SCIENCE

NAVIGATION
CSM DIGITAL AUTOPILOT
LM DIGITAL AUTOPILOT
LUNAR LANDING
ORBITS
COASTING FLIGHT NAVIGATION

The purpose of Coasting Flight Navigation is to estimate spacecraft position and velocity. The estimates are computed using orbital mechanics and navigation sightings to improve the accuracy of the orbital mechanics. The navigation sightings are incorporated into the position and velocity estimates using a modified Kalman filter which is a recursive optimal estimator that is characterized by the following computational procedure.

1. Extrapolate the state vector ahead to time $t_p$ using the best estimate of the state at time $t_{p-1}$

$$\hat{x}_p^* = \hat{x}_{p-1}^*$$

2. Extrapolate the error covariance matrix in a similar manner.

$$P_p^* = (\hat{x}_{p-1}^* - \hat{x}_{p-1})^T + P_{p-1}$$

3. Compute the optimal gain matrix.

$$K_p = P_p^* K_{p-1} (W_n + R_n)^{-1}$$

4. Calculate a measurement vector for time $t_p$.

$$\hat{z}_p = (W_n + R_n)^{-1} (Y_p - \hat{x}_p^*)$$

5. Update the state vector using the extrapolated state $\hat{x}_p^*$, the optimal gain $K_p$, the extrapolated measurement $\hat{z}_p^*$, and the actual measurement at time $t_p$.

$$\hat{x}_p = \hat{x}_p^* + K_p (\hat{z}_p - \hat{z}_p^*)$$

6. Update the error covariance matrix in a similar manner.

$$P_p = P_p^* W_n + R_n$$

This procedure is illustrated by the following block diagram.

STATE VECTOR DEFINITION

The state vector for coasting flight navigation is defined as the deviation of the spacecraft position and velocity from a reference conic. Deviations from the reference conic are assumed to be Gaussian distributed with a known mean and variance. The mean is estimated via the precision integration routine or obtained from MSFN. The variance is given by the error covariance matrix, which is precomputed and entered via erasable data load.

MODIFICATIONS TO THE BASIC KALMAN FILTER

The basic Kalman filter outlined on the previous page was modified for use in Coasting Flight Navigation in the following aspects.

1. Instead of extrapolating the error covariance matrix, the square root of the error covariance matrix is extrapolated. This ensures that the covariance matrix will always be positive semi-definite and avoids the usual difficulties that occur when computation techniques yield a negative error covariance matrix. The square root of the error covariance matrix is called the error transition matrix $W$.

2. The state vector and error transition matrix are extrapolated by integrating their respective second order differential equations via the coasting integration routine. This is done in lieu of the state transition approach and eliminates the necessity of computing a new time varying transition matrix for each measurement interval.

3. The update of the state vector after each measurement must first be displayed and approved by the astronaut before it is incorporated into the state vector. This eliminates the possibility of an erroneous update due to an improper mark.

4. Only one measurement is incorporated at a time. This reduces the dimension of the filter equations and changes the matrix inversion in the optimal gain equation to a scalar division.

CORRELATION BETWEEN BASIC KALMAN FILTER TERMINOLOGY AND COASTING FLIGHT TERMINOLOGY

The following table lists the correlation between basic Kalman filter terminology as used on the previous page and Coasting Flight Navigation terminology to be used on the following pages.

<table>
<thead>
<tr>
<th>CELESTIAL NAVIGATION TERMINOLOGY</th>
<th>KALMAN FILTER TERMINOLOGY</th>
<th>CORRELATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{b}$ = geometry vector of dimension $\mathbf{D}$</td>
<td>$[\mathbf{h}]$ = measurement matrix</td>
<td>$\mathbf{b} \Rightarrow [\mathbf{h}]$</td>
</tr>
<tr>
<td>$\mathbf{w}$ = weighting vector of dimension $\mathbf{D}$</td>
<td>$[\mathbf{k}^*]$ = optimal gain matrix</td>
<td>$\mathbf{w} \Rightarrow [\mathbf{k}^*]$</td>
</tr>
<tr>
<td>$[\mathbf{W}]$ = error transition matrix of dimension $\mathbf{D} \times \mathbf{D}$</td>
<td>$[\mathbf{P}]$ = error covariance matrix</td>
<td>$[\mathbf{W}] \Rightarrow [\mathbf{P}]$</td>
</tr>
<tr>
<td>$\mathbf{x}$ = state vector</td>
<td>$\mathbf{x}$ = state vector</td>
<td>$\mathbf{x} \Rightarrow \mathbf{x}$</td>
</tr>
<tr>
<td>$\mathbf{p}_0$ = a priori measurement error variance (scalar)</td>
<td>$[\mathbf{V}]$ = covariance of the measurement noise</td>
<td>$\mathbf{p}_0 \Rightarrow [\mathbf{V}]$</td>
</tr>
<tr>
<td>$\mathbf{Q}$ = measurement deviation (scalar)</td>
<td>$\mathbf{Q} = \mathbf{Q}$</td>
<td>$\mathbf{Q} \Rightarrow \mathbf{Q}$</td>
</tr>
</tbody>
</table>
The Coasting Integration routine is a standardized subroutine used to integrate spacecraft state vectors to specific times. It is used during each of the three navigation programs (PDE-Orbital Navigation, PDE-Berndorf-Orbital Navigation, and PDE-Dubna-Meadowood Navigation) to extrapolate the state vector and error transition matrix ahead to the measurement time by direct numerical integration of their differential equations.

**STATE VECTOR EQUATIONS**

The basic equation describing spacecraft motion is

\[
\frac{d}{dt} \begin{bmatrix} \dot{E} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{b} & 0 \end{bmatrix} \begin{bmatrix} \dot{E} \\ \dot{Z} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{b} \end{bmatrix} \begin{bmatrix} \dot{Y} \\ \dot{V} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{b} \end{bmatrix} \begin{bmatrix} \dot{A} \\ \dot{V} \end{bmatrix}
\]

where:
- \( E \) = spacecraft position vector
- \( Z \) = spacecraft velocity vector
- \( A \) = disturbance acceleration vector
- \( b \) = primary planet gravitational constant

When the disturbance, \( A \), is small, then Euler's method of differential accelerations can be used to solve Equation 1. Euler's method divides spacecraft motion into two parts: (1) constant or circular orbital motion which would result if \( A_b = 0 \) and (2) deviation from circular motion as a result of \( A_b \).

\[
E = E_{conic} + \Delta E \\
Z = Z_{conic} + \Delta Z
\]

where:
- \( E, Z \) = spacecraft position and velocity
- \( E_{conic}, Z_{conic} \) = conic position and velocity
- \( \Delta E, \Delta Z \) = deviations from conic position and velocity

Substitution of Equations 2 and 3 into the basic equation of motion, Equation 1, yields differential equations for the conic position and velocity and the deviation from conic position and velocity.

**Conic Motion Equation**

\[
\begin{bmatrix} \dot{E}_{conic} \\ \dot{Z}_{conic} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{b_{conic}} & 0 \end{bmatrix} \begin{bmatrix} E_{conic} \\ Z_{conic} \end{bmatrix}
\]

**Deviation Equation**

\[
\begin{bmatrix} \dot{\Delta E} \\ \dot{\Delta Z} \end{bmatrix} = \begin{bmatrix} \Delta E \\ \Delta Z \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{b_{conic}} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{b} \end{bmatrix} \begin{bmatrix} \dot{Y} \\ \dot{V} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{b} \end{bmatrix} \begin{bmatrix} \dot{A} \\ \dot{V} \end{bmatrix}
\]

where:
- \( C = \frac{b}{b_{conic}} \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_{conic} \\ Z_{conic} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{b} \end{bmatrix} \begin{bmatrix} \dot{Y} \\ \dot{V} \end{bmatrix} \)

The conic motion equation is solved explicitly using Kepler's subroutine. The deviation equation is solved by direct numerical integration and is called the "state equation." It can be written in the form

\[
\dot{\Delta E} = \begin{bmatrix} \Delta E \\ \Delta Z \end{bmatrix} = \begin{bmatrix} D_{x} & -D_{y} \\ D_{y} & 0 \end{bmatrix} \begin{bmatrix} \Delta E \\ \Delta Z \end{bmatrix}
\]

