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Space Guidance Analysis Memo #35-64

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SUBJECT: Optimization of Reentry "Up-Phase" Guidance

This memo presents the results of an attempt to optimize the "up-phase" portion of reentry guidance. The equations of motion are linearized about a reference trajectory in order to permit the computation of statistical results. An unsophisticated steepest descent procedure is used to find optimum values of control system gains. An optimum set of varying gains is found that is significantly better (less sensitive to altitude-rate errors) than the optimum set of constant gains.

1. Introduction

When the total reentry range is greater than approximately 2000 n. m. it is necessary to climb out of the atmosphere and perform a "skip" maneuver at velocities somewhat less than circular. The most critical task of reentry guidance is to steer the vehicle through this climbing maneuver, starting at a high-g, supercircular condition and achieving a suitable "exit" condition. The combination of velocity magnitude and rate-of-climb at exit should be such that when the vehicle enters the atmosphere the second time, the target is well within the remaining "footprint" defining the vehicle ranging capability at that time.

The measure of performance used in this study involves the projected range at exit, the sum of the range covered during the Keplerian portion prior to the second entry and the range covered after the second entry assuming a nominal value of  $L/D$ . An optimum up-phase system is one

which minimizes the mean squared value of the difference between the projected range and the actual range to go at exit.

The sources of error considered in this study are navigation errors such as incorrect knowledge of velocity and rate-of-climb and deviations in the scale height of the atmosphere.

It had previously been found that a computed reference trajectory scheme performed better during the up-phase portion than any other type of scheme tried. The system works as follows. During the high-g portion a pair of functions

$$v_{\text{ref}} = f_1(a_D)$$

$$\dot{r}_{\text{ref}} = f_2(a_D)$$

are generated within the AGC. These represent the way velocity and rate-of-climb are to vary as a function of acceleration-due-to-drag in order that the reference value of projected range is equal to the actual range-to-go. The control equation is then as follows:

$$(L/D)_{\text{comm}} = (L/D)_{\text{ref}} + K_1 (v_{\text{meas}} - v_{\text{ref}}) + K_2 (\dot{r}_{\text{meas}} - \dot{r}_{\text{ref}}) \quad (1)$$

In the investigation described below a control equation having the essential nature of Eq. (1) is assumed and values of the gains,  $K_1$  and  $K_2$ , (not necessarily constant) are sought which minimize  $\sigma_E$ , the root mean squared value of the deviation in projected range at exit. Note that the objective is not to optimize the trajectory, but to optimize the control system.

## 2. Analysis

The main steps of the analysis are outlined here; certain details are relegated to the appendix.

### A. Non-Linear Time-Dependent Equations

We assume that a particular reference trajectory has already been established and we are primarily interested in the actual time histories

of the following three state variables: altitude, velocity and rate-of-climb,  $\{r, v, c\}$ . Using time as the independent variable and assuming small flight path angles and an exponential atmosphere, we have the following approximate differential equations:

$$\dot{r} = c \quad (2)$$

$$\dot{v} = -a_D = - \left[ \frac{\rho_0 c_D A}{2 m} \right] e^{-\beta(r - r_0)} v^2 \quad (3)$$

$$\dot{c} = \ddot{r} = \frac{v^2}{r} - G + \frac{L}{m} \quad (4)$$

If we assume that a desired value of  $L/D$  can be obtained instantaneously (roll control system is fast compared to trajectory dynamics), we can insert Eq. (1) into Eq. (4) to get

$$\dot{c} = \frac{v^2}{r} - G + a_D \left[ \left( \frac{L}{D} \right)_{ref} + K_1 (v_{meas} - v_{ref}) + K_2 (c_{meas} - c_{ref}) \right] \quad (5)$$

Equations (2), (3) and (5) are a set of non-linear differential equations which approximately describe the behavior of the system.

#### B. Non-Linear Drag-Dependent Equations

We can derive an equivalent set of equations using  $a_D$ , rather than time, as the independent variable. Corresponding to Eqs. (2) and (3) we have ( $a = a_D$ )

$$r' = \frac{d r}{d a} = f(c, v, r, \beta) \quad (6)$$

and

$$v' = \frac{d v}{d a} = g(c, v, r, \beta) \quad (7)$$

where  $f$  and  $g$  are complicated, non-linear functions. Corresponding to Eq. (5) we have

$$\begin{aligned}
c' &= \frac{dc}{da} = \frac{dt}{da} \frac{dc}{dt} \\
&= \frac{1}{\dot{a}} \left\{ \frac{v^2}{r} - G + a \frac{L}{D}_{\text{ref}} \right\} + \frac{a}{\dot{a}} \left\{ K_1 (v_{\text{meas}} - v_{\text{ref}}) + K_2 (c_{\text{meas}} - c_{\text{ref}}) \right\} \quad (8)
\end{aligned}$$

which can be rewritten as

$$c' = c'_{\text{ref}} + k_1 (v_{\text{meas}} - v_{\text{ref}}) + k_2 (c_{\text{meas}} - c_{\text{ref}}) \quad (9)$$

where

$$k_1 = \frac{a}{\dot{a}} K_1 \quad \text{and} \quad k_2 = \frac{a}{\dot{a}} K_2$$

Equations (6), (7) and (9) are equivalent to Eqs. (2), (3) and (5). If we can find optimum values for  $k_1$  and  $k_2$ , it is a simple matter to compute corresponding values for  $K_1$  and  $K_2$  to be used in the real-world system.

### C. Linearized Equations

The next step is to linearize Eqs. (6), (7) and (9) about the reference trajectory. Thus we have

$$\delta r' = f_c \delta c + f_v \delta v + f_r \delta r + f_\beta \delta \beta \quad (10)$$

$$\delta v' = g_c \delta c + g_v \delta v + g_r \delta r + g_\beta \delta \beta \quad (11)$$

$$\delta c' = k_2 \delta c + k_1 \delta v + k_2 e_c + k_1 e_v \quad (12)$$

where  $\delta r$ ,  $\delta v$  and  $\delta c$  are deviations from the reference trajectory;  $\delta \beta$  is a deviation from the nominal value of  $\beta$  (reciprocal of scale height);  $e_c$  and  $e_v$  are errors in the navigation system's knowledge of rate-of-climb and velocity; and the subscripted f's and g's are partial derivatives.

We now have a set of linear differential equations with coefficients that vary as a function of the independent variable,  $a$ . (Expressions for the partial derivatives are given in the Appendix.)

