

Massachusetts Institute of Technology
Instrumentation Laboratory
Cambridge, Massachusetts

Space Guidance Analysis Memo #29-65

TO: SGA Distribution
FROM: Donald C. Fraser
DATE: 23 November 1965
SUBJECT: Recursive Filtering Applied to System Identification

I. Summary

This memo describes a quasi-linear technique for identifying unknown coefficients in a set of linear differential equations. The work was motivated by some recent interest in obtaining certain parameters of the CSM/LEM combination and this memo deals primarily with this example; the method, however, is generally applicable. The basic technique is to linearize the non-linear equations which result when a state vector is augmented with the unknown parameters. A Kalman type filter is then used with these linearized equations to minimize the uncertainty in the state variables.

II. The Equation of Motion

The example to be considered here will be that of a flexible vehicle driven in rotation by a set of reaction jets. The model used will consider only the rigid body mode plus the first bending mode. Experience has shown that these are far more important for the CSM/LEM combination than the higher bending modes and slosh modes.

Rigid body rotation is described by

$$I_{yy} \ddot{\theta} - u = 0$$

where

- I_{yy} = pitch moment of inertia
- θ = rigid body rotation angle
- u = moment applied by reaction jets
- σ_1 (RCS) = normalized slope of the first bending mode at the reaction jet station

The motion of the first normalized bending displacement, q_1 , is described by:

$$m_1 \ddot{q}_1 + 2m_1 \xi_1 \omega_1 \dot{q}_1 + m_1 \omega_1^2 q_1 - u \sigma_1 (\text{RCS}) = 0$$

where m_1 , ξ_1 , ω_1 are the normalized mass, damping ratio, and natural frequency of the first bending mode.

These equations can be written in vector form by make the definitions:

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ q_1 \\ \dot{q}_1 \end{bmatrix}$$

The result is:

$$\dot{\underline{y}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_1^2 & -2\xi_1 \omega_1 \end{bmatrix} \underline{y} + \begin{bmatrix} 0 \\ 1/I_{yy} \\ 0 \\ \sigma_1 (\text{RCS})/m_1 \end{bmatrix} u \quad (1)$$

III. State Augmentation

Now assume that none of the coefficients in Eq. (1) are known well enough to be useful. This is the worst case — if any of the coefficients are considered known the resulting filter equations can be simplified by reduction of the dimension of the state.

Augment the state by defining a new vector \underline{x} such that:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ -\omega_1^2 \\ -2\xi_1 \omega_1 \\ 1/I_{yy} \\ \sigma_1 (\text{RCS})/m_1 \end{bmatrix}$$

If, for this memo, we consider the coefficients of Eq. (1) to be constants then the new state equation can be written in the form:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_5 & x_6 & 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underline{x} \quad (2)$$

Note the nonlinear character of this equation caused by products of state variables.

IV. Linearization

For convenience write Eq. (2) in the form:

$$\dot{\underline{x}} = F \underline{x} \quad (3)$$

The corresponding estimate of \underline{x} , $\hat{\underline{x}}$, is described by

$$\dot{\hat{\underline{x}}} = \hat{F} \hat{\underline{x}} \quad (4)$$

Now define the difference between \underline{x} and its estimate as:

$$\delta \underline{x} = \underline{x} - \hat{\underline{x}} \quad (5)$$

Similarly with the matrix F:

$$\delta F = F - \hat{F} \quad (6)$$

Substituting Eq. (5) and (6) into Eq. (3) gives:

$$\underline{\dot{\hat{x}}} + \delta \underline{\dot{x}} = (\hat{F} + \delta F) (\underline{\hat{x}} + \delta \underline{x})$$

Expanding, neglecting second order terms, and subtracting Eq. (4) gives:

$$\delta \underline{\dot{x}} = \hat{F} \delta \underline{x} + \delta F \underline{\hat{x}} \quad (7)$$

By assuming \hat{F} and $\underline{\hat{x}}$ known and constant over any given sample period we may rewrite Eq. (7) as:

$$\delta \underline{\dot{x}} = (\hat{F} + \delta F') \delta \underline{x} \quad (8)$$

If, in addition, the applied control magnitude, u , is considered to be the sum of the commanded magnitude u^* and additive noise with mean zero such that:

$$u = u^* + \delta u$$

then Eq. (8) becomes

$$\delta \underline{\dot{x}} = (F + \delta F') \delta \underline{x} + \underline{n}$$

