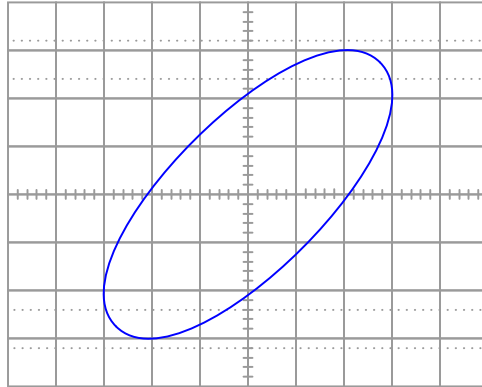
Notes 23

This question is nothing more than an exercise in Lissajous figure interpretation.

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Question 24

Calculate the amount of phase shift indicated by this Lissajous figure:



file 03576

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Answer 24

$$\Theta \approx 44.4^\circ$$

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Notes 24

This question is nothing more than an exercise in Lissajous figure interpretation.