This set of linear equations can be integrated (from  $a_0$  to  $a_f$ ) to solve the following deterministic problem.

Given:

1. A particular reference trajectory  $\{f\text{'s and } g\text{'s tabulated as a function of } a\}$
2. A particular set of initial deviations  $\{\delta v_0, \delta c_0\}$
3. A particular set of error sources  $\{\delta \beta, e_c, e_v\}$
4. A particular set of gains  $\{k_1(a), k_2(a)\}$

Determine:

1. The final (exit) deviations  $\{\delta v_f, \delta c_f\}$
2. The "error" in projected range at exit

$$E = \lambda_c \delta c_f + \lambda_v \delta v_f$$

where  $\lambda_c$  and  $\lambda_v$  are "exit sensitivities" describing the effect of deviations in exit conditions on the projected range.

#### D. Statistical Equations

So far we have not gained much since the same deterministic problem could have been solved using the original set of non-linear equations. But the fact that we have linear equations permits us to proceed to a statistical analysis. We can then directly compute statistical results pertaining to an ensemble of runs.

From the linear set, Eqs. (10), (11) and (12), we derive a set of 15 linear equations relating various statistical quantities. These are given in the Appendix. They can be integrated to solve the following statistical problem.

Given:

1. A particular reference trajectory  $\{f\text{'s and } g\text{'s tabulated as a function of } a\}$
2. Statistics of initial deviations  $\{\overline{\delta v_0^2}, \overline{\delta c_0^2}\}$

3. Statistics of error sources  $\{\overline{\delta \beta^2}, \overline{e_c^2}, \overline{e_v^2}\}$
4. A particular set of gains  $\{k_1(a), k_2(a)\}$

Determine:

1. The statistics of exit deviations  $\{\overline{\delta v_f^2}, \overline{\delta c_f^2}, \overline{\delta v_f \delta c_f}\}$
2. The measure of performance,

$$\sigma_E = \sqrt{\lambda_c^2 \overline{\delta c_f^2} + 2 \lambda_c \lambda_v \overline{\delta c_f \delta v_f} + \lambda_v^2 \overline{\delta v_f^2}}$$

### 3. Computer Programs

#### A. UPREFFILE294BC

This program simulates a particular type of reference trajectory and generates a file of partial derivatives to be used by the other three programs.

#### B. DETERM294BC

This program integrates Eqs. (10), (11) and (12) to solve the deterministic problem described on page 5. One run simulates a particular case and results in a particular error in projected range at exit.

#### C. RMSCOMP294BC

This program integrates the 15 equations given in the Appendix to solve the statistical problem described above. One run generates ensemble averages and produces a measure of performance,  $\sigma_E$ , for a particular set of gains.

#### D. STEEPDSCT294BC

This program uses the previous one as a sub-program and is essentially an iterative procedure which continually modifies the gains to improve performance. Each trial set of gain modifications is generated by means of a steepest descent procedure. The number of parameters to be modified is 2, 4, 6 etc. depending upon whether the gains are assumed to be constant, linearly varying with  $a_D$ , parabolically varying, etc.

#### 4. Results and Discussion

##### A. Expected Performance with "Balanced" Gains

Consider first the performance we might expect from one obvious choice of gains. Most of our results to date pertain to an up-phase reference trajectory which is part of a typical 5000 nm. range trajectory. The exit sensitivities for this trajectory are

$$\lambda_v = 2.99 \text{ nm/fps}$$

and

$$\lambda_c = 1.97 \text{ nm/fps} .$$

Thus, for example, if the exit deviations  $\delta c_f$  and  $\delta v_f$  are both + 100 fps, the error in projected range would be

$$E = 2.99 (100) + 1.97 (100) = 496 \text{ nm}.$$

We choose a pair of constant gains which are high enough to achieve fairly tight control of the indicated exit conditions and select the ratio of the gains so as to balance the effects of  $\delta v_f$  and  $\delta c_f$ . That is, the gain ratio is

$$\frac{k_1}{k_2} = \frac{\lambda_v}{\lambda_c} = \frac{2.99}{1.97} = 1.52$$

and we call these balanced gains. In this case we expect the projected-range error to be mainly a function of the navigation errors,  $e_v$  and  $e_c$ . Assuming that these two are uncorrelated and that

$$\sigma_{e_v} = 10 \text{ fps}$$

and

$$\sigma_{e_c} = 30 \text{ fps}$$

we expect the following performance:

$$\sigma_E = \sqrt{[2.99 (10)]^2 + [1.97 (30)]^2} = 66.2 \text{ nm.}$$

This value is a useful standard of comparison for what follows. The significance of this value is that it defines the required ranging ability of the final, subcircular, glide phase of reentry. In a  $3\sigma$  case, for example, the range to be covered during the final glide is approximately 200 nm different from what would occur if the glide were flown open-loop with a nominal  $L/D$ .

### B. Constant Gain Performance

With constant gains there are only two parameters to vary and we can get a complete picture of the effects of various gain combinations. Such a picture is shown in Fig. 1 which plots constant  $\sigma_E$  contours on the  $k_1 - k_2$  plane. Figure 1 pertains to a 5000 nm trajectory and the following data.

#### Reference Exit Conditions:

$$\begin{aligned} v_{\text{ref}} &= 24,700 \text{ fps} \\ c_{\text{ref}} &= 1,055 \text{ fps} \\ \lambda_v &= 2.99 \\ \lambda_c &= 1.97 \end{aligned}$$

#### Initial Deviation Statistics:

$$\begin{aligned} \sigma_{\delta v_o} &= 250 \text{ fps} \\ \sigma_{\delta c_o} &= 100 \text{ fps} \end{aligned}$$

#### Error Source Statistics:

$$\begin{aligned} \sigma_{\delta\beta} &= 10\% \text{ of nominal} \\ \sigma_{e_v} &= 10 \text{ fps} \\ \sigma_{e_c} &= 30 \text{ fps} \end{aligned}$$

these 5 quantities  
are assumed to be  
uncorrelated

The dashed line in Fig. 1 is a locus of balanced gains for the given exit sensitivities. We see that along a considerable portion of this line the performance is very close to the expected value of 66.2 nm.

We see also that there is a narrow region somewhat to the right of the balanced-gains line where the performance is somewhat better; namely, about 62 nm. These are the optimum constant gains.