**ERROR TRANSITION MATRIX EQUATIONS**

The accuracy of the state vector estimation process is characterized by the error covariance matrix which expresses the mean squared error of each state vector element in matrix form.

\[
\begin{align*}
\Delta \mathbf{E} & = \mathbf{E} - \hat{\mathbf{E}} \\
\Delta \mathbf{E} & = \mathbf{E} - \hat{\mathbf{E}} \\
\Delta \mathbf{E} & = \mathbf{E} - \hat{\mathbf{E}}
\end{align*}
\]

where:
- \( \Delta \mathbf{E} \) = estimation error
- \( \hat{\mathbf{E}} \) = estimated state vector
- \( \mathbf{E} \) = actual state vector
- \( \mathbf{E} \) = error covariance matrix

\[
\begin{align*}
\Delta \mathbf{E} & = \mathbf{E} - \hat{\mathbf{E}} \\
\Delta \mathbf{E} & = \mathbf{E} - \hat{\mathbf{E}} \\
\Delta \mathbf{E} & = \mathbf{E} - \hat{\mathbf{E}}
\end{align*}
\]

where:
- \( \mathbf{F} \) = error covariance matrix
- \( \Delta \mathbf{E} \) = "expected value"

If the state vector is described by Equation 6, then the error covariance matrix is described by

\[
\Delta \mathbf{E} = \mathbf{F} \Delta \mathbf{E} \mathbf{F}^T (\text{predicting process noise} \) \tag{9}
\]

and the error transition matrix is described by

\[
\Delta \mathbf{E} = \mathbf{F} \Delta \mathbf{E} \mathbf{F}^T \tag{10}
\]

**SUMMARY**

<table>
<thead>
<tr>
<th>TIME DOMAIN EQUATIONS</th>
<th>STATE TRANSITION</th>
<th>NUMERICAL INTEGRATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{E} = Fe + \Delta E )</td>
<td>( \dot{E} = Fe + \Delta E \mathbf{F}^T )</td>
<td>( \dot{E} = \mathbf{F} \Delta \mathbf{E} \mathbf{F}^T )</td>
</tr>
<tr>
<td>( \dot{E} = Fe + \Delta E \mathbf{F}^T )</td>
<td>( \dot{E} = \mathbf{F} \Delta \mathbf{E} \mathbf{F}^T )</td>
<td>( \dot{E} = \mathbf{F} \Delta \mathbf{E} \mathbf{F}^T )</td>
</tr>
</tbody>
</table>
MEASUREMENT INCORPORATION ROUTINE

The Measurement Incorporation routine, like the Coasting Integration routine, is used by all three navigation programs. Its purpose is to compute a state vector update based on the information obtained from the navigation sighting. The procedure for updating the state vector can be divided into three parts:

1. Compute a geometry vector, \( \mathbf{g} \), based on the state vector and type of navigation sighting that is being made.
2. Compute a state vector update.
3. Update the state vector and error transition matrix if the astronaut approves of the incorporation.

GEOMETRY VECTOR

The geometry vector relates the measurement, \( \mathbf{Y} \), to the state vector, \( \mathbf{x} \), according to Equation 11

\[
\mathbf{g} = \frac{1}{r_L \sin \Delta} \begin{bmatrix} r_L \cos \Delta & \Delta \end{bmatrix}
\]

where

\[ r_L = \text{the expected measurement (a scalar)} \]
\[ \mathbf{L} = \text{geometry vector (n \times 1 column vector)} \]
\[ \Delta = \text{state vector extrapolated to the measurement time via the Coasting Integration routine} \]

The state vector for coasting flight navigation is a deviation from a reference (osculating) orbit and the measurement variable can also be thought of as a deviation from a reference or nominal. Thus the measurement vector can be determined by taking the derivative of the measurement with respect to the state vector. This can be illustrated for the case of custom on-course navigation, where the navigation sighting is the angle between a known near planetary landmark and a star as shown below:

![Diagram of geometry vector](image)

![Diagram of state vector update](image)

Taking partials

\[
\delta g = \frac{1}{r_L \sin \Delta} \begin{bmatrix} r_L \cos \Delta & \Delta \end{bmatrix} \delta \mathbf{L} = \begin{bmatrix} r_L \cos \Delta & \Delta \end{bmatrix} \delta r_L \sin \Delta
\]

By definition

\[
\delta r_L = \frac{\delta g}{r_L \sin \Delta}
\]

Therefore

\[
\mathbf{g} = \frac{1}{r_L \sin \Delta} \begin{bmatrix} r_L \cos \Delta & \Delta \end{bmatrix}
\]

and

\[
\delta \mathbf{g} = \begin{bmatrix} r_L \cos \Delta & \Delta \end{bmatrix} \begin{bmatrix} \delta r_L \cos \Delta & \delta \Delta \end{bmatrix}
\]

UPDATE THE STATE VECTOR UPDATE

The computation of the state vector update encompasses Steps 3, 4, and 5 of the basic Kalman filtering procedure.

1. Compute the optimal gain matrix

\[
\mathbf{K}_n = \mathbf{P}_n^{-1} \mathbf{H}_n \mathbf{R}^{-1} \mathbf{H}_n^T \mathbf{P}_n^{-1}
\]

2. Compute the expected measurement based on the extrapolated state vectors

\[
\hat{\mathbf{Y}}_n = \mathbf{H}_n \mathbf{x}_n^E
\]

3. Compute the state vector update

\[
\mathbf{x}_n^E = \mathbf{x}_n^E + \mathbf{K}_n (\mathbf{Y}_n - \hat{\mathbf{Y}}_n)
\]

UPDATE THE STATE VECTOR AND ERROR TRANSITION MATRIX

Before the state vector update, \( \mathbf{x}_n^E \), is incorporated into the state vector, it is displayed to the astronaut for his approval. This is to prevent erroneous tracking data, such as improperly identified stars or landmarks, from being used to update the state. When astronaut approval has been issued, the state vector and error transition matrix are updated as follows:

UPDATE THE STATE VECTOR

\[
\mathbf{x}_n^E = \mathbf{x}_n^{\text{EST}} + \delta \mathbf{x}
\]

where

\[ \mathbf{x}_n^{\text{EST}} = \text{best estimate of state at time } t_n \]
\[ \mathbf{x}_n^{E} = \text{state vector extrapolated ahead from time } t_{n-1} \text{ to } t_n \]
\[ \delta \mathbf{x} = \text{state vector update} \]

UPDATE THE ERROR TRANSITION MATRIX

The equation governing the updating of the error covariance matrix is given in Step 6 of the basic Kalman filtering procedure.

\[
\mathbf{P}_n = \mathbf{P}_n^{\text{EST}} + \mathbf{K}_n \mathbf{R} \mathbf{K}_n^T \mathbf{P}_n^{\text{EST}}^{-1}
\]
This can be rewritten in terms of coasting flight navigation terminology

\[ WW^T = W^T W = W^T W^T - \frac{\xi^T}{1 + \xi^T} \]

Equation 18 can be separated to obtain an equation for updating only the error transition matrix \([W]\) instead of the product \([WW^T]\).

\[ [W] = [W] - \frac{\xi^T}{1 + \xi^T} \]

where

- \([W]\) = updated error transition matrix
- \([W]\) = extrapolated error transition matrix
- \(\xi\) = optimal gain vector
- \(\xi^T\) = measurement noise covariance

**SUMMARY**

The total spacecraft position and velocity is kept current through extrapolation, updating, and rectification. Rectification is a process used to redefine the reference trajectory by adding the deviations, \(A_T\) and \(A_V\), in the osculating elements, \(E_{conic}\) and \(E_{conic}\), thereby redefining the osculating orbit and reducing the deviation state vector to zero.

\[
\begin{bmatrix}
L_{conic} \\
E_{conic}
\end{bmatrix} = \begin{bmatrix}
L_{conic} \\
E_{conic}
\end{bmatrix} + \begin{bmatrix}
\xi^T \\
\xi
\end{bmatrix}
\]

This process is used to preserve the efficiency of Kepler's method and is illustrated below.