It may be verified that the result of these operations is:

$$\delta \underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u^* & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{x}_5 & \hat{x}_6 & \hat{x}_3 & \hat{x}_4 & 0 & u^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \delta \underline{x} + \begin{bmatrix} 0 \\ \hat{x}_7 \delta u \\ 0 \\ \hat{x}_8 \delta u \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (9)$$

or in condensed form

$$\delta \underline{\dot{x}} = G \delta \underline{x} + \underline{n} \quad (10)$$

Equation (10) is a linear, constant coefficient perturbation equation over any given sample period. Its coefficients change from sample period to sample period as $\underline{\hat{x}}$ changes.

V. The Filter Equations

This section will merely summarize the computation steps required by the filter. The procedure is an application of the linear recursive filter theory which is well documented elsewhere.

The following definitions are used in the expression of the filter equations:

$\Psi(t, t_0)$ = state transition matrix associated with Eq. (3). (See Appendix for details).

$\Phi(t, t_0)$ = state transition matrix associated with Eq. (10). (See Appendix for details).

$N = \underline{\underline{n}} \underline{\underline{n}}^T$ = covariance of noise

E = covariance matrix

σ^2 = variance of measurement noise

\tilde{m} = actual measurement taken

' = denotes parameter before incorporation of measurement

\wedge = denotes estimate

t_0, t_1 = two times $t_1 > t_0$ (The absence of a prime indicates the measurement has been incorporated.)

For this example the following additional vector is necessary:

$$\underline{\underline{h}}^T = [1, 0, \sigma_1(x_A), 0, 0, 0, 0, 0]$$

where $\sigma_1(x_A)$ = normalized slope of the first bending mode at the IMU station.

The sequence of calculations is then:

1. Extrapolate the state:

$$\underline{\underline{\hat{x}}}'(t_1) = \Psi(t_1, t_0) \underline{\underline{\hat{x}}}(t_0)$$

2. Extrapolate the covariance matrix:

$$E'(t_1) = \Phi(t_1, t_0) E(t_0) \Phi^T(t_1, t_0) + \int_{t_0}^{t_1} \Phi(t_1, t) N \Phi^T(t_1, t) dt$$

3. Compute the scalar $a(t_1)$

$$a(t_1) = \underline{h}^T E'(t_1) \underline{h} + \sigma^2$$

4. Compute the weighting vector

$$\underline{w}(t_1) = E'(t_1) \underline{h} / a(t_1)$$

5. Incorporate the measurement:

$$\hat{\underline{x}}(t_1) = \underline{\hat{x}}'(t_1) + \underline{w}(t_1) [\tilde{m}(t_1) - \underline{h}^T \underline{\hat{x}}'(t_1)]$$

6. Update the covariance matrix:

$$E(t_1) = E'(t_1) - a(t_1) \underline{w}(t_1) \underline{w}^T(t_1)$$

7. Update the state transition matrices $\Phi(t_1, t_0)$ and $\Psi(t_1, t_0)$ by evaluating them for the new value of $\hat{\underline{x}}(t_1)$.