The fact that the system can achieve better performance than that achieved by balanced gains means that it is taking advantage of the dynamic relationship between  $\delta v$  and  $\delta c$  described by the equations of motion. Since  $\delta v$  is known more accurately than  $\delta c$ , the control equation gives relatively more weight to  $\delta v$  than the balanced-gains concept calls for. As a result of using optimum gains the vehicle exits with an "indicated" error in projected range, but it relies on the total system dynamics (with optimum gains) to produce a final rate-of-climb which is better than that produced by balanced gains. In other words, for a given exit velocity, the difference between the actual rate-of-climb and the ideal rate-of-climb. The above discussion applies even more strongly to the varying gain combinations which achieve still better performance.

### C. Linearly Varying Gain Performance

With linearly varying gains there are four parameters to vary. We take these to be the initial and final values of the two gains,  $k_{1_o}$ ,  $k_{1_f}$ ,  $k_{2_o}$  and  $k_{2_f}$ . Here the steepest descent procedure proves to be very useful and gives surprisingly good results.

Table 1 illustrates the action of the steepest descent program. Starting with a set of constant gains ( $\sigma_E = 169$  nm) the program took 41 iterations to reach a performance of  $\sigma_E = 43$  nm. At this point it was suspected that better performance might be obtained with  $k_{2_f} = 0$ . The program was restarted with this assumption and took only 16 iterations to go from  $\sigma_E = 141$  nm to  $\sigma_E = 37.2$  nm. This is believed to be very close to the best performance possible with linearly varying gains.

Each iteration cycle involves integration of the equations 5 times; one nominal run and 4 perturbed runs, each with a small test variation in one parameter. The next set of parameter changes are chosen in proportion to the effects of the test variations (steepest descent procedure). If the performance worsens instead of improving the parameter changes are cut in half.

The "path" of the gains corresponding to the last row of Table 1 are plotted as a dotted straight line on the  $k_1 - k_2$  plane (Fig. 1). Note that in the early portion rate-of-climb ( $k_2$ ) is emphasized while toward the end of the trajectory velocity ( $k_1$ ) is emphasized. An interesting result is that no single point along the path represents a good set of constant gains. In fact the beginning and ending points would be very poor choices as constant gains ( $\sigma_E > 300$  nm in both cases).

#### D. Optimum Gains

It was discovered that significantly better performance could be achieved without increasing the number of parameters. This is done by making  $k_2(a)$  the product of allinear  $k_1(a)$  and another straight line representing the ratio,  $k_2(a)/k_1(a)$ . Thus,  $k_2(a)$  becomes a parabola, but we still have just four parameters, the initial and final values of  $k_1$  and  $k_2$ .

Figure 2 plots two sets of gain variations which have the form described above. One set is believed to be nearly optimum for the 5000 nm case, and the other for a 3000 nm case. These were found by means of the steepest descent program.

The best performance achieved in the 5000 nm case is 31.7 nm. Since the theoretical limit is believed to be 29.9 nm ( $\lambda_v = 2.99$ ,  $\sigma_{e_v} = 10$ ), no attempt was made to use more sophisticated gain variations. Similarly, the best performance achieved in the 3000 nm case is 6.4 nm while the theoretical limit is 5 nm ( $\lambda_v = 0.5$ ,  $\sigma_{e_v} = 10$ ).

The operation of the system with nearly-optimum gains is illustrated in Figs. 3 and 4. These are phase-plane plots showing  $\{\delta v, \delta c\}$  histories for several specific cases generated by computer program DETERM294BC. All of the runs plotted in Fig. 3 assume perfect information ( $e_v = e_c = 0$ ). The four runs shown at the top also assume a standard atmospheric scale height and simply show how the trajectory achieves the nominal exit conditions regardless of initial deviations. The straight line labeled "target line" is a locus of acceptable exit conditions which balance off the effects of deviations in velocity and rate-of-climb. The slope of this line is simply the negative of the ratio of the exit sensitivities. The two runs plotted at the bottom of Fig. 3 show the effect of scale height deviations of 10%. The exit conditions are far from the origin (reference values) but

do lie along the target line and are, therefore, acceptable. That is, the error in projected range is small.

Figure 4 compares the operation of the nearly-optimum gains with that of balanced gains when errors in the knowledge of rate-of-climb are present. In each case three runs are plotted with  $e_c = +30, 0$  and  $-30$ fps, respectively. Points of equal acceleration are connected with dotted lines. The difference between the two systems is very striking in the later part of the trajectory. With optimum gains, which are comparatively low toward the end, the trajectories slowly "drift" in to the target line. The effect of a rate-of-climb error is to displace the exit point along the target line, making the final projected range very insensitive to the rate-of-climb error. With the higher, balanced gains, on the other hand, the final indicated values of  $\delta v$  and  $\delta c$  are driven to a point along the target line, with the result that the actual exit point is above or below the line by an amount equal to the negative of the error,  $e_c$ .

#### E. Conclusions

For any particular up-phase reference trajectory a set of varying gains can be found which achieve good performance by making the projected range insensitive to errors in knowledge of rate-of-climb. The performance achieved is mainly a function of the accuracy in knowledge of velocity magnitude. The following table summarizes the results found to date. (Assuming  $\sigma_{e_v} = 10$  fps,  $\sigma_{e_c} = 30$  fps.)

Performance ( $\sigma_E$ ) with:	<u>5000 nm case</u>	<u>3000 nm case</u>
1. Balanced Gains	66.2 nm	30.4 nm
2. Optimum Constant Gains	62 nm	not found
3. Best Linear Gains Found	37.2 nm	not found
4. Best Gains Found ( $k_1$ linear, $k_2$ parabolic)	31.7 nm	6.4 nm
Theoretical Performance Limit ( $\sigma_{e_v} \times \lambda_v$ )	29.9 nm	5 nm

The improvement from 66.2 nm to 31.7 nm may seem unimportant since a performance of  $\sigma_E = 66.2$  nm ought to be well handled by the final glide phase. If, however, the rate-of-climb errors become larger than those assumed above the improvement could become much more striking and very important. Just how large the rate-of-climb errors could be before linearity and the predicted insensitivity breaks down can only be determined by extensive simulation of the complete, non-linear problem.