---

**INITIALIZING THE COASTING FLIGHT NAVIGATION PROGRAMS**

Each of the navigation programs is initialized prior to use by specifying the measurement variance, \(S^2\), and the initial error transition matrix, \(W_0\). The measurement variance gives a confidence level for the navigation routine by specifying the variance of all the error sources associated with the instrument. The \(W\) matrix gives a confidence level for the initial estimate of the state vector by specifying the mean squared error in the position and velocity estimates.

\(W_0\) is initialized as a diagonal matrix which says that initially the position and velocity errors are independent.

<table>
<thead>
<tr>
<th>MEASUREMENT VARIANCE</th>
<th>INITIAL (W^T) MATRIX</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CIRCULAR MIDSURFACE NAVIGATION ROUTINE</strong></td>
<td></td>
</tr>
<tr>
<td>(S^2 = Var_{X_{ LCS}}^2) + (Var_{Y_{ LCS}}^2) + (Var_{Z_{ LCS}}^2)</td>
<td></td>
</tr>
</tbody>
</table>
| \(\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}\) |
| **ORBITAL NAVIGATION ROUTINE** |
| \(S^2 = Var_{V_{ LCS}}^2\) |
| \(\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}\) |
| **RENDEZVOUS NAVIGATION PROGRAM (LEM)** |
| \(S^2 = Var_{R_{ LCS}}^2\) + \(Var_{V_{ LCS}}^2\) |
| \(\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}\) |
| **RENDEZVOUS NAVIGATION PROGRAM (CSM)** |
| \(S^2 = Var_{R_{ LCS}}^2\) + \(Var_{V_{ LCS}}^2\) |
| \(\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}\) |

**FOR RENDEZVOUS**

<table>
<thead>
<tr>
<th>RANGE MEASUREMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S^2 = Max\left(Var_{R_{ LCS}}^2, Var_{V_{ LCS}}^2\right))</td>
</tr>
</tbody>
</table>

**FOR LUNAR SURFACE NAVIGATION**

<table>
<thead>
<tr>
<th>VHF RANGING</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S^2 = Max\left(Var_{R_{ LCS}}^2, Var_{V_{ LCS}}^2, Var_{S_{ LCS}}^2\right))</td>
</tr>
</tbody>
</table>

**ALTERNATE LOS**

| \(S^2 = Var_{R_{ LCS}}^2\) + \(Var_{V_{ LCS}}^2\) |

\(Var_{S_{ LCS}}\) is stored in erasable memory.
ERASABLE DATA LOAD PARAMETERS

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>DATA LOAD MNEMONIC</th>
<th>VALUE</th>
<th>PARAMETER</th>
<th>DATA LOAD MNEMONIC</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Varr (LM)</td>
<td>RANGEVAR</td>
<td>1.1111 \times 10^{-5}</td>
<td>W_{mg} (CSM)</td>
<td>WMIDPOS</td>
<td>30,000 ft</td>
</tr>
<tr>
<td>Varr min (LM)</td>
<td>EVARMIN</td>
<td>66 m²</td>
<td>W_{mv} (CSM)</td>
<td>WMIDVEL</td>
<td>30 ft/s</td>
</tr>
<tr>
<td>Varr (LM)</td>
<td>RATTVAR</td>
<td>1.8777 \times 10^{-5}</td>
<td>W_{rr} (CSM)</td>
<td>WORPPOS</td>
<td>0.0</td>
</tr>
<tr>
<td>Varr min (LM)</td>
<td>VVARMIN</td>
<td>0.017445 m²/s²</td>
<td>W_{rr} (CSM)</td>
<td>WORPVEL</td>
<td>0.0</td>
</tr>
<tr>
<td>Varβ (LM)</td>
<td>SHAFTVAR</td>
<td>(1 mrad)²</td>
<td>W_{β} (CSM)</td>
<td>S2WSUBL</td>
<td>10,000 m</td>
</tr>
<tr>
<td>Varβ (LM)</td>
<td>TURVAR</td>
<td>(1 mrad)²</td>
<td>W_{β} (LM)</td>
<td>WRENDPOS</td>
<td>10,000 ft</td>
</tr>
<tr>
<td>Varβ (LM)</td>
<td>ENTVAR</td>
<td>(14 m)²</td>
<td>W_{β} (LM)</td>
<td>WRENDVEL</td>
<td>10 ft/s</td>
</tr>
<tr>
<td>Var R (CSM)</td>
<td>RVAR</td>
<td>0.0</td>
<td>W_{β} (LM)</td>
<td>WSHAFT</td>
<td>15 mrad</td>
</tr>
<tr>
<td>Var R (CSM)</td>
<td>VARMIN</td>
<td>(200 ft)²</td>
<td>W_{β} (LM)</td>
<td>WTRUN</td>
<td>15 mrad</td>
</tr>
<tr>
<td>Var ALT (CSM)</td>
<td>ALTVAR</td>
<td>(0.9 mrad)²</td>
<td>W_{β} (LM)</td>
<td>WSURFPOS</td>
<td>0.0</td>
</tr>
<tr>
<td>Var ALT (CSM)</td>
<td>ALTVAR</td>
<td></td>
<td>W_{β} (LM)</td>
<td>WSURFVEL</td>
<td>0.0</td>
</tr>
<tr>
<td>Var ALT (CSM)</td>
<td>ALTVAR</td>
<td></td>
<td>W_{β} (LM)</td>
<td>WRENDPOS</td>
<td>10,000 ft</td>
</tr>
<tr>
<td>Var ALT (CSM)</td>
<td>ALTVAR</td>
<td></td>
<td>W_{β} (LM)</td>
<td>WRENDVEL</td>
<td>10 ft/s</td>
</tr>
</tbody>
</table>

During close-in/flight navigation, the angle between the lines of sight to a known star and a planetary landmark is measured. This angle measurement is then used to update the state vector via the measurement incorporation routines. The geometry relating the angular measurement to the deviation state vector is given below.

STAR/LANDMARK GEOMETRY

The relationship between the state vector and measurement variable can be determined by redrawing part of the preceding figure.

\[
\delta A = \frac{|T|}{E_{C-L}} = \frac{T_{x} x_{s} + T_{y} y_{s} + T_{z} z_{s}}{E_{C-L}}
\]

where \( \delta A \) is a unit vector perpendicular to \( E_{C-L} \).
During orbital navigation, the initial line of sight from the spacecraft to a planetary landmark is measured by recording the optical shaft and trunnion angles, the IMU gimbal angles, and the time of the marks. Up to five marks are made on each landmark before the Measurement Incorporation routine is called to update the state vector.

Classical celestial mechanics says that if the angles between a planetary landmark and two different stars are measured then the line of sight from the spacecraft to the landmark can be determined, or that two star/landmark measurements are equivalent to one line-of-sight measurement. Orbital navigation uses this continuity to incorporate the line-of-sight measurements by treating it as two star/landmark measurements. The two fictitious star/landmark measurements are then incorporated in the same manner as real star/landmark measurements in P25.

The dimension of the state vector for orbital navigation is expanded from six to nine to include the landmark position vector. If the concept of "deviation state vector" is carried over to include the landmark, then

\[
\begin{bmatrix}
\delta x \\
\delta y \\
\delta z \\
\delta \theta_x \\
\delta \theta_y \\
\delta \theta_z \\
\delta x_l \\
\delta y_l \\
\delta z_l
\end{bmatrix}
\]

where:

- \( \delta x, \delta y, \delta z \): deviation of spacecraft position from the reference cone
- \( \delta \theta_x, \delta \theta_y, \delta \theta_z \): deviation of spacecraft velocity from the reference cone
- \( \delta x_l, \delta y_l, \delta z_l \): deviation of landmark position from the nominal

The geometry vector, \( b \), can be determined using a procedure similar to that used for P25.

The expected line of sight is \( \mathbf{r}_{CL} \).

