Because sextant and horizon biases may be important in midcourse navigation, and since non-linear problems do not seem to be of major consequence in midcourse, the following derivation has been carried out. With the following equations it should be possible to avoid doing most Monte Carlo (MC) simulations, although as a final check, MC studies are always desirable.

We know the error in the state vector after the $K$ th measurement is

$$e_K = e_K' - w_K (b_K e_K' - (a_K + \beta_K))$$  (1)

where ' indicates before the measurement and $\alpha_K$ is the random (from measurement to measurement) measurement error; i.e., $\alpha_j \alpha_K = 0$, and $\beta_K$ is the correlated (colored) part of the measurement noise; $\beta_j \beta_K$ can be non-zero. The vectors $w_K$ and $b_K$ are the weighting and geometry vectors.

Taking the transpose of $e_K$ and forming the covariance matrix, we have

$$E_K = e_K e_K^T = (I - w_K b_K^T) \Gamma_K^T (I - w_K b_K^T)^T + w_K w_K^T \frac{1}{2} \frac{\alpha_K^2}{\sigma_a^2} + \frac{\beta_K^2}{\sigma_b^2}$$

$$+ (I - w_K b_K^T) e_K^T \Gamma_K e_K + w_K \beta_K e_K^T (I - w_K b_K^T)$$  (2)
Since we desire to know what the true uncertainties are (neglecting non-linear effects), \( w_K \), the weighting vector which must be used in Eq. 2 is

\[
w_K = \frac{E_{OB,K}^{t} b_K}{b^T_K E_{OB,K}^{t} b_K + \sigma^2_{OB,K}}
\]

where \( E_{OB} \) is the on-board covariance matrix. This \( w_K \) must be used in the calculation of \( E \) (the true covariance). The on-board covariance matrix nominally commences with the approximation

\[
E_{OB} = \begin{pmatrix}
P & O \\
P & P \\
O & P \\
O & V \\
O & V \\
V & V \\
\end{pmatrix}
\]

The true matrix, \( E \), will commence with the true injection uncertainties. Also, the true \( \sigma^2 \) may be different than the on-board \( \sigma^2_{OB} \). Hence, everything is known and can be calculated in Eq. 2 except for the last two terms which involve the unknown vector, \( e_{K}^{t} \beta_{K}^{t} \). Since \( \beta_{K} \) is the colored part of the measurement noise, it can possibly be correlated with all of the previous \( \beta \)'s that are implicitly included in \( e_{K}^{t} \).

The equation for the error before the \( K^{th} \) measurement is from Eq. 1

\[
e_{K}^{t} = \Phi_{K,K-1} \left[ (I - w_{K-1} b_{K-1}^{T}) e_{K-1}^{t} + w_{K-1} (\sigma_{K-1}^{2} + \beta_{K-1}^{2}) \right]
\]

which is just \( e_{K-1}^{t} \) propagated up to the time, \( K \), with the transition matrix, \( \Phi \) (assuming no driving noise).
If we let $e_0$ be the injection error, then just before the first measurement

$$e'_1 = \Phi_{1,0} e_0 \tag{4}$$

After the first measurement the error is:

$$e_1 = e'_1 + w_1 (\alpha_1 + \beta_1 - b^T_{-1} e'_1)$$

and using Eq. (4):

$$e_1 = (I - w_1 b^{T}_{-1}) \Phi_{1,0} e_0 + w_1 (\alpha_1 + \beta_1)$$

Extrapolating to the second measurement gives

$$e'_2 = \Phi_{2,1} e_1 = \Phi_{2,1} \left[ (I - w_1 b^{T}_{-1}) \Phi_{1,0} e_0 + w_1 (\alpha_1 + \beta_1) \right]$$

The error before the next measurement is

$$e'_3 = \Phi_{3,2} \left[ (I - w_2 h^{T}_{-2}) \Phi_{2,1} \left( (I - w_1 b^{T}_{-1}) \Phi_{1,0} e_0 + w_1 (\alpha_1 + \beta_1) \right) 
\quad + w_2 (\alpha_2 + \beta_2) \right] ;$$

etc.