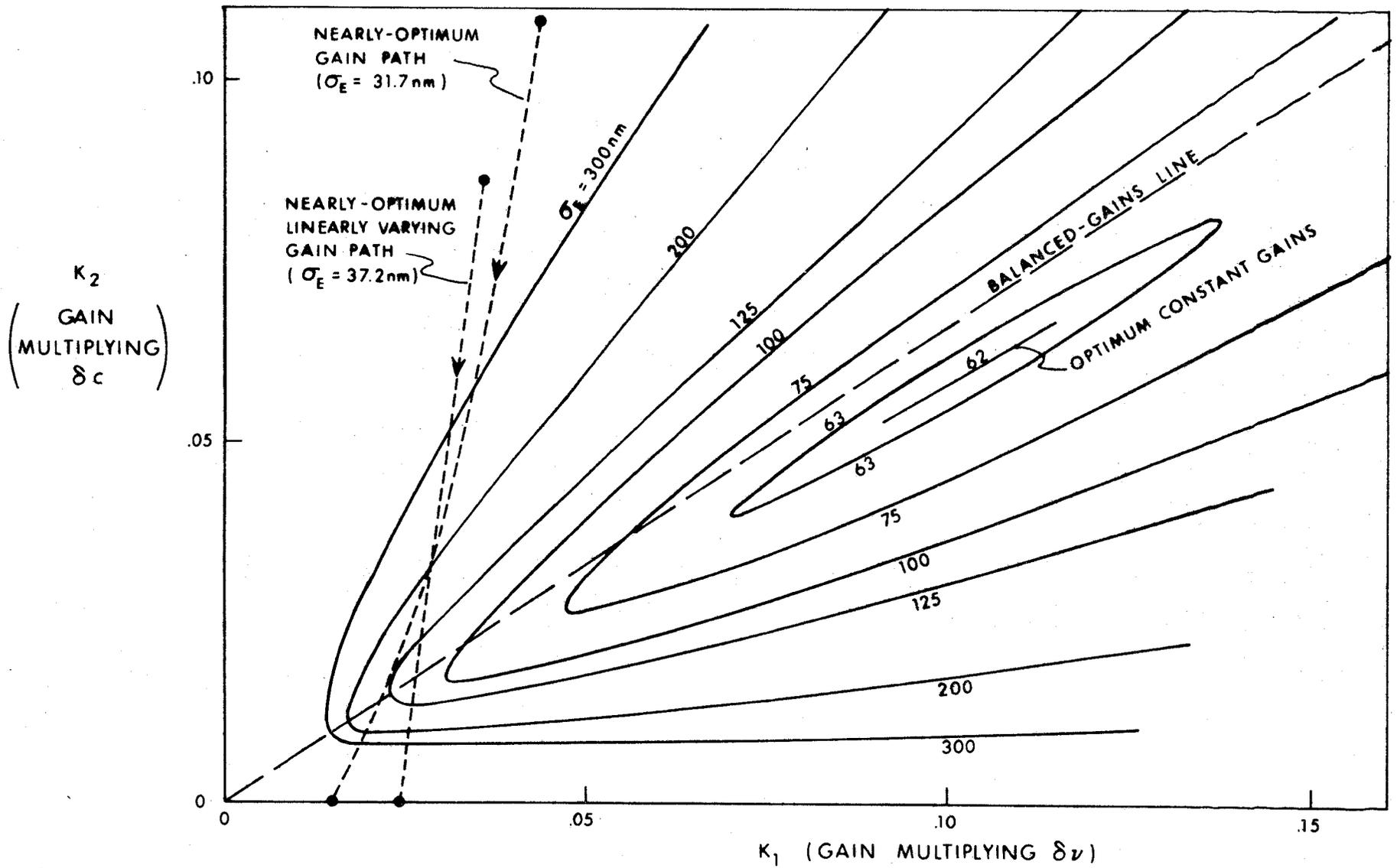


Fig. 1 Performance Contours for Constant Gains (5000 nm case)

Table 1  
Operation of Steepest Descent Program with Linearly Varying Gains

effect of 0.1% change in parameter

Iteration	$k_1(a_0)$	effect of 0.1% change in parameter			$k_1(a_f)$	effect of 0.1% change in parameter			$\frac{k_2}{k_1}(a_0)$	effect of 0.1% change in parameter			$\frac{k_2}{k_1}(a_f)$	% change in parameter with largest effect	$\sigma_E$ (nm)
		(nm)	(nm)	(nm)		(nm)	(nm)	(nm)		(nm)					
Start	.03000	-.0019	.03000	-.0326	1.0000	+.0439	1.0000	+.233	--	169.2					
1	.03004	+.0059	.03067	-.0409	.9698	+.0287	.8400	+.194	16	129.8					
2	.02989	+.0061	.03170	-.0442	.9468	+.0010	.7056	+.132	16	99.1					
3	.02967	-.0098	.03341	-.0346	.9456	-.0388	.5927	+.0376	16	81.8					
4	.03087	-.0377	.03817	-.0064	1.0970	-.0568	.5007	-.0378	16	70.5					
5	.03415	-.0028	.03886	-.0292	1.2720	-.0109	.5541	+.0685	16	64.3					
10	.03821	-.0120	.04006	+.0187	1.5750	-.0255	.4462	-.0126	8	53.3					
20	.03816	-.0043	.03761	+.0235	1.9540	-.0213	.3766	-.0057	4	47.9					
30	.03574	+.0002	.03517	+.0191	2.1690	-.0159	.3344	-.00004	2	45.0					
40	.03326	-.0038	.03370	+.0297	2.3300	-.0198	.3063	-.00560	2	43.3					
41	.03335	+.0124	.03302	-.0089	2.3300	+.0032	.3063	+.01490	2	43.0					
Restart	.03330	-.1377	.03300	+.3332	2.3300	-.1472	0	0	--	141.6					
1	.03550	-.1173	.02772	+.3019	2.4950	-.1616	0	0	16	64.4					
2	.03771	+.1102	.02328	-.2676	2.7090	+.1866	0	0	16	55.0					
3	.03647	-.0217	.02515	+.0736	2.5570	-.0362	0	0	8	38.1					
10	.03623	+.0078	.02450	-.0017	2.5460	+.0095	0	0	0.5	37.4					
15	.03579	-.0007	.02432	+.0187	2.5130	-.0040	0	0	0.5	37.25					
16	.03580	+.0007	.02420	-.0020	2.5160	+.0089	0	0	0.5	37.22					