VI. Simulation Results

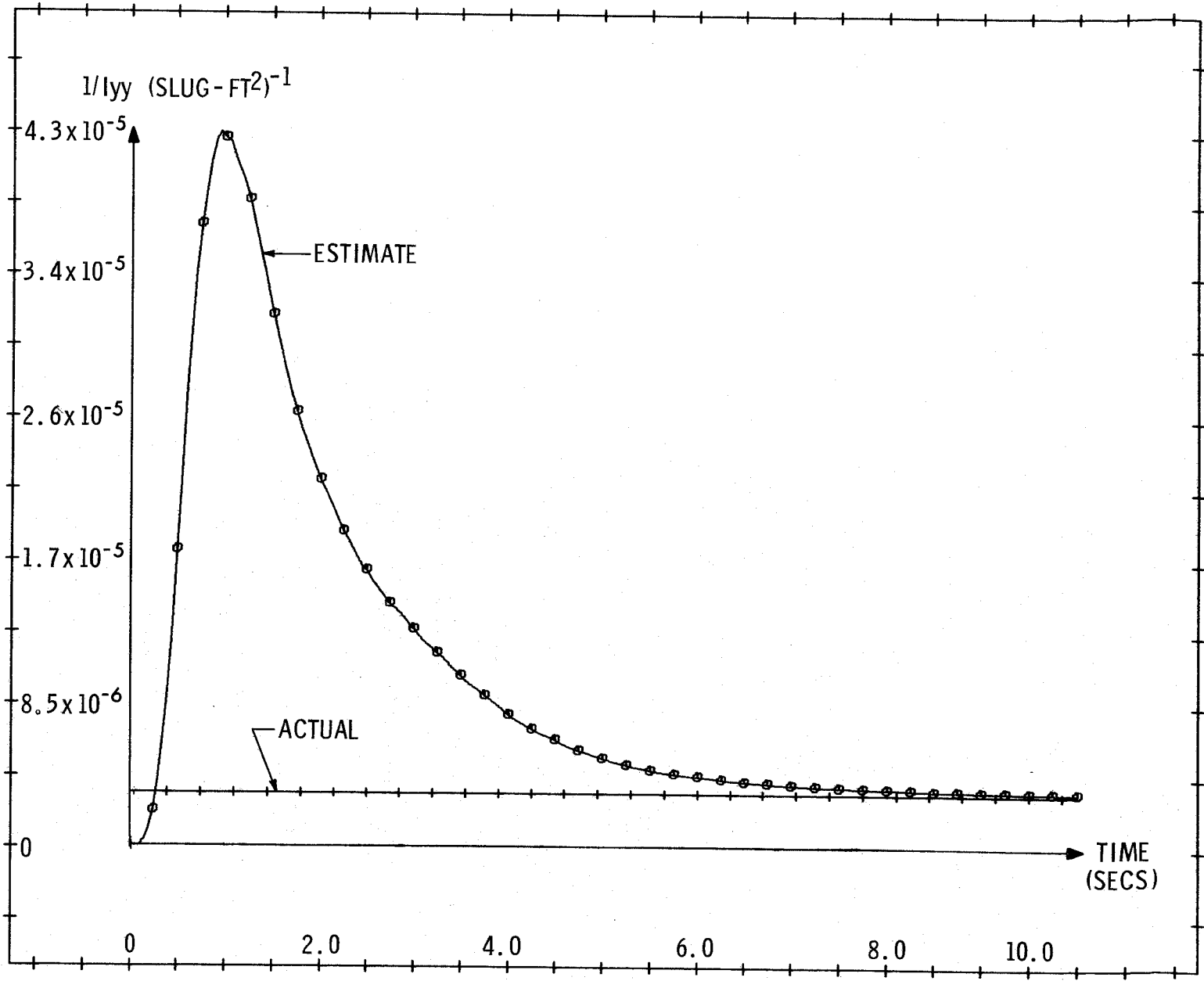
A MAC language computer program has been written to evaluate the use of these equations. The initial results show that the method does indeed work and could be used for ground based parameter identification. Major simplifications (which may be possible for specific applications) would be necessary for airborne operation.

Figures 1 through 5 illustrate some preliminary results. No attempt has been made thus far to improve these results — they are merely initial runs. All have noise on both the measurement and control and have a basic sample period of 50 milliseconds. The nominal control $u^*(t)$ is a constant for all runs.

VII. Conclusions

This technique can be used to identify some of the important parameters needed in the design of the CSM autopilot. While much more work is necessary to adapt it for an AGC size computer it could be used as it is to analyze test flight data. Using the entire recursive procedure recursively on a set of recorded gimbal angle measurements should yield good estimates of the parameters discussed here. The procedure could be adapted, if necessary, to the identification of other parameters, or to finding time varying parameters.

