

Massachusetts Institute of Technology
Instrumentation Laboratory
Cambridge, Massachusetts

MEMO

TO: Distribution
FROM: A. R. Klumpp
DATE: January 7, 1966
SUBJECT: The Coriolis Equation

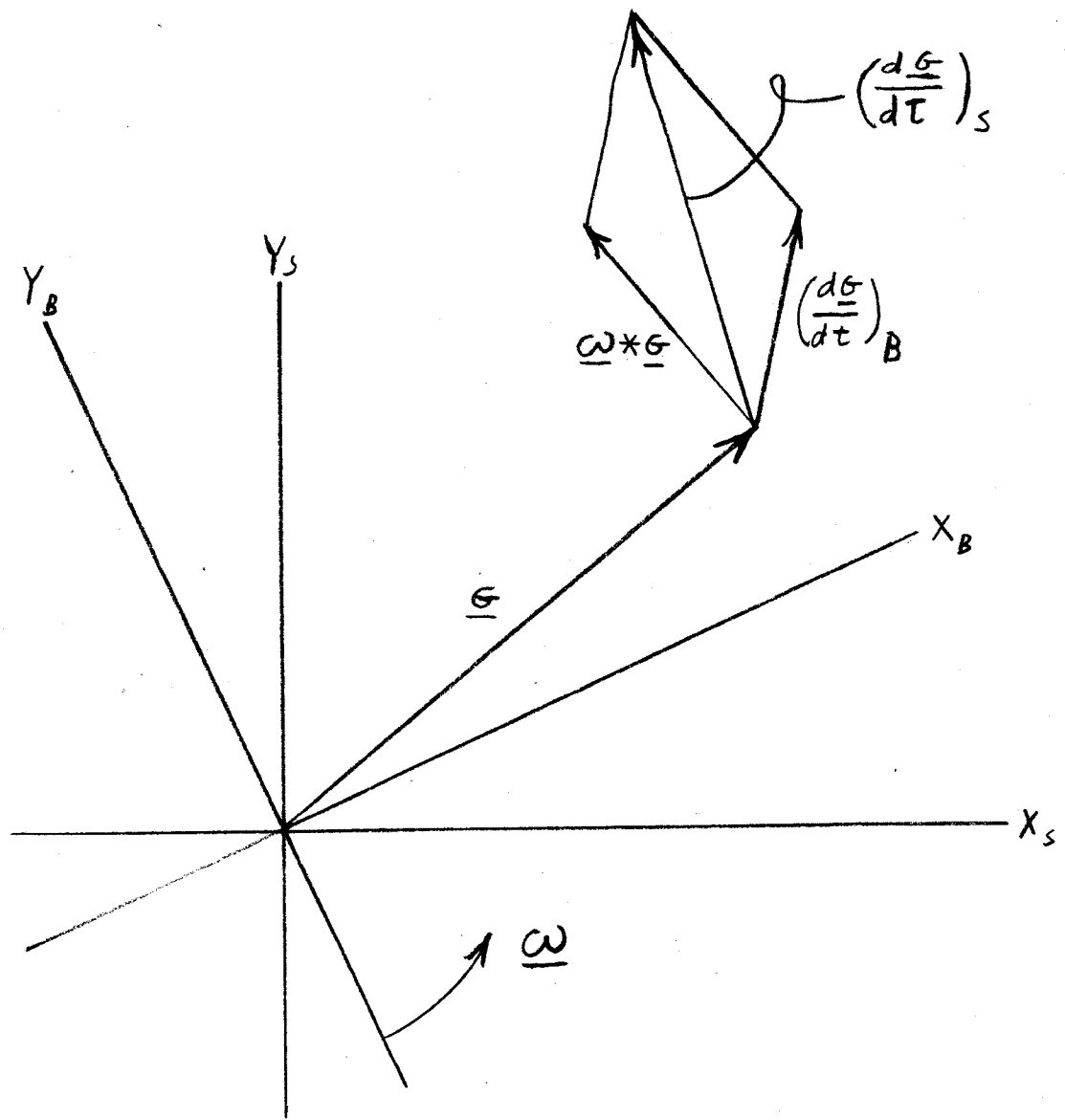
This memo uses the Coriolis equation and certain concepts to arrive at a contradiction. The Coriolis principle is not questioned, but the need for caution in applying it is illustrated. The concepts used to reach the contradiction appear inherent in some formulations of the Coriolis principle. These concepts are illuminated along the way to the contradiction, Eq. (8). Then the Coriolis equation and the second derivative equation are derived by an approach which seems especially straight forward and free from pitfalls. The suggested approach automatically yields the Coriolis, centripetal, and all other terms regardless of how many derivatives are taken, and it yields results in a form directly applicable to numerical analysis.

As written by Goldstein in "Classical Mechanics", the Coriolis equation reads

$$\left(\frac{d\underline{G}}{dt}\right)_S = \left(\frac{d\underline{G}}{dt}\right)_B + \underline{\omega} * \underline{G} \quad (4-100)$$

where \underline{G} is any vector, the subscripts S and B indicate that the derivatives are with respect to the space frame and the body frame respectively, and $\underline{\omega}$ is the angular rate of the body frame with respect to the space frame.

Figure 1 is a vector diagram of the vector Eq. (4-100). Evidently the principle is being used that "a vector is the same vector irrespective of the coordinate frame in which it is viewed". In particular, Eq. (4-100) uses the vector $(d\underline{G}/dt)_B$, which is a vector observed in the body frame, as a component of $(d\underline{G}/dt)_S$, a vector observed in the space frame.