The angle between the star LOS and landmark LOS is

\[
A = A_0 + \Delta A
\]

where

- \( A_0 \): Star/landmark LOS angle
- \( \Delta A \): Part of angle due to deviations from cone or nominal positions

From the above figure

\[
\Delta A = \frac{1}{||\mathbf{r}_{CL}||} \left[ \mathbf{r}_{CL}^T (\mathbf{r} - \mathbf{r}_{CL}) \right]
\]

\[
\Delta A = \frac{1}{||\mathbf{r}_{CL}||} \begin{bmatrix}
\delta x \\
\delta y \\
\delta z
\end{bmatrix}
\]

\[
\begin{bmatrix}
\delta x \\
\delta y \\
\delta z
\end{bmatrix} = \begin{bmatrix}
\delta x_l \\
\delta y_l \\
\delta z_l
\end{bmatrix}
\]

The Rendezvous Navigation Program (P20)

The Rendezvous Navigation program, P20, is used during the rendezvous phases of flight. Both the CSM and LM computers have Progress P20 so that the CSM can do its rendezvous navigation by tracking the LLR, or the LM can navigate by tracking the CSM. The CSM can navigate by measuring the line of sight to the LLR using the STS or the COAS, and/or the VHF transmit link. The LM uses the Rendezvous Ranging to measure range, range rate, LLR shaft angle bias, and LLR trunnion angle bias.

The state vectors for rendezvous navigation are

\[
\begin{bmatrix}
\delta x_{CSM} \\
\delta y_{CSM} \\
\delta z_{CSM} \\
\delta \theta_{x,CSM} \\
\delta \theta_{y,CSM} \\
\delta \theta_{z,CSM}
\end{bmatrix}
= \begin{bmatrix}
\delta x_{LM} \\
\delta y_{LM} \\
\delta z_{LM} \\
\delta \theta_{x,LM} \\
\delta \theta_{y,LM} \\
\delta \theta_{z,LM}
\end{bmatrix}
\]

where

- \( \delta x, \delta y, \delta z \): deviations from cone position
- \( \delta \theta_x, \delta \theta_y, \delta \theta_z \): deviations from cone velocity
- \( \delta \theta_x, \delta \theta_y, \delta \theta_z \): LLR shaft angle bias
- \( \delta \theta_x, \delta \theta_y, \delta \theta_z \): LLR trunnion angle bias

The Measurement Incorporation routine is used to incorporate the rendezvous measurements, and the geometry vectors are determined as follows:

- Line-of-Sight Measurements Made with the STS or COAS. The line-of-sight measurements are incorporated into the state vector by adapting two fictitious star/landmark sightings just as with the LOS measurements made during orbital navigation.
- VHF Range Measurements Made by the CSM. The geometry vector for VHF ranging is derived in the same manner as the RR range measurement vector. The only difference is the dimension of the CSM which is only 6 versus 9 for the LLR.

The geometry relating the RR measurements to the state vector is described on the following pages.
RENDEZVOUS RANGE RATE $\dot{r}_m$ MEASUREMENT

Rendezvous range rate measurement has a term due to deviation from conic velocity $V_\text{conic}$ and a term due to deviation from conic position $V_\text{conic}$. The term due to deviation from conic velocity:

$$
\dot{r}_m = V_{\text{conic}} - V_0
$$

where $V_{\text{conic}}$ is the component of range rate due to conical vectors known with zero uncertainty.

The Kalman filter measurement variable:

$$
\dot{r}_m = (V_{\text{conic}} - V_0) \cdot \text{Kalman filter measurement variable}
$$

RENDEZVOUS RANGE RATE $\dot{r}_m$ MEASUREMENTS

The rendezvous range rate term due to position deviation is:

$$
\dot{r}_m = \frac{\Delta V_{\text{conic}} \cdot \Delta V_{\text{conic}}}{|\Delta V_{\text{conic}}|^2} \cdot \Delta \dot{r}_m
$$

RENDEZVOUS RADAR SHAFT ANGLE $\phi_\text{rad}$ MEASUREMENTS

The rendezvous radar shaft angle term is defined in the $x_N$-y_N coordinate plane. The effect of $x_N$ and $y_N$ position vectors on the shaft angle can be determined by including the projection of these vectors in the equation:

$$
\phi_\text{rad} = \phi_x + \phi_y
$$

where $\phi_x$ and $\phi_y$ are the components of the shaft angle due to $x_N$ and $y_N$, respectively.

The Kalman filter measurement variable:

$$
\phi_\text{rad} = \begin{bmatrix}
\phi_x \\
\phi_y
\end{bmatrix} \cdot \text{Kalman filter measurement variable}
$$

RENDEZVOUS RADAR TRANSLATION ANGLE $\gamma_\text{rad}$ MEASUREMENTS

The rendezvous radar translation angle is defined in the $x_R$-y_R plane and can be illustrated by projecting the spacecraft position vectors onto this plane. Like the shaft angle, it is defined into two components, a nominal one due to spacecraft positions and a deviation term $\gamma_\text{dev}$ which is an element of the state vector.

$$
\gamma_\text{rad} = \gamma_x + \gamma_y + \gamma_\text{dev}
$$

where $\gamma_x$ and $\gamma_y$ are the components of the translation angle due to $x_R$ and $y_R$, respectively.

The Kalman filter measurement variable:

$$
\gamma_\text{rad} = \begin{bmatrix}
\gamma_x \\
\gamma_y \\
\gamma_\text{dev}
\end{bmatrix} \cdot \text{Kalman filter measurement variable}
$$

$$
\begin{bmatrix}
\phi_x \\
\phi_y \\
\gamma_x \\
\gamma_y \\
\gamma_\text{dev}
\end{bmatrix} \cdot \text{Kalman filter measurement variable}
$$
CSM DIGITAL AUTOPILOT

The CSM digital autopilot (DAP) provides a primary stabilization and control function and attitude error display for CSM or CSM/LM coasting flight, CSM or CSM/LM powered flight and CM entry, as well as a backup SATURN takeover function.

COASTING FLIGHT

The CSM coasting flight autopilot or reaction control system autopilot (RCS DAP) provides attitude and translation control in three CSM axes during noncoasting phases of flight. The RCS DAP has three major modes of operation, Auto, Hold and Free, as commanded by the S/C CONTROL switch.

The Hold mode maintains or holds the spacecraft (S/C) at a desired attitude within the limits of an attitude deadband specified by the crew. The Hold mode is commanded by the Hold mode selector switch (HMSW).

The Auto mode enables rate and attitude commands from the steering routines to be processed by the DAP for maneuvering the S/C to a desired attitude at a specified rate. The Auto mode is commanded by the Auto mode selector switch (AMSW). RCS commands are processed in the Auto mode as discrete commanded rates and automatic maneuvering is terminated upon release of the DAP.

The Free mode of operation releases the S/C from all maneuvers and attitude hold commands, other than minimum impulse commands, and allows the S/C to drift freely. Minimum impulse commands are single 14-millisecond control jet firings which are commanded by the RHC or, if there are no RHC commands, by the minimum-impulse controller (MIC).

Translation hand controller (THC) commands are processed in any mode and are combined with rotation commands for the desired maneuver. When a combination rotation and translation is not possible, due to a quad failure, the rotation command has priority.

ERROR DISPLAY

The DAP also provides attitude error display to the crew via the FDAI attitude error meters. There are three types of attitude error displays available:

1. To provide a monitor of autopilot performance, the autopilot following errors or phase plane errors in control axis coordinates can be displayed by keying V61E.
2. To aid the crew in executing a manual maneuver, the total attitude error with respect to the desired maneuver angles in N22 can be displayed by keying V62E.
3. Total Astronaut attitude error with respect to preloaded N17 angles can be displayed by keying V63E providing another manual maneuver aid. N17 can be loaded with a snapshot of the present DU angles by keying V60E.

POWERED FLIGHT

The powered flight autopilot stabilizes and controls the attitude of the spacecraft and maintains thrust vector control with the steering program to position the thrust vector along a desired thrust direction.

Entry

The entry autopilot provides attitude control of the command module (CM) from separation from the service module (SM) to deployment of the drogue chute. The DAP has an extra-atmospheric phase and an atmospheric phase. The extra-atmospheric phase provides three-axis spacecraft control for the trajectory segment prior to 0.05 g and accepts attitude commands from the entry guidance program to orient the S/C for the coast of 0.06 g. The atmospheric phase provides attitude control, after 0.05 g about entry roll or about the vector direction of S/C velocity relative to the air mass, to steer the S/C along the entry trajectory. The DAP accepts steering commands from the entry guidance program.