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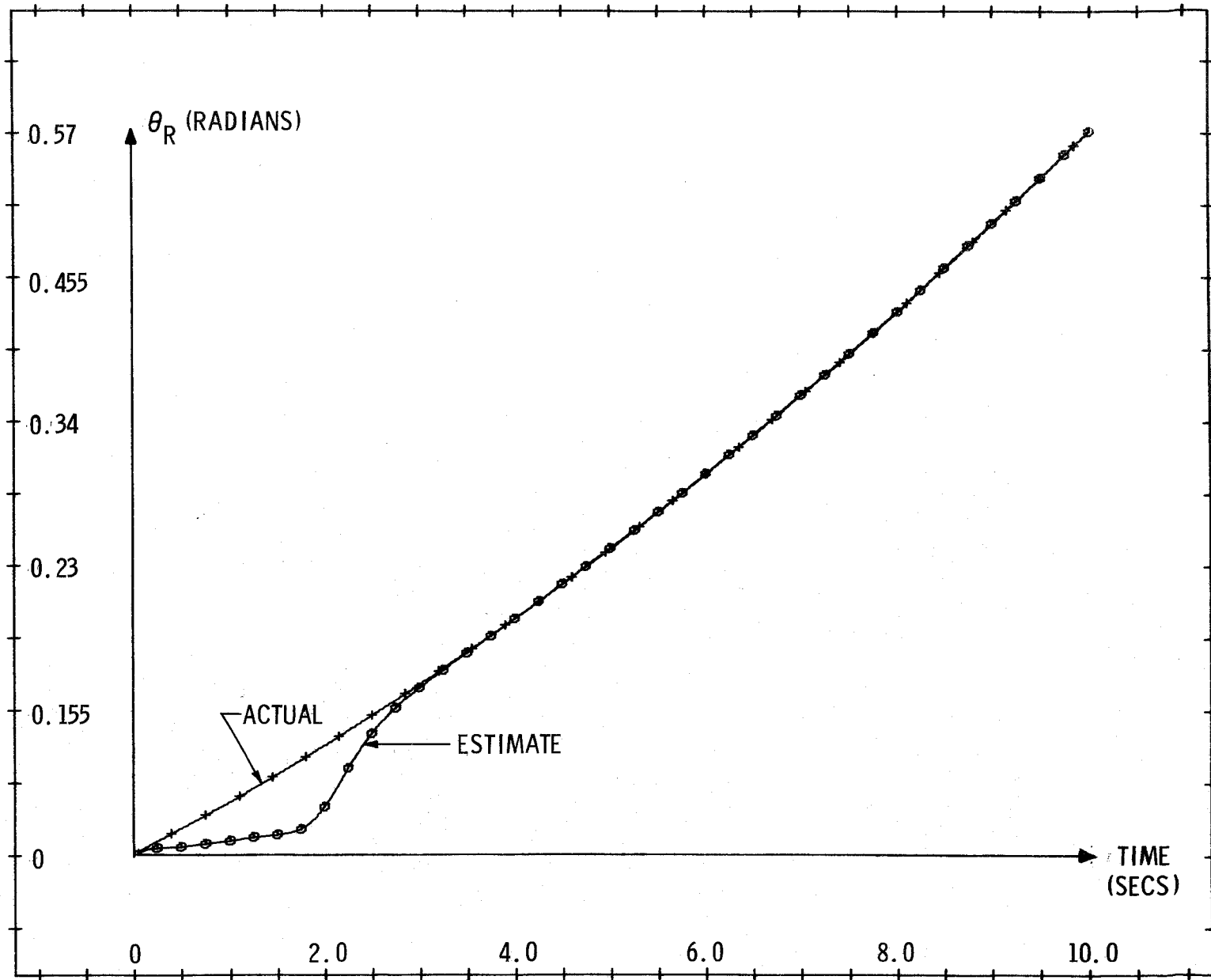


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Fig. 1 Estimate of moment of inertia.