VECTOR DIAGRAM OF THE CORIOLIS EQUATION

FIG 1

The principle of the irrelevance of the coordinate frame can be symbolically expressed as follows,

$$\underline{V}^S = \underline{V}^B \quad (1)$$

where \underline{V} is any vector, and the superscripts indicate the frame in which the vector \underline{V} is viewed; they do not indicate that components are taken, Eq. (1) is a so called "vector equation". The superscripts are not to be confused with the subscripts used in Eq. (4-100) which indicate the coordinate frame with respect to which the derivatives are observed.

Applying the above principle to the vector \underline{G} at two times T_1 and T_2 we have

$$\underline{G}^S|_{t=T_1} = \underline{G}^B|_{t=T_1} \quad (2)$$

and

$$\underline{G}^S|_{t=T_2} = \underline{G}^B|_{t=T_2} \quad (3)$$

But the left member of Eq. (3) is given by

$$\underline{G}^S|_{t=T_2} = \underline{G}^S|_{t=T_1} + \int_{T_1}^{T_2} \left(\frac{d\underline{G}}{dt} \right)_S dt. \quad (4)$$

Similarly the right member of Eq. (3) is given by

$$\underline{G}^B|_{t=T_2} = \underline{G}^B|_{t=T_1} + \int_{T_1}^{T_2} \left(\frac{d\underline{G}}{dt} \right)_B dt. \quad (5)$$

Substituting Eqs.(4) and (5) into Eq. (3) yields

$$\underline{G}^S|_{t=T_1} + \int_{T_1}^{T_2} \left(\frac{d\underline{G}}{dt} \right)_S dt = \underline{G}^B|_{t=T_1} + \int_{T_1}^{T_2} \left(\frac{d\underline{G}}{dt} \right)_B dt. \quad (6)$$

Subtracting Eq. (2) from Eq.(6) yields

$$\int_{T_1}^{T_2} \left(\frac{d\underline{G}}{dt} \right)_S dt = \int_{T_1}^{T_2} \left(\frac{d\underline{G}}{dt} \right)_B dt. \quad (7)$$

Substituting Eq. (4-100) into the left side of Eq. (7) yields

$$\int_{T_1}^{T_2} \left(\left(\frac{d\underline{G}}{dt} \right)_B + \underline{\omega} * \underline{G} \right) dt = \int_{T_1}^{T_2} \left(\frac{d\underline{G}}{dt} \right)_B dt, \quad (8)$$

which is not an equality.

The difficulty obviously arises from the literal application of two apparently conflicting concepts which are:

1. A vector is the same vector irrespective of the coordinate frame in which it is viewed.
2. The derivative of a vector is different in coordinate frames which have relative rotation rates.

Both concepts appear to be inherent in Eq. (4-100).

No difficulty will arise if Eq. (1) is permitted only when modified to include the appropriate transformation.

If this is done, any number of derivatives of Eq. (1) may be taken and the coriolis, centripetal, and other terms will automatically appear. Furthermore the resulting equations will be in a form appropriate for numerical analysis, which Eq. (4-100) generally is not.

Thus modifying Eq. (1),

$$\underline{V}_S = C \underline{V}_B, \quad (9)$$

where the subscripts S and B indicate observation in the space and body frame respectively, and C is the transformation to the space frame. Taking derivatives yields

$$\dot{\underline{V}}_S = C \dot{\underline{V}}_B + \dot{C} \underline{V}_B. \quad (10)$$

It may be proven by taking limits, without using the Coriolis equation, that the derivative of a direction cosine matrix C is given by

$$\dot{C} = C [\underline{\omega}_B] \quad (11)$$

where $[\underline{\omega}_B]$ is the cross product matrix of the vector $\underline{\omega}_B$ defined such that

$$\underline{A} [\underline{\omega}_B] = \underline{A} * \underline{\omega}_B \quad (12)$$

where \underline{A} is any row vector, or

$$[\underline{\omega}_B] \underline{A} = \underline{\omega}_B * \underline{A} \quad (13)$$

where \underline{A} is any column vector.

Substituting Eq. (11) into Eq. (10) yields

$$\dot{\underline{V}}_S = C (\dot{\underline{V}}_B + [\underline{\omega}_B] \underline{V}_B). \quad (14)$$

Using Eq. (13) yields the Coriolis Equation

$$\dot{\underline{V}}_S = C (\dot{\underline{V}}_B + \underline{\omega}_B * \underline{V}_B). \quad (15)$$

Taking another derivative, using Eqs. (15) and (11) yields

$$\ddot{\underline{V}}_S = C (\ddot{\underline{V}}_B + \underline{\omega} * \dot{\underline{V}}_B + \dot{\underline{\omega}} * \underline{V}_B) + C [\underline{\omega}_B] (\dot{\underline{V}}_B + \underline{\omega}_B * \underline{V}_B). \quad (16)$$

Using Eq. (13) yields

$$\ddot{\underline{V}}_S = C (\ddot{\underline{V}}_B + \underline{\omega}_B * \dot{\underline{V}}_B + \dot{\underline{\omega}}_B * \underline{V}_B) + C (\underline{\omega}_B * \dot{\underline{V}}_B + \underline{\omega}_B * (\underline{\omega}_B * \underline{V}_B)). \quad (17)$$

Collecting terms yields finally

$$\ddot{\underline{V}}_S = C (\ddot{\underline{V}}_B + 2\underline{\omega}_B * \dot{\underline{V}}_B - \underline{\omega}_B * \underline{V}_B * \underline{\omega}_B + \dot{\underline{\omega}}_B * \underline{V}_B). \quad (18)$$

Equation (18) displays the Coriolis and centripetal accelerations and the acceleration due to the angular acceleration of the body frame.