SATURN TAKEOVER

In the event of a Saturn instrumentation unit (IU) fail, a capability is provided for the CMIC to issue angular rate steering commands to the IU autopilot. CMIC takeover of Saturn control, which is accomplished by means of the LV GUIDANCE switch, may be an automatic or manual steering mode.

The manual or stick mode is available by keying V46E, which terminates computation of automatic mode attitude errors. Discrete rate commands, based on erasable parameters, are initiated by means of the RHC and transmitted to the IU autopilot.

DAP DATA

The DAP registers containing the variable parameters which determine DAP selection and desired DAP performance are accessible by keying V48E.

V48E

Flash V04 N16

Register 1:

<table>
<thead>
<tr>
<th>CONFIG</th>
<th>XTAC</th>
<th>XTBBD</th>
<th>DB</th>
<th>RATE</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Flash V04 N46

Register 1: CSM weight in pounds

Register 2: LM weight in pounds

Flash V04 N48

Register 1: Pitch-trim gimbal offset, in 1/100 degree

Register 2: Yaw-trim gimbal offset, in 1/100 degree

AUTOMATIC MODE

ATTITUDE HOLD MODE

1. Automatic three-axis rotation.
3. Attitude hold to program or manual defined attitude.

Flash V04 N47

Register 1: CSM weight in pounds

Register 2: LM weight in pounds

Flash V04 N48

Register 1: Pitch-trim gimbal offset, in 1/100 degree

Register 2: Yaw-trim gimbal offset, in 1/100 degree

Attitude Hold Mode

2. RHC produces a rotational rate as specified by N46 while out of deflection range.
3. Attitude hold to attitude selected via hold controller.
Transform Error to Control Axes:
- Error is zero when RHC not activated.
- Error is zero when CMC Free Mode.

Controller Inputs:
- N46 specifies manual rate - Minimum Impulse if Free Mode from RHC or CMC.
- N4 specifies auto maneuver.

Automatic Attitude Position Control:
- 6/1 Control
- RCS Thrusters
- Spacecraft

Rotational Hand Controller Inputs:
- Update Desired Global Angles
- Update Desired Local Angles

Automatic Attitude Position Control:
- 6/1 Control
- RCS Thrusters
- Spacecraft

Error Display IMU Error Counters:
- *IMU resolver angle display is on the same FDI selected for error display.
LM DAP CONTROL

Flashing V04 N46

<table>
<thead>
<tr>
<th>CONFIG</th>
<th>ACC</th>
<th>ACA</th>
<th>DB</th>
<th>RATE</th>
</tr>
</thead>
</table>
| CONFIG - Vehicle Configuration
| 1 = Ascent stage only
| 2 = Ascent and descent stages
| 3 = LM and CSM docked
| ACC - Acceleration Code
| 0 = Two-jet translation (RCS System A)
| 1 = Two-jet translation (RCS System B)
| 2 = Four-jet translation (RCS System A - minimum impulse)
| 3 = Four-jet translation (RCS System B - minimum impulse)
| ACA - ACA Scaling
| 0 = Docked (1 deg/s, max. rate)
| 1 = Normal (20 deg/s, max. rate)
| DB - Deadband
| 0 = 0.3 degree
| 1 = 1.0 degree
| 2 = 5.0 degrees
| RATE - Maneuver Rate (Automatic Mode)
| 0 = 0.2 deg/s
| 1 = 0.5 deg/s
| 2 = 1.0 deg/s
| 3 = 10.0 deg/s

Flashing V06 N47

Register 1: LM weight in pounds
Register 2: CSM weight in pounds

Flashing V06 N48

Register 1: Pitch-trim engine gimbal angle, in 0.01 degree
Register 2: Roll-trim engine gimbal angle, in 0.01 degree

AUTOMATIC MODE ATITUDE HOLD MODE

1. Automatic three-axis rotation and translation.
4. Attitude hold to program defined attitude.
5. Automatic rate damping.

V77 - Used to provide a manual rate command. Commanded rotational rate is proportional to hand controller (ACA) deflection. Maximum commanded rotational rate is either 4 deg/s or 20 deg/s as chosen in DAP Data Load routine.

V76 - Used to provide a minimum impulse command. Releases Attitude Hold mode and allows vehicle to drift freely. One impulse is produced for each hand controller (ACA) deflection greater than 2.5 degrees.
PHASE-PLANE FUNDAMENTALS

The equations describing spacecraft attitude errors and attitude rate errors in the phase plane are derived as follows:

**ATTITUDE RATE ERRORS**

\[
\dot{\theta} = 4M - p_\omega \quad (1)
\]

where

\[
(1) = \text{Attitude rate error} = \text{Desired spacecraft rate:
\]

\[
= \dot{\omega}_D = \omega_0 + \int (\dot{\omega}_0 + p_\omega) dt
\]

\[
(2) = \text{Actual spacecraft rate} = \dot{\omega}
\]

\[
(3) = \text{Controlled disturbance acceleration} = g_\omega
\]

Equations 2 and 1 above can be combined to eliminate \( \omega_0 \) and solve for the phase plane equation of \( \theta \):

\[
\frac{2g_\omega}{24} = \theta_0 - \omega_0 \quad (9)
\]

Equation 5 describes a parabola that goes clockwise in the phase plane direction for negative initial rate and in a counterclockwise direction for positive rate. Equations 6 and 7 together describe a typical limit cycle trajectory that traverses a clockwise direction about the origin of the error plane as shown below.

OPTIMAL PHASE-PLANE CONTROLLERS

- **TIME-OPTIMAL CONTROLLER**
  - (double integral plant with maximum control acceleration = 9)

- **FUEL-OPTIMAL CONTROLLER**
  - (same as above)

- **COMBINATION CONTROLLER**
  - (same as above)

PURPOSE: To bring \( \theta \) and \( \dot{\theta} \) to zero in minimum time.

LM UNDOCKED PHASE-PLANE

The purpose of the LM Undocked Phase-Plane logic is to compute and issue commands to the RCS thrusters in order to null spacecraft attitude and attitude rate errors. The thruster commands are in the form of a signed time, where the sign indicates the sense of rotation (positive or negative) and the time is the duration of thruster required to reach the phase-plane switching boundary. In order to achieve these ends, the phase-plane is divided into two principal regions, ROUGHLAW and FINE LAW.

ROUGHLAW

The ROUGHLAW phase-plane logic is used for coarse control of the spacecraft when the phase-plane errors are greater than 11.25 degrees and/or 1.25 degrees/second. The division between the positive and negative thrusting regions are shown in the diagram on the right.

FINE LAW

The FINE LAW phase-plane logic is used whenever the phase-plane errors are less than 11.25 degrees and 1.25 degrees/second. The FINE LAW phase-plane is configured differently for powered and coasting flight to accommodate offset accelerations caused by the EPS or EPS engine not thrusting through the LM center of gravity.

- **DRIFTING FLIGHT**
  - The drifting flight phase-plane is set up to achieve a minimum impulse limit cycle. When the phase-plane errors are in the minimum impulse zone, a 1-millisecond jet firing will be commanded. This should be sufficient to reverse the sign on \( \theta \) and cause the spacecraft to drift back to the other deadband.

- **POWERED FLIGHT**
  - When the EPS trim gimbal system is operating during powered flight, the offset accelerations (ACS) will be nullified and the phase-plane logic is the same as for drift flight except that the minimum impulse zone is eliminated.

The switching curves are then established based on ACS (changing acceleration). The switching curves are then established based on ACS (changing acceleration). The DB is selected as 0.3, 1, or 5 degrees dependent upon the mission program and the LM's orientation.
The LM/CSM docked phase-plane is used to provide feedback control mechanism. If the docking configuration during #1 "protection" attitude control, stability is determined rather than performance. Simplicity is achieved as follows:

- The piecewise-linear phase-plane is used for all three control cases.
- Only target thrusting is allowed for each case.
- Dots are turned on or off at constant sample intervals rather than computing the thrusting time required to reach the target sticking point.
- The piecewise-linear phase-plane is used by striping lines from regions.