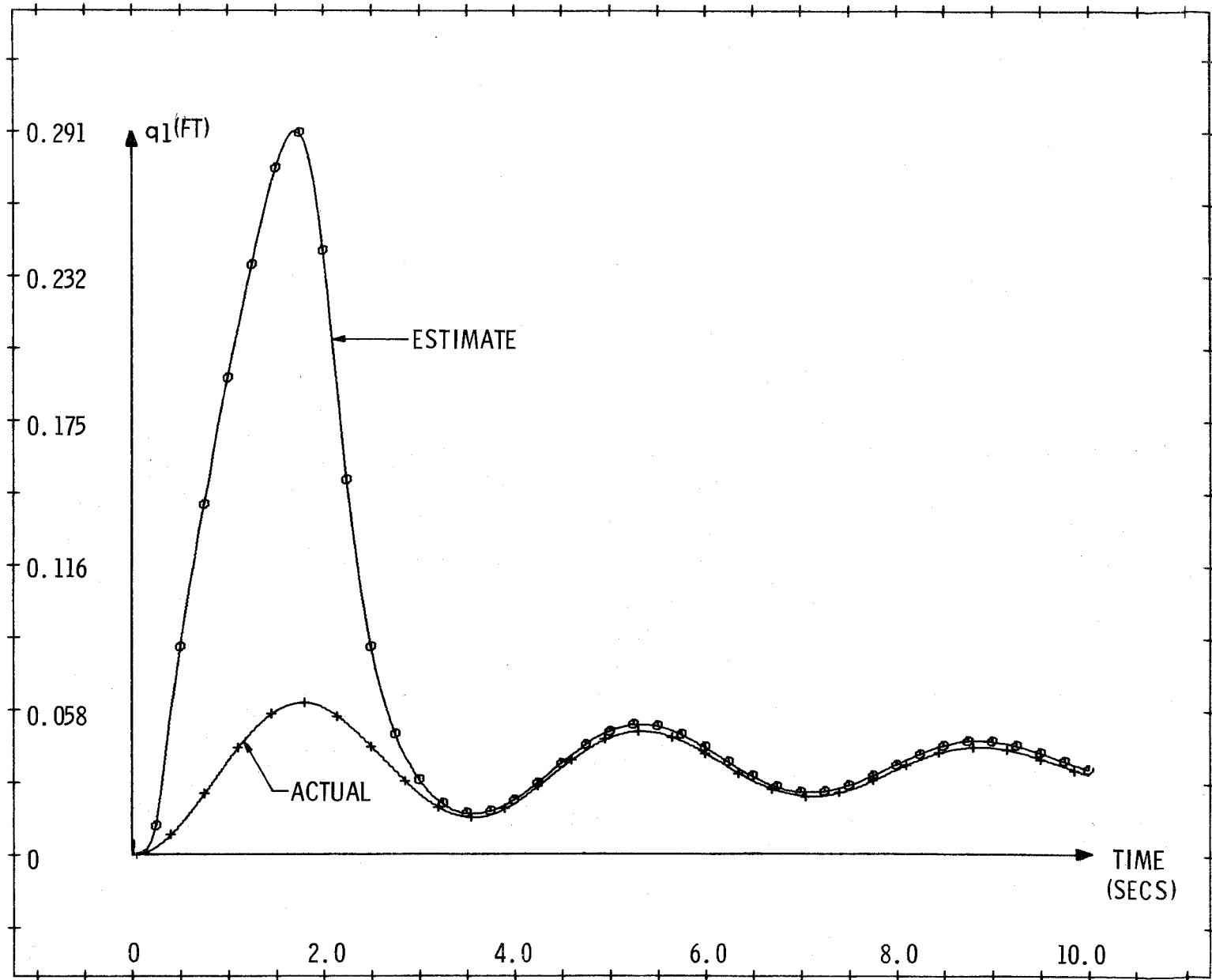
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Fig. 2 Rigid body angle estimate $\frac{\sigma_1(x_{RCS})}{m_1}$ initially in error.

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Fig. 3 Bending displacement estimate $\frac{\sigma_1(x_{RCS})}{m_1}$ initially in error.