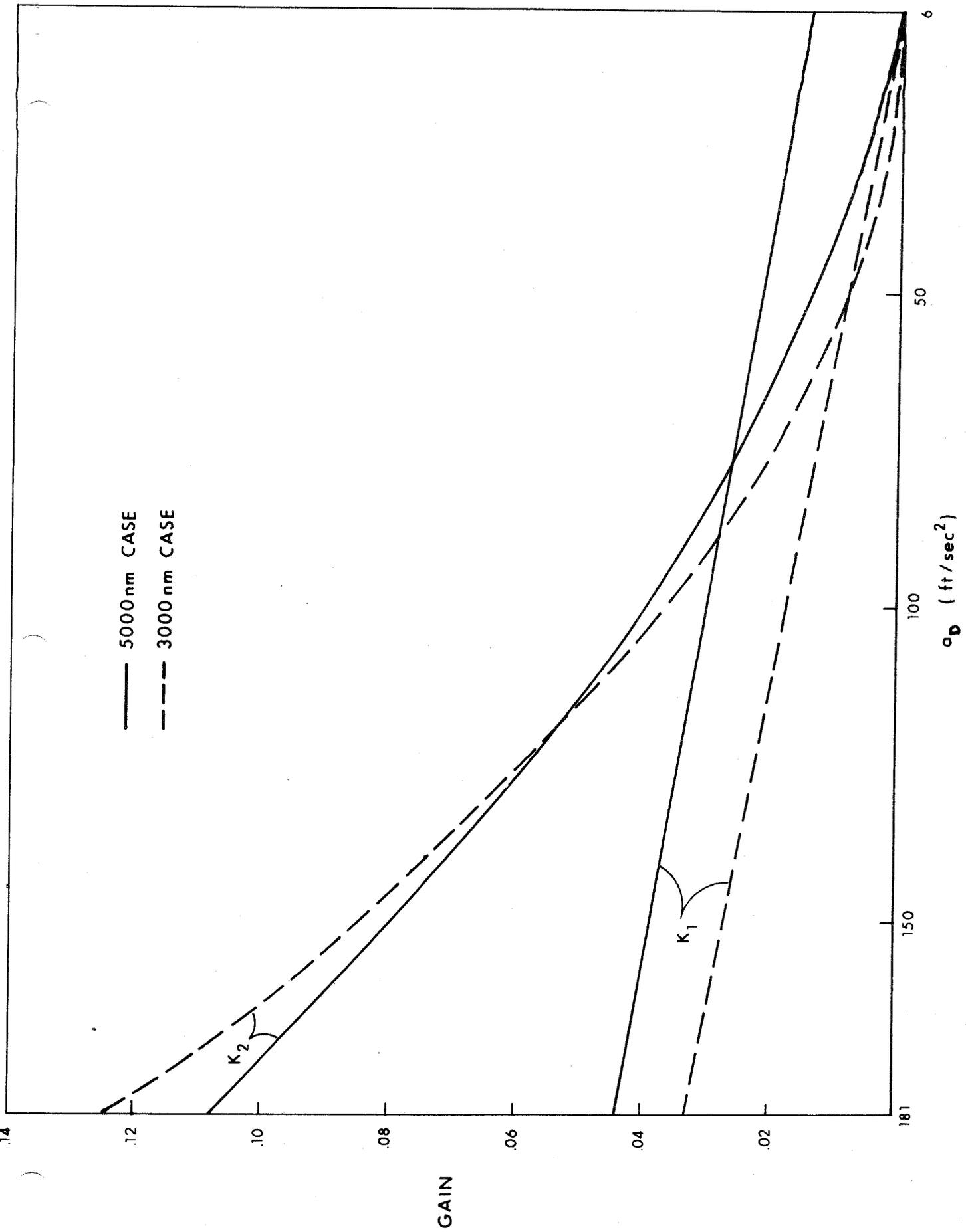


Fig. 2 - Nearly Optimum Gains

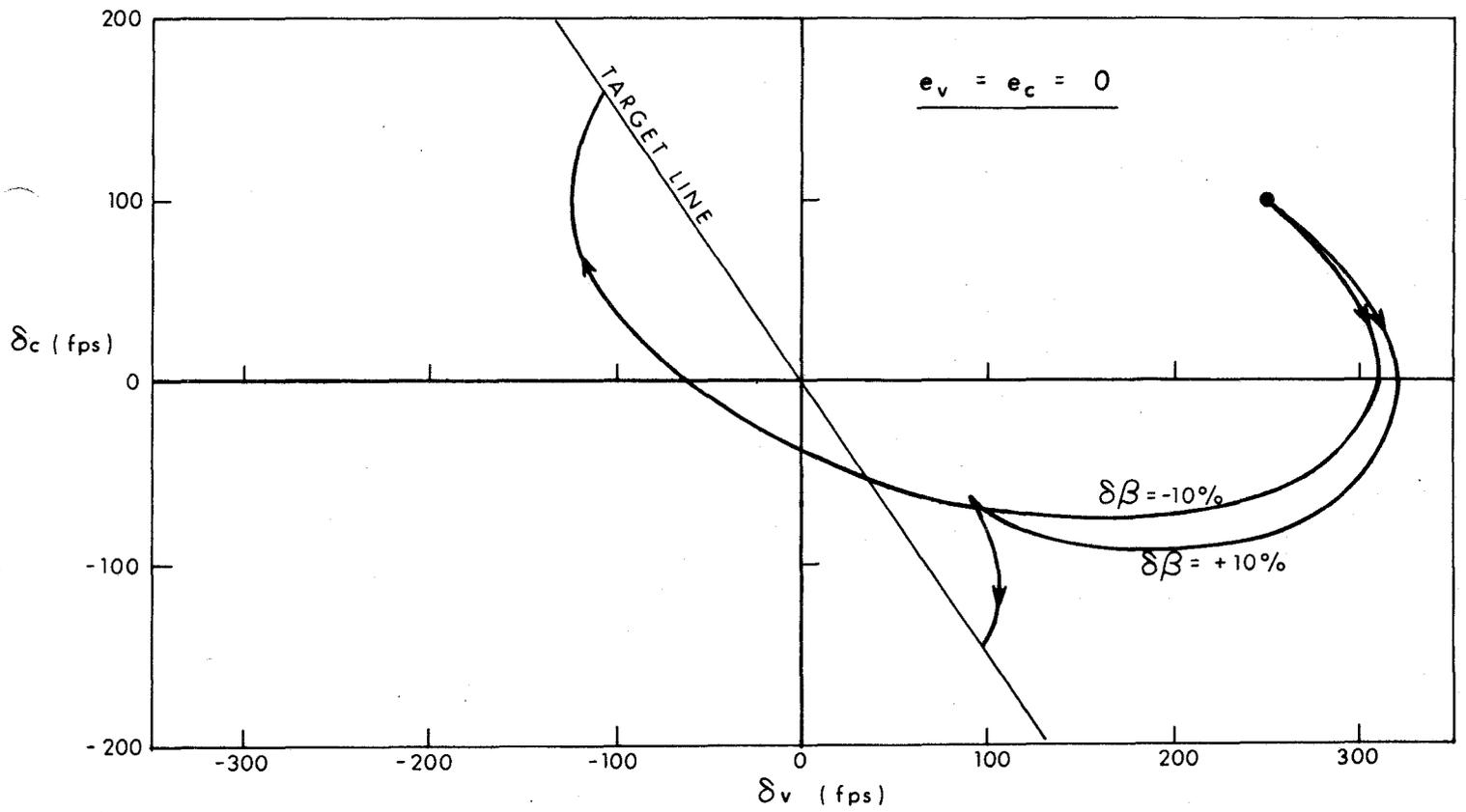