The design objective of the simplified conceptual diagram is to:

- Enable low rate attitude maneuvers or attitude hold during drifting flight.
- Enable RCS attitude control in conjunction with the LM (Gimbal Trim System) during docked flight.
- Reduce the probability of hard mode excitations in cases of conflicting (nominal) load changes.
- Provide the above task sequencing in cases of hard mode excitations, for example, from a control maneuver.

**RATE LIMITING**

Rate limiting is incorporated to prevent the RCS jets from causing attitude rate errors exceeding 1.125 degrees/second. This number was chosen to allow the largest configuration to reach a limit cycle with zero crossings.

**TARGET RATES**

Target rates of ±1 degree/second for positive thrusting and ±0.1 degree/second for negative thrusting are incorporated during normal flight. Target rates ±1.0 degree/second for drifting flight. This allows a steady state limit cycle to be reached in minimum time by bypassing the phase-plane logic when the jets are on and thrusting toward the target rate.

**JET INHIBITION LOGIC**

Jet inhibition logic operates to prevent high-speed attitude excursions by preventing jet commands from exceeding the maximum allowed rate. Target rates ±1 degree/second for drifting flight. This allows a steady state limit cycle to be reached in minimum time by bypassing the phase-plane logic when the jets are on and thrusting toward the target rate.

**EFFECT OF CONTROL EFFECTIVENESS**

The gimbal trim system (GTS) is controlled by the LOC which issues only on or off commands. The GTS responds to these commands by enabling the RCS engine to have larger (than a constant gain, K). The resulting difference of the RCS engine rate and the target rate is computed as a control signal. The relationship between the RCS engine rate and the control signal is chosen as shown in Figure 1.

\[
F = K (\theta - \dot{\theta})
\]

where:
- \( F \) = apparent angular acceleration
- \( \dot{\theta} \) = apparent angular rate
- \( K \) = torque due to RCS engine (scaling)
- \( FL_{max} = FL_{max}(\theta) \) (angular rate approximation)
- \( T \) = RCS engine thrust
- \( R \) = trim control (imposed on center of gravity)
- \( \theta \) = trim angle

The Gimbal Drive Trim Deflection control law is shown in Figure 2. The GTS control law has two modes of operation: acceleration nulling and attitude control.

**ACCELERATION NULLING MODE**

This mode is used when the GTS phase-plane moves outside the null zone and the RCS jets are not thrusting. The acceleration nulling mode was chosen for two reasons:
- It is simple and robust, particularly for the RCS jets, in case of large errors or unexpected disturbances.
- The control law is designed to provide smooth and continuous control actions, which is important for maintaining stability during maneuvers.

**ATTITUDE CONTROL MODE**

The attitude control mode uses a trim-optimal control law to position the attitude error and error rates are small. This mode is selected when the RCS phase-plane moves inside the null zone and the RCS jets are thrusting. The trim-optimal control law ensures that the angle of attack and the angular rate are minimized to maintain stability during maneuvers. The control law is designed to provide smooth and continuous control actions, which is important for maintaining stability during maneuvers.

**EFFECT OF CONTROL EFFECTIVENESS GAIN ON ATTITUDE CONTROL RESPONSE**

The effect of control effectiveness gain on attitude control response is shown in Figure 3. The response time is shown to be significantly shorter with a higher gain, indicating improved performance and stability during maneuvers.
The State Estimator is mechanized as a two-part process: State Vector Extrapolation and State Vector Updating.

**State Vector Extrapolation**

The state vector is extrapolated using Equation 4 where the state transition matrix \( \Phi(t) \) is defined in Equation 8 and the driving matrix \( \Gamma(t) \) in Equation 7. The process noise is assumed to be Gaussian with zero mean so that the best estimate of its effect on the state vector is zero.

\[
\begin{align*}
\Delta\hat{\mathbf{x}}(t) &= \Phi(t) \Delta\hat{\mathbf{x}}(t) + \Gamma(t) \Delta\mathbf{u}(t) \\
\Delta\hat{\mathbf{x}}(t) &= \Phi(t) \Delta\hat{\mathbf{x}}(t) + \Gamma(t) \Delta\mathbf{u}(t) \\
\end{align*}
\]

**State Vector Updating**

The state vector is updated based on the weighting functions and the measurement residual.

\[
\begin{align*}
\Delta\hat{\mathbf{x}}(t) &= \Theta(t) + K(t) (\mathbf{z}(t) - \Delta\hat{\mathbf{x}}(t)) \\
\end{align*}
\]

The weighting functions use a threshold logic to filter out CDU quantizing noise. This means that if the measurement residual is less than a preselected threshold, the weighting function is zero and the measurement residual is ignored. When the measurement residual is above the threshold, the gain is not zero and the measurement residual is used to update the state vector. In other words, small variations between the measured and expected attitudes are assumed to be due to CDU quantizing and large variations due to spacecraft maneuvering.

The weighting function also varies as a function of the time since the measurement residual last exceeded the threshold level.

**Weighting Functions**

- **Attitude Gain**
  \[
  \begin{align*}
  K_{\theta \theta} &= \begin{cases} 
  0 & \text{for } |\mathbf{z}(t)\mathbf{e}_\theta| > \theta_{\text{max}} \\
  \frac{1}{\theta_{\text{max}}} & \text{otherwise}
  \end{cases} 
  
  \end{align*}
  \]

- **Angular Acceleration Gain**
  \[
  \begin{align*}
  K_{\dot{\phi} \dot{\phi}} &= \begin{cases} 
  0 & \text{for } |\mathbf{z}(t)\mathbf{e}_{\dot{\phi}}| > \dot{\phi}_{\text{max}} \\
  \frac{1}{\dot{\phi}_{\text{max}}} & \text{otherwise}
  \end{cases} 
  \end{align*}
  \]
LUNAR LANDING GUIDANCE & NAVIGATION

The quadratic guidance equations generate thrust & attitude commands based on the present vehicle state vector and the desired aim point conditions. The thrust & attitude commands are executed by the spacecraft to achieve the desired trajectory.

LUNAR LANDING NAVIGATION

Navigation is performed by the state vector update routine which uses a modified Kalman filter to estimate the vehicle state vector based on PIPA & landing radar measurements.
PRIMARY CONSTRAINTS ON LUNAR LANDING

THE TERMINAL POINT IS CONSTRAINED IN POSITION AND VELOCITY
\[ r(t_f) = 0 \quad \text{(landing site)} \]
\[ v(t_f) = 0 \quad \text{(soft landing)} \]

THE INITIAL POINT IS CONSTRAINED IN POSITION AND VELOCITY
\[ r(t_0) = 0 \quad \text{(initial position)} \]
\[ v(t_0) = 0 \quad \text{(initial velocity)} \]

BASIC GUIDANCE SCHEME

EXPRESS LEM POSITION AS A TAYLOR SERIES IN TIME EXPANDED ABOUT THE TERMINAL TIME \( t_f \)
\[ r(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} r^{(n)}(t_f) \]

EXPRESS VELOCITY AND ACCELERATION AT THE INITIAL AND TERMINAL TIMES
\[ v(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} v^{(n)}(t_f) \]
\[ a(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} a^{(n)}(t_f) \]

COMPUTE THRUST COMMANDS FROM THE ACCELERATION PROFILE AND VELOCITY
\[ T(t) = m(t) \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} a^{(n)}(t_f) \]

CAPABILITY OF THE BASIC GUIDANCE SCHEME

THE FORM OF THE ACCELERATION PROFILE IS DETERMINED BY THE NUMBER OF TERMS INCLUDED IN THE TAYLOR SERIES EXPANSION OF \( r(t) \)

