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MIT Instrumentation Laboratory

DG Memo No. 637
(Revision A)

TO: Distribution

FROM: P. C. Watson

DATE: 27 January 1966

SUBJECT: Interpretation of Vector Quantities in the Coriolis Equation

REF: Memo by A. R. Klumpp, January 7, 1966

The reference memo discusses the following form of the Coriolis equation:

$$(d\underline{G}/dt)_s = (d\underline{G}/dt)_b + \underline{w}X\underline{G}, \qquad (4\text{-}100)$$

where the subscripts indicate the frames with respect to which the derivatives are to be taken. The reference then states that "a vector is the same vector irrespective of the coordinate (1) frame in which it is viewed", and uses that statement to obtain a contradiction by integrating.

The present memo presents a method of resolving these difficulties. The first step is an interpretation of the Coriolis equation given above:

A. An Interpretation of (4-100)

A better result is obtained if it is recognized that a vector cannot exist independently of any frame, and that any vector has to be specified in terms of at least one frame. Using this approach, it is possible to show that the Coriolis equation as given (4-100) can be used to obtain correct results in any frame.

As an example, consider a vector G, which is coordinatized in frame s. Also consider that another frame b exists with direction cosine transformation matrix C_s^b. Then it is true that

$$G^b = C_s^b \, G^s \qquad (1)$$

In this notation, which is used in the MIT Aero Department, the superscripts refer to the frame in which the vectors are coordinatized. Subscripts define the operation. The subscripts and superscripts on the C matrix are defined as in (1). Equation (1) can now be differentiated with respect to t, yielding:

$$dG^b/dt = C_s^b\,(dG^s/dt) + (dC_s^b/dt)\,G^s \qquad (2)$$

We now use the known result: $dC_s^b/dt = C_s^b\,W_{bs}^{sk}$ (3)

In this equation, W_{bs}^{sk} means the Gibbs cross-product form of W_{bs} coordinatized in the S-frame, where W_{bs} is the angular velocity of the S-frame in respect to the b-frame. Substituting (3) into (2) yields:

$$dG^b/dt = C_s^b\,(dG^s/dt) + C_s^b\,W_{bs}^{sk}\,G^s \qquad (4)$$

Using the notation p = d/dt, and using subscripts to indicate the frame in which differentiation is carried out, we obtain from (4) by performing the indicated operations:

$p_b\,G^b = p_S\,G^b + (w_{bs}XG)^b$, which is the Coriolis theorem (5) as given in 4-100, coordinatized in the b-frame.

Equation (4) can also be obtained in the s-frame by multiplying by C_b^s.

The result is:

$$C_b^s\,(dG^b/dt) = C_b^s\,C_s^b\,(dG^s/dt) + C_b^s\,C_s^b\,W_{bs}^{sk}\,G^s \qquad (6)$$

Performing the indicated operations yields:

$$(p_bG)^s = p_s\,G^s + (w_{sb}\;X\;G)^s, \text{ or} \qquad (7A)$$

$$p_sG^s = p_bG^s + (w_{sb}\;X\;G)^s \qquad (7B)$$

Comparing Equations (7A) and (5) shows that they both have the same form, but that one is coordinatized in the s-frame, while the other is coordinatized in the b-frame.

The conclusion is that the Coriolis theorem is correct in any frame, but that errors could occur if the theorem is applied with some of the terms evaluated in one frame while others are evaluated in some other frame.

B. <u>The integral of an integrand consisting of a vector times the differential of a scalar.</u>

The derivation given in the reference can be analyzed as follows: Equation numbers are those used in the reference. Equation (1) states:

". . $\underline{V}^s = \underline{V}^b$ where $\underline{V}$ is any vector and the superscripts indicate the frame in which the vector V is viewed. They do not indicate that components are taken. Equation (1) is a so called 'vector equation'. The superscripts are not to be confused with the subscripts used in Equation (4-100), which indicate the coordinate frame with respect to which the derivatives are observed".

It should be clear that this equation cannot be substantiated, except if it can be coordinatized to yield a consistent result. If this is done, it would have to be given in coordinate form by multiplying each vector by the appropriate transformation. Similar objections apply to equations (2) and (3) of the reference. Equation (4) of the reference now introduces integration of a derivative of a vector. The derivative is taken in a specified frame and the integral is taken in respect to a scalar parameter, t. If equation (4) is to be made correct, it should be written:

$$\begin{bmatrix} G_{xs}(T_2) \\ Gys(T_2) \\ Gzs(T_2) \end{bmatrix} = \begin{bmatrix} G_{xs}(T_1) \\ Gys(T_1) \\ Gzs(T_1) \end{bmatrix} + \begin{bmatrix} \int_{t1}^{t2} \frac{dGxs}{dt}\, dt \\ \int_{t1}^{t2} \frac{dGys}{dt}\, dt \\ \int_{t1}^{t2} \frac{dGzs}{dt}\, dt \end{bmatrix}$$

If integration as defined above is carried out, it can be shown that the integral is associated with a frame. It can also be shown that the integral of the same vector, carried out in two different frames, yields two different results. A further result is that the integral taken in one frame, and compared to the integral of the same vector taken in another frame and transformed into the first frame will be found to be different.

This can be justified by recognizing that the vector integral contains a history of the vector, and that the transformation of each element of the integral is different as the two frames move in respect to each other. In particular, the vectors representing the two end points of the integral are not connected by the same direction cosine matrix transformation.

Appendix

The revision to the reference contains an attempt to couple equation 4-100 with equation (1) of the reference. The arguments given include items 1, 2, and 3. These items cannot be upheld for the following reasons:

1. Equation 4-100 can be coordinatized in any coordinate frame, using the appropriate transformations, and will yield a correct result.

2. In general, vector equations must contain terms which are all either taken in the same coordinate frame or include enough transformations to make all of the terms interpretable in terms of a single coordinate frame. As an example, $A = B + C$ does not, in general, remain true if only one or two of the three terms are transformed to different frames.

3. An argument attacking equation (1) has already been made. This argument is that equation (1) has no meaning because it cannot be coordinatized. Equation 4-100, however, can be coordinatized, as has been demonstrated.

4. There is nothing in equation 4-100 that implies that a vector has meaning independent of a frame.

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