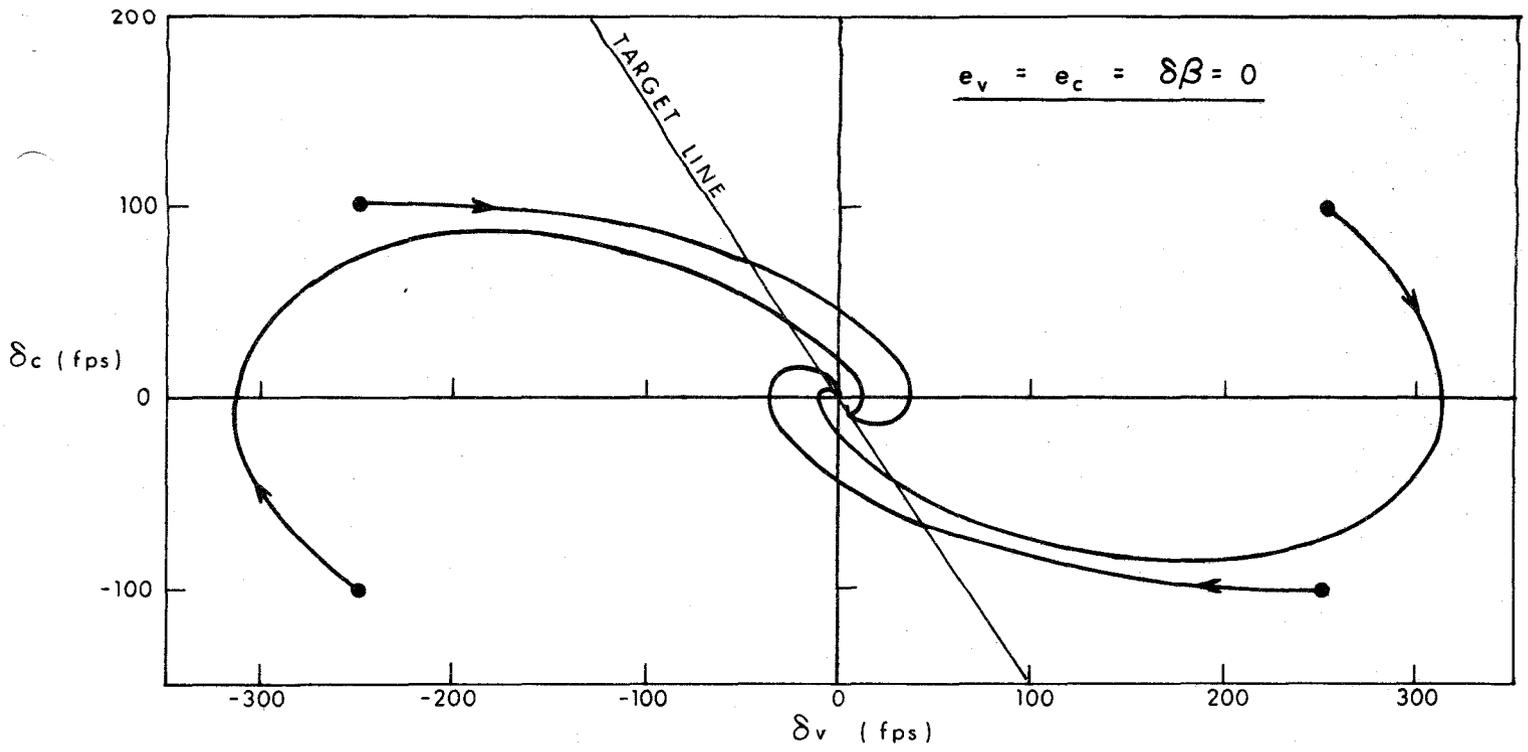


Fig. 3 Operation with Optimum Gains (5000 n. m. case)