<table>
<thead>
<tr>
<th>ORDER OF EXPANSION</th>
<th>N-2</th>
<th>N-3</th>
<th>N+4</th>
</tr>
</thead>
<tbody>
<tr>
<td>POSITION EQUATION</td>
<td>( r(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} r^{(n)}(t_f) )</td>
<td>( r(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} r^{(n)}(t_f) )</td>
<td>( r(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} r^{(n)}(t_f) )</td>
</tr>
<tr>
<td>VELOCITY EQUATION</td>
<td>( v(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} v^{(n)}(t_f) )</td>
<td>( v(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} v^{(n)}(t_f) )</td>
<td>( v(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} v^{(n)}(t_f) )</td>
</tr>
<tr>
<td>ACCELERATION EQUATION</td>
<td>( a(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} a^{(n)}(t_f) )</td>
<td>( a(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} a^{(n)}(t_f) )</td>
<td>( a(t) = \sum_{n=0}^{N} \frac{(t-t_f)^n}{n!} a^{(n)}(t_f) )</td>
</tr>
</tbody>
</table>

EFFECT OF INITIAL CONSTRAINTS ON THE ACCELERATION PROFILE

<table>
<thead>
<tr>
<th>INITIAL CONSTRAINTS</th>
<th>POSITION</th>
<th>VELOCITY</th>
<th>ACCELERATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a(t) = 0 )</td>
<td>( v(t) = 0 )</td>
<td>( r(t) = 0 )</td>
<td></td>
</tr>
</tbody>
</table>

1. BRAKING PHASE
   THIS PHASE BRAKES THE LEM DOWN FROM ORBITAL VELOCITY AND IS TARGETED SUCH THAT THE LANDING SITE VISIBILITY CONSTRAINTS CAN BE MET DURING THE VISIBILITY PHASE.

2. VISIBILITY PHASE (APPROACH PHASE)
   THIS PHASE IS FLOWN WITH THE LEM ORIENTED SUCH THAT THE LANDING SITE IS VISIBLE FOR LANDING SITE DESIGNATION. TARGETING IS CHOSEN TO YIELD THE PROPER INITIAL CONDITIONS FOR VERTICAL DESCENT CONSISTENT WITH THE VISIBILITY CONSTRAINTS.

3. VERTICAL DESCENT PHASE
   THIS PHASE REDUCES THE SPACECRAFT HORIZONTAL VELOCITY TO ZERO AND VERTICAL VELOCITY TO \( \pm 5 \text{ ft/sec} \). QUADRATIC GUIDANCE IS NOT USED.

SECONDARY CONSTRAINTS OF LUNAR LANDING

- REACH TERMINAL CONDITIONS WITH HIGH ACCURACY
- VERTICAL DESCENT FOR LAST 200 ft. AT A RATE OF \(-5 \text{ ft/sec}\).
- LANDING SITE VISIBILITY FOR 200 sec. BEFORE TOUCHDOWN

MODIFICATIONS TO BASIC GUIDANCE SCHEME

THE FOLLOWING MODIFICATIONS WERE MADE TO THE BASIC QUADRATIC GUIDANCE SCHEME IN ORDER TO ACHIEVE THE SECONDARY CONSTRAINTS OF LUNAR LANDING

- THE DESIRED Trajectory is redefined on every iteration cycle (2 sec.) such that the initial conditions \( r(t), v(t), a(t) \) are the current best estimate of vehicle state as determined by the state vector update routine. This enables point accuracy by eliminating cumulative errors in the implementation of the basic scheme.
- THE LANDING SEQUENCE IS DIVIDED INTO THREE PHASES EACH WITH ITS OWN FINAL CONDITIONS OR AIM POINTS.
  1. BRAKING PHASE
     THIS PHASE BRAKES THE LEM DOWN FROM ORBITAL VELOCITY AND IS TARGETED SUCH THAT THE LANDING SITE VISIBILITY CONSTRAINTS CAN BE MET DURING THE VISIBILITY PHASE.
  2. VISIBILITY PHASE (APPROACH PHASE)
     THIS PHASE IS FLOWN WITH THE LEM ORIENTED SUCH THAT THE LANDING SITE IS VISIBLE FOR LANDING SITE DESIGNATION. TARGETING IS CHOSEN TO YIELD THE PROPER INITIAL CONDITIONS FOR VERTICAL DESCENT CONSISTENT WITH THE VISIBILITY CONSTRAINTS.
  3. VERTICAL DESCENT PHASE
     THIS PHASE REDUCES THE SPACECRAFT HORIZONTAL VELOCITY TO ZERO AND VERTICAL VELOCITY TO \( \pm 5 \text{ ft/sec} \). QUADRATIC GUIDANCE IS NOT USED.

IGNITION
\[ V_1 = 6,500 \text{ ft/sec.} \]
\[ H_1 = 40,000 \text{ ft.} \]
\[ V_2 = 500 \text{ ft/sec.} \]
\[ H_2 = 7,500 \text{ ft.} \]

GATE
\[ V_1 = 350 \text{ ft/sec.} \]
\[ V_2 = 5 \text{ ft/sec.} \]
\[ H_1 = 200 \text{ ft.} \]
\[ H_2 = 6 \text{ ft/sec.} \]
DERIVATION OF QUADRATIC GUIDANCE LAW

EXPRESS THE POSITION, VELOCITY & ACCELERATION EQUATIONS IN MATRIX FORM

\[
\begin{bmatrix}
\dot{q}(t) \\
Q(t)
\end{bmatrix} =
\begin{bmatrix}
T_p & T_p & 1 & 0 \\
0 & T_p & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{q}(t) \\
q(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

WHERE: \( T_p = t - t_f \)

REARRANGE THE MATRIX EQUATION SUCH THAT \( q(T_p, t) \) & \( q(T_p, t) \) ARE EXPRESSED IN TERMS OF \( q(T_p, t) \) & \( q(T_p, t) \) \( q(T_p, t) \) \( q(T_p, t) \), \( q(T_p, t) \) \( q(T_p, t) \) \( q(T_p, t) \) \( q(T_p, t) \), \( q(T_p, t) \) \( q(T_p, t) \) \( q(T_p, t) \) \( q(T_p, t) \)

SOLVE THE ABOVE EQUATION FOR \( q(T_p, t) \) \( q(T_p, t) \)

\[
\begin{bmatrix}
\dot{q}(t) \\
q(t)
\end{bmatrix} =
\begin{bmatrix}
6/T_p & -3/T_p & 0 & 1 \\
24/T_p & -72/T_p & 12/T_p & 0
\end{bmatrix}
\begin{bmatrix}
\dot{q}(t) \\
q(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{q}(t) \\
q(t)
\end{bmatrix} +
\begin{bmatrix}
\dot{q}(t) \\
q(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

QUADRATIC ACCELERATION EQUATION

\[
q(T_p) = q(T_p) + \frac{1}{2} \frac{d^2 q(T_p)}{dt^2} \]

WHERE:

\( q(T_p) \) \( q(T_p) \) \( q(T_p) \) \( q(T_p) \)

LEADTIME COMPENSATION

\[
T_p = T_p + \left( \frac{1}{2} \frac{d^2 q(T_p)}{dt^2} \right) t_f
\]

TIME-TO-GO COMPUTATION

TIME-TO-GO IS COMPUTED TO SATISFY THE REQUIRED FINAL VALUE OF DOWNRANGE SEEK.

FROM THE ABOVE EQUATIONS:

\[
q(T_p) = q(T_p) + \frac{1}{2} \frac{d^2 q(T_p)}{dt^2} \]

TO FIND \( q(T_p) \):

INITIAL GUESS = \( q(T_p) \)

NEW APPROXIMATION = \( q(T_p) \)

TO FIND TIME-TO-GO:

INITIAL GUESS = \( q(T_p) \)

NEW APPROXIMATION = \( q(T_p) \)

RADIAL GUIDANCE

IF THE DESIRED ACCELERATION IS GREATER THAN MAX. ENGINE THRUST, THEN DOWNRANGE THRUST IS LIMITED.
BASIC MECHANIZATION (CONT.)

3. ALIGN THE GUIDANCE COORDINATE FRAME

The guidance frame is aligned so that the cross-range (y-component) of jerk is zero at phase terminus. The y-component of jerk is given by the quadratic guidance equations:

\[ \ddot{y}_g = \text{unit} \left( \frac{\ddot{y}_g}{\gamma} \right) \]

The y-components of the aim-point vectors are specified as zero:

\[ \ddot{y}_g = \text{unit} \left( f_1 \times \left( f_1 \times f_1 \right) \right) \]
\[ \ddot{y}_g = \text{unit} \left( f_1 \times f_1 \right) \]

Thus, the guidance frame is aligned with its y-axis perpendicular to the vector \( f_1 \).