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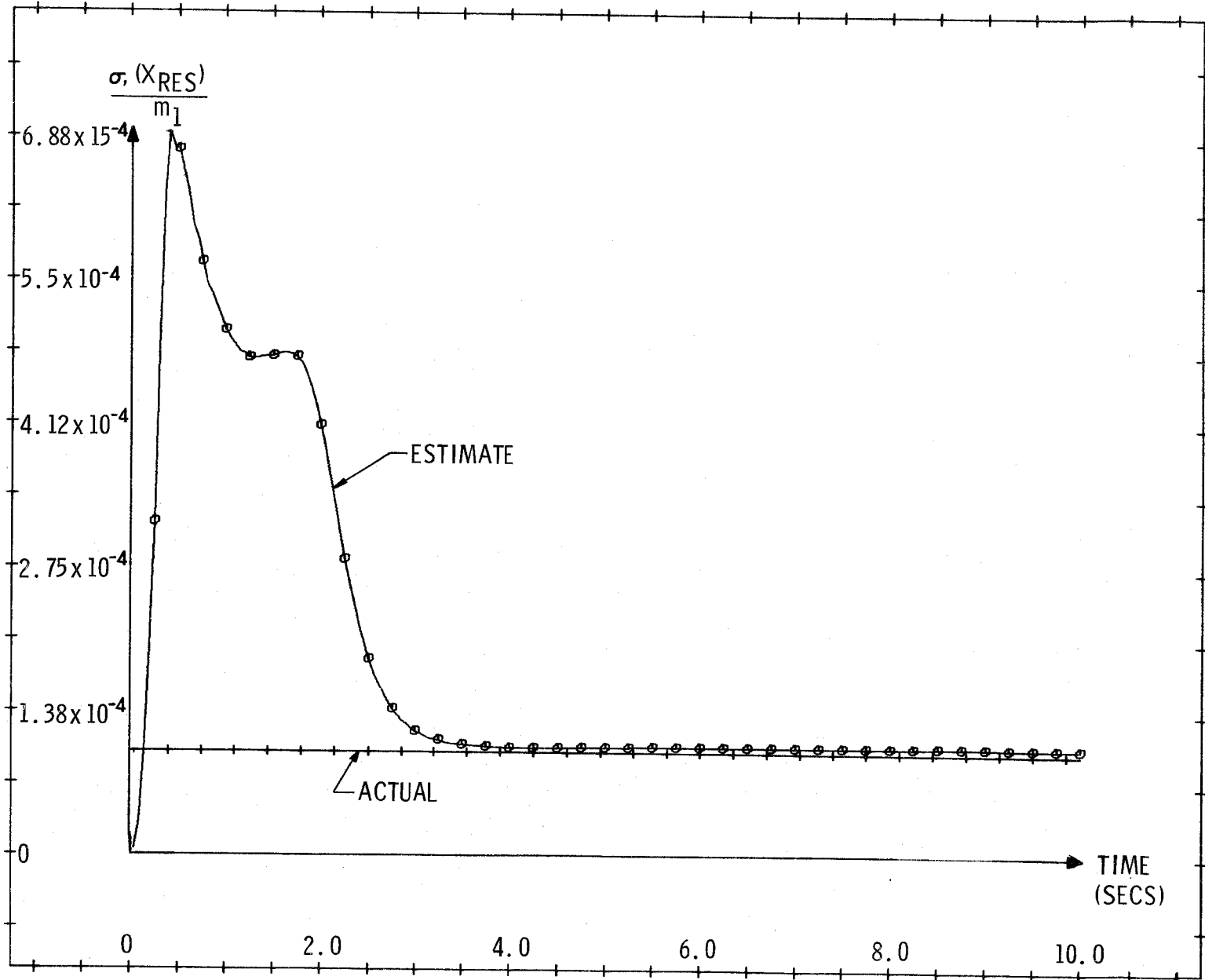


Fig. 4 Estimate of bending slope to mass ratio.