From these above equations and Eq. 3, there follows
\[e^I_K = \Phi_{K,K-1} \left\{ (I - \frac{w_K}{b_{K-1}}) e_{K-1,K-2} \left[ (I - \frac{w_{K-2}}{b_{K-2}}) \ldots \right. \right. \right. \]

\[\left. \left. \left. \left. \Phi_{3,2} \left\{ (I - \frac{w_2}{b_2}) \Phi_{2,1} \left[ (I - \frac{w_1}{b_1}) \Phi_{1,0} + \frac{e_0}{w_1} (\alpha_1 + \beta_1) \right] \right. \right. \right. \]

\[\left. \left. \left. \left. \left. + \frac{w_2}{w_1} (\alpha_2 + \beta_2) \right] + \ldots + \frac{w_{K-2}}{w_{K-1}} (\alpha_{K-2} + \beta_{K-2}) \right] + \frac{w_{K-1}}{w_{K-2}} (\alpha_{K-1} + \beta_{K-1}) \right\} \right\} \]

or

\[e^I_K = \prod_{I=1}^{K-1} \Phi_{I+1,I} \left( I - \frac{w_1}{b_{I-1}} \right) \Phi_{I,0} + \Phi_{I,0} + \Phi_{K,K-1} \frac{w_{K-1}}{w_{K-1}} (\alpha_{K-1} + \beta_{K-1}) \]

\[\sum_{I=1}^{K-2} \prod_{J=I+1}^{K-1} \Phi_{J+1,J} \left( I - \frac{w_J}{b_J} \right) \Phi_{I+1,I} \frac{w_I}{w_I} (\alpha_I + \beta_I) \]

where in the products, a factor with \( I \) or \( J = M-1 \) is to the right of the factor with \( I \) or \( J = M \).

Post multiplying by \( \beta_K \) and taking the expectation gives

\[\frac{\bar{e}_K^I \beta_K}{\overline{\beta}_K} = C_1 \overline{e_0 \beta_K} + \overline{\phi_1,0 e_0 \beta_K} + \Phi_{K,K-1} \frac{w_{K-1}}{w_{K-1}} \left[ \frac{\alpha_{K-1} \beta_K}{\overline{\beta}_K} + \frac{\beta_{K-2} \beta_K}{\overline{\beta}_K} \right] \]

\[+ C_2 \left[ \frac{\alpha_I \beta_K}{\overline{\beta}_K} + \frac{\beta_I \beta_K}{\overline{\beta}_K} \right]. \tag{5} \]

The first two terms contain \( \overline{e_0 \beta}_K \) and are zero since \( e_0 \) is assumed to be independent of all measurement errors. Also, since \( \alpha_I \) is assumed to be completely random, \( \alpha_I \beta_K = 0 \) for all \( I \) and \( K \). For our present
study, we shall make another simplification and assume $\beta_1$ is not dependent on $I$ (on time); therefore, $\beta_1 \beta_K = \beta^2 = \sigma^2_\beta$ where $\sigma_\beta$ is the standard deviation. Other models for the time dependence of $\beta$ are possible. A common one is of the form $e^{-at}$ where $a$ is a positive constant.

So, Eq. 5 becomes

$$e_{-K}^\beta K = \sum_{K>1} (\prod_{J=K-1}^{\infty} \Phi_{J+1,J} (I - W_{J,B_J}) \Phi_{I+1,I} W_{I}) \sigma_\beta^2$$

If there are both instrument and horizon biases, the variance can be written as

$$\sigma^2_\beta = \sigma^2_{IB} + \sigma^2_{HB}$$

where $\sigma_{IB}$ is the instrument bias standard deviation and $\sigma_{HB}$ is the horizon bias standard deviation of the planet. We have assumed no time dependence of either bias and the independence of each from the other.

At MCC times, if the direction and magnitude of the acceleration measurement uncertainties due to the burns are known, then these can be added to the true covariance matrix ($E$) at each MCC.

A simple recursion equation can be obtained from Eq. 6 if it is written in a compiler language instead of as an algebraic expression. In MAC, at the $K$th measurement, the equation for $e_{-K}^\beta K$ can be written as
\[ A = \Phi \left( (I - W_P B_P) A + W_P \text{BIAS}^2 \right) \]  

(7)

where \( \bar{A} \) on the left side is \( e_K^1 \beta_K \), and \( \bar{A} \) on the right side is \( e_{K-1}^1 \beta_{K-1} \). \( \Phi \) is \( \Phi_{K,K-1} \) and \( W_P \) and \( B_P \) are the weighting and geometry vectors from the previous (K-1) measurement. \( \text{BIAS} \) is the standard deviation of the bias. Equation 7 is valid at all measurements except the first one. For the first measurement, \( A = 0 \).