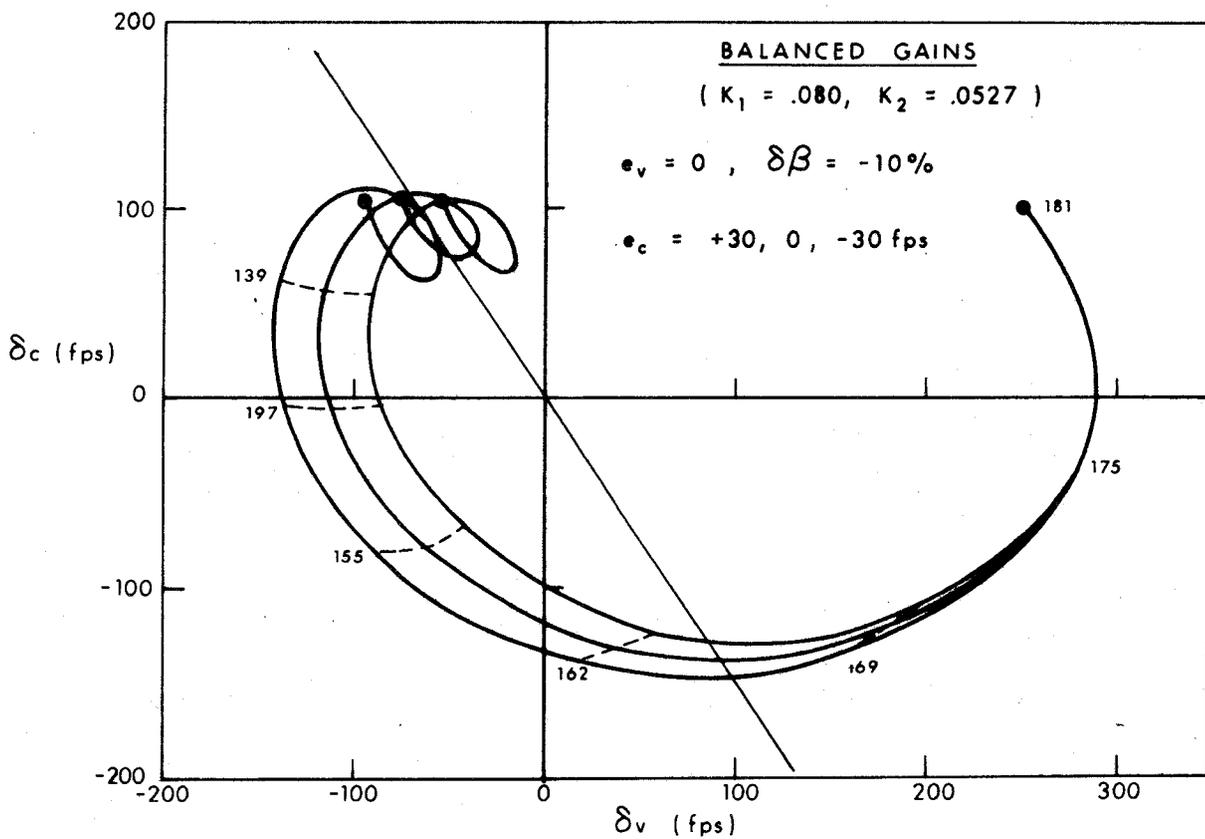
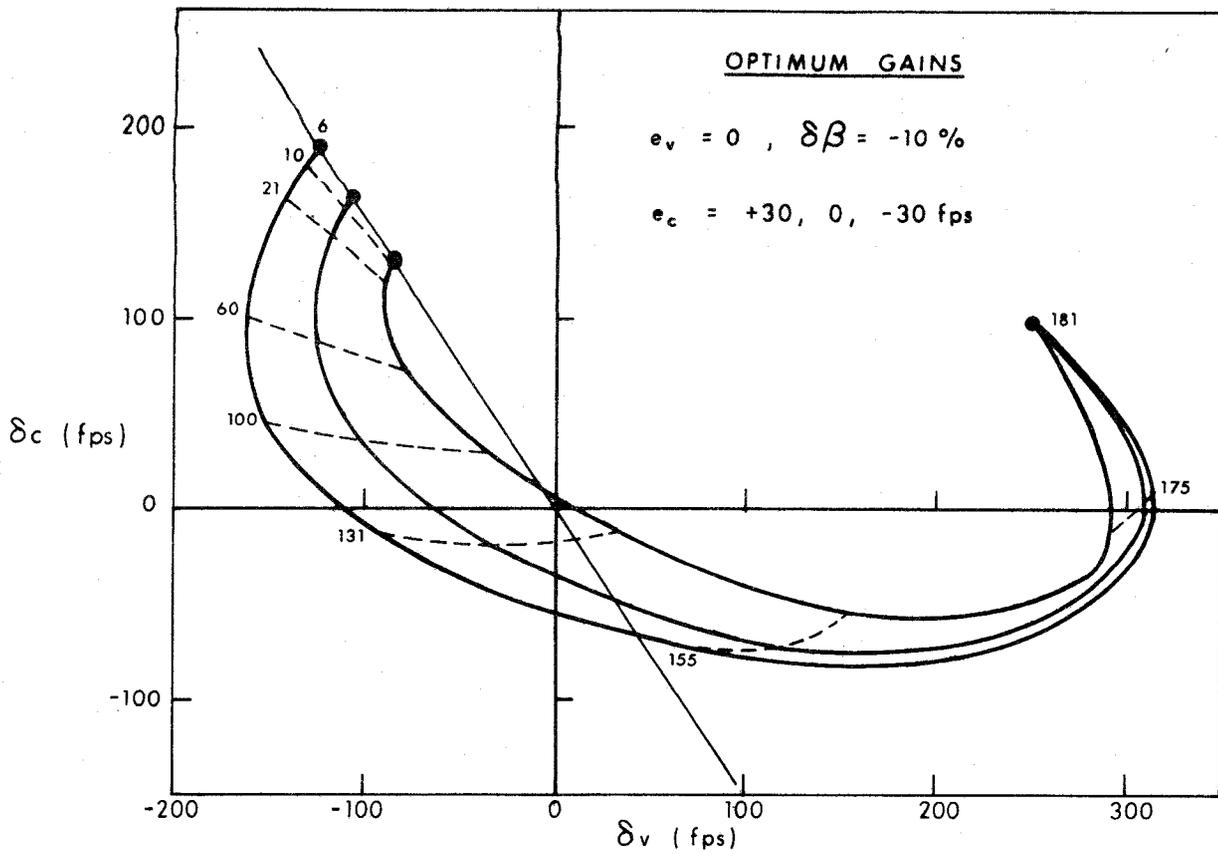


Fig. 4 Comparison of Optimum-Gains Operation with Balanced-Gains Operation

## APPENDIX

### 1. Partial Derivatives

The partial derivatives used in the linearized drag-dependent equations (Page 4) are derived in two steps. First, some partial derivatives pertaining to the time-dependent equations are derived. Second, expressions relating these to the desired partial derivatives are obtained.

With the following changes in notation

$$z = r - r_0$$

$$C = \frac{\rho_0 c_D A}{2m}$$

we have the following time-dependent equations.

$$\frac{dv}{dt} = d(v, z, \beta) = -a = -C e^{-\beta z} v^2$$

$$\frac{da}{dt} = h(c, v, z, \beta) = \frac{\partial a}{\partial z} \frac{dz}{dt} + \frac{\partial a}{\partial v} \frac{dv}{dt}$$

$$= \frac{\partial a}{\partial z} c - \frac{\partial a}{\partial v} a$$

$$= C \beta v^2 c e^{-\beta z} - 2C v^3 e^{-2\beta z}$$

The following partial derivatives are then easily obtained.

$$\frac{\partial d}{\partial v} = d_v = -2C v e^{-\beta z}$$

$$d_z = C \beta v^2 e^{-\beta z}$$

$$d_\beta = C z v^2 e^{-\beta z}$$

$$h_c = -C\beta v^2 e^{-\beta z}$$

$$h_v = -2C\beta v c e^{-\beta z} - 6C^2 v^2 e^{-2\beta z}$$

$$h_z = C\beta v^2 c e^{-\beta z} + 4C^2 \beta v^3 e^{-2\beta z}$$

$$h_\beta = (\beta z - 1)Cv^2 c e^{-\beta z} + 4C^2 z v^3 e^{-2\beta z}$$

The drag-dependent differential equations may be expressed as follows:

$$\begin{aligned} r' &= \frac{dr}{da} = \frac{dr}{dt} \frac{dt}{da} = \frac{c}{h(c, v, z, \beta)} \\ &= f(c, v, z, \beta) \end{aligned}$$

$$\begin{aligned} v' &= \frac{dv}{da} = \frac{dv}{dt} \frac{dt}{da} = \frac{d(v, z, \beta)}{h(c, v, z, \beta)} \\ &= g(c, v, z, \beta) \end{aligned}$$