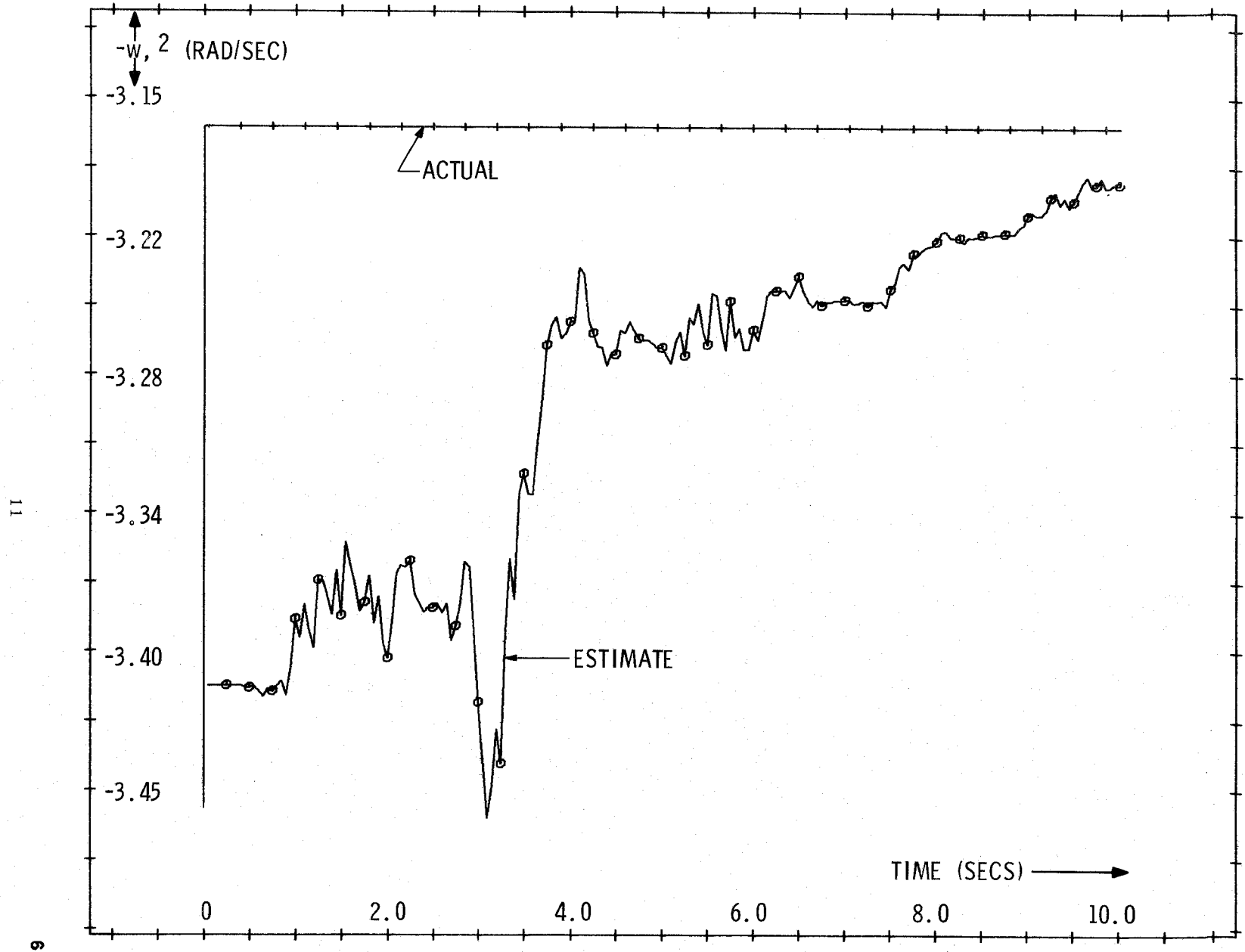


Fig. 5 Estimate of bending frequency.

Appendix

State Transition Matrix Elements

The state transition matrices involved in the filter equations are somewhat complex. They are listed here for reference.

The following notation is used:

$$\omega = \hat{x}_5^{1/2}$$

$$\xi = -\hat{x}_6 / 2\omega$$

$$\omega_D = \omega(1 - \xi^2)^{1/2}$$

$$DT = t_1 - t_0$$

$$\text{EXP} = e^{-\xi\omega DT}$$

$$\text{Sin} = \text{Sin}(\omega_D DT)$$

$$\text{Cos} = \text{Cos}(\omega_D DT)$$

Then if ϕ_{ij} = the i 'th row j 'th column element of $\Phi(t_1, t_0)$.

The matrix elements are:

$$\psi_{11} = \phi_{11} = 1$$

$$\psi_{12} = \phi_{12} = DT$$

$$\psi_{17} = \phi_{17} = 1/2 u^* DT^2$$

$$\psi_{22} = \phi_{22} = 1$$

$$\psi_{27} = \phi_{27} = u^* DT$$

$$\psi_{44} = \phi_{44} = 1$$

$$\psi_{55} = \phi_{55} = 1$$

$$\psi_{66} = \phi_{66} = 1$$

$$\psi_{77} = \phi_{77} = 1$$

$$\psi_{88} = \phi_{88} = 1$$

The remainder of the elements depend upon the discriminant:

$$D = \hat{x}_6^2 + 4 \hat{x}_5$$

If D is negative

$$\begin{aligned} \psi_{33} = \phi_{33} &= \text{EXP} \left(\text{Cos} + \frac{\xi \omega}{\omega_D} \text{Sin} \right) \\ \psi_{34} = \phi_{34} &= \text{EXP Sin} / \omega_D \\ \phi_{35} &= \frac{\hat{x}_3}{\omega^2} \left[1 - \text{EXP} \left(\text{Cos} + \frac{\xi \omega \text{Sin}}{\omega_D} \right) \right] \\ \phi_{36} &= \frac{\hat{x}_4}{\omega^2} \left[1 - \text{EXP} \left(\text{Cos} + \frac{\xi \omega \text{Sin}}{\omega_D} \right) \right] \\ \phi_{38} &= \frac{u^*}{\omega^2} \left[1 - \text{EXP} \left(\text{Cos} + \frac{\xi \omega \text{Sin}}{\omega_D} \right) \right] \\ \psi_{43} = \phi_{43} &= - \frac{\omega^2}{\omega_D} \text{EXP Sin} \\ \psi_{44} = \phi_{44} &= \text{EXP} \left(\text{Cos} - \frac{\xi \omega}{\omega_D} \text{Sin} \right) \\ \phi_{45} &= \frac{\hat{x}_3}{\omega_D} \text{EXP Sin} \\ \phi_{46} &= \frac{\hat{x}_4}{\omega_D} \text{EXP Sin} \\ \phi_{48} &= \frac{u^*}{\omega_D} \text{EXP Sin} \end{aligned}$$

If D is zero:

$$\text{define } r_1 = \frac{\hat{x}_6}{2} \quad \text{and} \quad \text{EX} = e^{r_1 DT}$$

Then

$$\begin{aligned}
\psi_{33} &= \phi_{33} = \text{EX} \left[1 + (r_1 - \hat{x}_6) \text{DT} \right] \\
\psi_{34} &= \phi_{34} = \text{EX DT} \\
\phi_{35} &= \frac{\hat{x}_3}{r_1^2} \left[1 + \text{EX} (r_1 \text{DT} - 1) \right] \\
\phi_{36} &= \frac{\hat{x}_4}{r_1^2} \left[1 + \text{EX} (r_1 \text{DT} - 1) \right] \\
\phi_{38} &= \frac{u^*}{r_1^2} \left[1 + \text{EX} (r_1 \text{DT} - 1) \right] \\
\psi_{43} &= \phi_{43} = \hat{x}_5 \text{EX DT} \\
\psi_{44} &= \phi_{44} = \text{EX} (1 + r_1 \text{DT}) \\
\phi_{45} &= \hat{x}_3 \text{EX DT} \\
\phi_{46} &= \hat{x}_4 \text{EX DT} \\
\phi_{48} &= u^* \text{EX DT}
\end{aligned}$$

If D is positive

$$\text{define } r_1 = \frac{1}{2} \left[\hat{x}_6 + (\hat{x}_6^2 + 4 \hat{x}_5)^{1/2} \right]$$

$$r_2 = \frac{1}{2} \left[\hat{x}_6 - (\hat{x}_6^2 + 4 \hat{x}_5)^{1/2} \right]$$

$$\text{EX}_1 = e^{r_1 \text{DT}}$$

$$\text{EX}_2 = e^{r_2 \text{DT}}$$

$$\begin{aligned}
\psi_{33} &= \phi_{33} = \frac{1}{r_1 - r_2} \left[EX_1(r_1 - \hat{x}_6) - EX_2(r_2 - \hat{x}_6) \right] \\
\psi_{34} &= \phi_{34} = \frac{1}{r_1 - r_2} \left[EX_1 - EX_2 \right] \\
\phi_{35} &= \frac{\hat{x}_3}{r_1 r_2 (r_1 - r_2)} \left[r_1 - r_2 + EX_1 r_2 - EX_2 r_1 \right] \\
\phi_{36} &= \frac{\hat{x}_4}{r_1 r_2 (r_1 - r_2)} \left[r_1 - r_2 + EX_1 r_2 - EX_2 r_1 \right] \\
\phi_{38} &= \frac{u^*}{r_1 r_2 (r_1 - r_2)} \left[r_1 - r_2 + EX_1 r_2 - EX_2 r_1 \right] \\
\psi_{43} &= \phi_{43} = \frac{\hat{x}_5}{r_1 - r_2} \left[EX_1 - EX_2 \right] \\
\psi_{44} &= \phi_{44} = \frac{1}{r_1 - r_2} \left[r_1 EX_1 - r_2 EX_2 \right] \\
\phi_{45} &= \frac{\hat{x}_3}{r_1 - r_2} \left[EX_1 - EX_2 \right] \\
\phi_{46} &= \frac{\hat{x}_4}{r_1 - r_2} \left[EX_1 - EX_2 \right] \\
\phi_{48} &= \frac{u^*}{r_1 - r_2} \left[EX_1 - EX_2 \right]
\end{aligned}$$

Any elements of either state transition matrix which has not been defined above is zero.