The desired partial derivatives then become:

$$f_c = \frac{h - ch_a}{h^2}$$

$$g_c = -\frac{dh_c}{h^2}$$

$$f_v = -\frac{ch_v}{h^2}$$

$$g_v = \frac{hd_v - dh_v}{h^2}$$

$$f_z = -\frac{ch_z}{h^2} = f_r$$

$$g_z = \frac{hd_z - dh_z}{h^2} = g_r$$

$$f_\beta = -\frac{ch_\beta}{h^2}$$

$$g_\beta = \frac{hd_\beta - dh_\beta}{h^2}$$

The un-subscripted quantities,  $d$  and  $h$ , are simply the time rates of change of velocity and acceleration-due-to-drag,  $\dot{v}$  and  $\dot{a}$ , along the reference trajectory.

## 2. Statistical Differential Equations

The following equations are easily derived from Eqs. (10), (11) and (12) on Page 4.

$$\overline{\delta c^2}' = 2\left\{f_c \overline{\delta c^2} + f_v \overline{\delta c \delta v} + f_r \overline{\delta c \delta r} + f_\beta \overline{\delta c \delta \beta}\right\}$$

$$\overline{\delta v^2}' = 2\left\{g_v \overline{\delta v^2} + g_c \overline{\delta c \delta v} + g_r \overline{\delta v \delta r} + g_\beta \overline{\delta v \delta \beta}\right\}$$

$$\overline{\delta r^2}' = 2\left\{f_r \overline{\delta r^2} + f_c \overline{\delta c \delta r} + f_v \overline{\delta v \delta r} + f_\beta \overline{\delta r \delta \beta}\right\}$$

$$\begin{aligned} \overline{\delta c \delta v}' &= g_c \overline{\delta c^2} + k_1 \overline{\delta v^2} + (k_2 + g_v) \overline{\delta c \delta v} + g_r \overline{\delta c \delta r} + g_\beta \overline{\delta c \delta \beta} \\ &\quad + k_2 \overline{\delta v e_c} + k_1 \overline{\delta r e_v} \end{aligned}$$

$$\begin{aligned} \overline{\delta c \delta r}' &= f_c \overline{\delta c^2} + f_v \overline{\delta c \delta v} + (k_2 + f_r) \overline{\delta c \delta r} + k_1 \overline{\delta v \delta r} + f_\beta \overline{\delta c \delta \beta} \\ &\quad + k_2 \overline{\delta r e_c} \end{aligned}$$

$$\begin{aligned} \overline{\delta v \delta r}' &= f_v \overline{\delta v^2} + g_c \overline{\delta c^2} + f_c \overline{\delta c \delta v} + g_c \overline{\delta c \delta r} + (f_r + g_v) \overline{\delta v \delta r} \\ &\quad + f_\beta \overline{\delta v \delta \beta} + g_\beta \overline{\delta v \delta \beta} \end{aligned}$$

$$\overline{\delta c e_c}' = k_2 \overline{\delta c e_c} + k_1 \overline{\delta v e_c} + k_2 \overline{e_c^2}$$

$$\overline{\delta c e_v}' = k_2 \overline{\delta c e_v} + k_1 \overline{\delta v e_v} + k_1 \overline{e_v^2}$$

$$\overline{\delta c \delta \beta}' = k_2 \overline{\delta c \delta \beta} + k_1 \overline{\delta v \delta \beta}$$

$$\overline{\delta v e_c}' = g_c \overline{\delta c e_c} + g_v \overline{\delta v e_c} + g_r \overline{\delta r e_c}$$

$$\overline{\delta v e_v}' = g_c \overline{\delta c e_v} + g_r \overline{\delta v e_v} + g_r \overline{\delta r e_v}$$

$$\overline{\delta v \delta \beta}' = g_c \overline{\delta c \delta \beta} + g_r \overline{\delta v \delta \beta} + g_r \overline{\delta r \delta \beta} + g_\beta \overline{\delta \beta^2}$$

$$\overline{\delta r e_c}' = f_c \overline{\delta c e_c} + f_v \overline{\delta v e_c} + f_r \overline{\delta r e_c}$$

$$\overline{\delta r e_v'} = f_c \overline{\delta c e_v} + f_v \overline{\delta v e_v} + f_r \overline{\delta r e_v}$$

$$\overline{\delta r \delta \beta'} = f_c \overline{\delta c \delta \beta} + f_v \overline{\delta v \delta \beta} + f_r \overline{\delta r \delta \beta} + f_\beta \overline{\delta \beta^2}$$

We have assumed the error sources to be constant over a particular trajectory and uncorrelated with each other. Therefore,

$$\overline{e_c^2}' = \overline{e_v^2}' = \overline{\delta \beta^2}' = 0$$

and

$$\overline{e_c e_v} = \overline{e_c \delta \beta} = \overline{e_v \delta \beta} = 0$$

Massachusetts Institute of Technology  
Instrumentation Laboratory  
Cambridge, Massachusetts

Errata Sheet to Space Guidance Analysis Memo #35-64

TO: SGA Distribution  
FROM: Bard S. Crawford  
DATE: September 8, 1964  
SUBJECT: Optimization of Reentry "Up-Phase" Guidance

On Page 9, paragraph 1 should read:

The fact that the system can achieve better performance than that achieved by balanced gains means that it is taking advantage of the dynamic relationship between  $\delta v$  and  $\delta c$  described by the equations of motion. Since  $\delta v$  is known more accurately than  $\delta c$ , the control equation gives relatively more weight to  $\delta v$  than the balanced-gains concept calls for. As a result of using optimum gains the vehicle exits with an "indicated" error in projected range, but it relies on the total system dynamics (with optimum gains) to produce a final rate-of-climb which is better than that produced by balanced gains. In other words, for a given exit velocity, the difference between the actual rate-of-climb and the ideal rate-of-climb is less (on the average) than the error in knowledge of rate-of-climb. The above discussion applies even more strongly to the varying gain combinations which achieve still better performance.