4. COMPUTE THE WINDOW POINTING VECTOR

The x-body axis is aligned along the desired thrust vector \( \mathbf{F}_d \).

The y-body axis is aligned in the plane containing \( \mathbf{F}_d \) and \( \mathbf{U}_w \).

5. COMPUTE THRUST MAGNITUDE

Actual throttle settings are limited by engine considerations to not exceed 57% of full scale or less than 60%.

ERASABLE DATA LOAD PARAMETERS
LOCITY 5995 496 = 57%
MAXIMUM 6415 485 = 65%

MANUAL THRUST LEVEL
0% 20% 40% 60% 80% 100%
DESIRED THRUST 1841 (%)
STATE VECTOR EXTRAPOLATION

State vector extrapolation is accomplished by an Average G routine at 2-second intervals coincident with PIPA ΔV processing.

LM position vector \( \vec{r}_p \) is extrapolated assuming constant acceleration over the 2-second interval

\[
\vec{r} = \vec{r}_0 + \vec{v}_0 \Delta t + \frac{1}{2} \vec{a} \Delta t^2
\]

where

- \( \vec{r}_0 = \) position vector (\( \vec{r}_p \)) at end of previous interval
- \( \vec{v}_0 = \) velocity vector (\( \vec{v}_p \)) at end of previous interval
- \( \vec{a} \) = accumulated PIPA ΔV pulses during 2-second interval
- \( \vec{g}_0 \) = lunar gravitational acceleration at end of previous interval

LM velocity vector \( \vec{v}_p \) is extrapolated using PIPA ΔV pulses and the average gravitational acceleration over the 2-second interval

\[
\vec{v} = \vec{v}_0 + \vec{a} \Delta t
\]

where

- \( \vec{v}_0 = \) velocity (\( \vec{v}_p \)) at end of previous interval
- \( \vec{a} = \) accumulated PIPA ΔV pulses over 2-second interval
- \( \vec{g}_0 = \) lunar gravitational acceleration at end of previous interval
- \( \vec{g} = \) lunar gravitational acceleration at end of present interval

In addition to the state vector update, the following terms are computed:

- Altitude \( h' = r_p - r_LS \)
  where
  - \( r_p = \) magnitude of position, \( \vec{r}_p \)
  - \( r_LS = \) magnitude of landing site, \( \vec{r}_LS \)
- Velocity \( v' = \frac{h'}{r_p} \)
- Mass \( m_n = m_{n-1} - \frac{|\Delta v_p|}{V_e} \) (\( V_e = \) Exhaust Velocity Constant)
- Velocity Increment \( \Delta V = \frac{|\Delta v_p|}{r_p} \)
UPDATE THE STATE VECTOR USING LR VELOCITY DATA

The Landing Radar has three velocity components. They are used (one during each 2-second interval) to update state according to the timeline shown below.

\[
\begin{align*}
\Delta v_{n-1} & \quad 2 \text{ seconds} \quad \Delta v_{n+1} \\
V_{x} \quad V_{x} & \quad V_{y} \quad \text{PIPA Processing Times}
\end{align*}
\]

LR Velocity Read Times

\[
\begin{align*}
\Delta v_{u} = & \quad \Delta v_{x} \\
\Delta v_{y} = & \quad \Delta v_{y} \\
\Delta v_{z} = & \quad \Delta v_{z}
\end{align*}
\]

- Compute the measurement residual corresponding to the time that the velocity data was read \( t_{u} \),

\[
\begin{align*}
\Delta q_{u} = & \quad \tilde{q}_{u} - q_{u} \\
q_{u} = & \quad \text{LR velocity component read at time } t_{u} \\
q_{u}^{'} = & \quad \text{estimated component of LM relative velocity in the direction of } \tilde{q}_{u} \\
q_{u}^{' *} = & \quad \omega_{p} \times \tilde{v}_{p} \cdot \Delta v_{u}
\end{align*}
\]

where

- \( \omega_{p} \times \tilde{v}_{p} \) = velocity of lunar surface
- \( \Delta v_{u} \) = unit vector in direction of LM velocity data
- \( \Delta v_{u} = \omega_{p} \times \tilde{v}_{p} = \text{velocity of LM relative to lunar surface} \)
- \( \Delta v_{u}^{' \prime} = \text{estimated LM velocity at time } t_{u} \)
- \( \Delta v_{u}^{' \prime} = \Delta v_{u} + \Delta v_{u}^{\prime \prime} \) \( t_{u} = t_{h-1} \)

where

- \( \Delta v_{u}^{\prime \prime} = \text{LM velocity at end of previous update cycle} \)
- \( \Delta v_{u} = \text{PIPA } \Delta v \text{ read at time of LR velocity data } t_{u} \)
- \( \Delta v_{u}^{\prime \prime} = \text{lunar gravitational acceleration at end of previous cycle} \)

- Update the LM velocity vector at time \( t_{u} \), using measurement residual and extrapolate velocity \( v_{u}^{'} \)

\[
\begin{align*}
\Delta v_{u} = & \quad \tilde{q}_{u} \cdot \Delta v_{u}^{' \prime \prime} \\
\Delta v_{u}^{' \prime \prime} = & \quad \text{PIPA } \Delta v \text{ read at time of LR velocity data } t_{u} \\
\Delta v_{u}^{\prime \prime} = & \quad \text{lunar gravitational acceleration at end of previous cycle} \)
\end{align*}
\]
A. ORBITAL RELATIONSHIPS

Assuming a spherical planet, the equation of motion for a satellite is given by

\[ \frac{d^2 r}{dt^2} = -\frac{\mu}{r^3} \tag{1} \]

where

\[ \mu = (M_{\text{planet}} + M_{\text{satellite}}) \times G \]

\[ G \text{ = gravitational constant} \]

The solution of Equation 1 is:

\[ r = \frac{h^2 / \mu}{1 + e \cos f} \tag{2} \]

where

\[ h = \text{angular momentum of the satellite} \]

\[ e = \text{eccentricity of the orbit} \]

\[ f = \text{true anomaly} \]

which is the polar equation of a conic. The conic will be an ellipse, a parabola, or a hyperbola. Treating the parabola as a special case of the ellipse and considering only the hyperbola and ellipse, the following relationships are obtained:

**Angular Momentum**

\[ h = r \times v = r^2 \dot{r} \text{ = constant} \]

**Velocity**

\[ v = \mu (2/r - 1/a) \]

**Apogee**

\[ r_a = a (1 + e) \]

**Semilatus Rectum**

\[ p = h^2 / \mu \]

**Semimajor Axis**

\[ a = r_p / (1 - e^2) \] Negative for hyperbola

**Eccentricity**

\[ e = (1 - h^2 / \mu a)^{1/2} \]

**Perigee**

\[ r_p = a (1 - e) \]

**True Anomaly**

\[ \cos f = \frac{p}{r e} = \frac{1}{e} \sin f = \frac{h}{p \cos f} \]

**Elipse Only**

**Period**

\[ P = 2 \pi \left( a^{3/2} / \mu^{1/2} \right) \]

**Mean Motion**

\[ n = \mu^{1/2} a^{-3/2} \]

**Mean Anomaly**

\[ M = n (t - t_p) \]

\[ t_p = \text{time of perigee passage} \]

**Eccentric Anomaly**

\[ E - e \sin E = M \] (Kepler's Equation)

\[ \tan (E/2) = \sqrt{(1 - e) / (1 + e)} \]

**Hyperbola Only**

**Mean Motion**

\[ \gamma = \mu / a^2 \]

**Mean Anomaly**

\[ M = \gamma (t - t) \]

\[ t = \text{time of perigee passage} \]

**Eccentric Anomaly**

\[ H - e \sinh H = M \]

\[ \tanh \left( (H/2) \right) = (e - 1) / (e + 1) \]

From these relationships, given an initial position and velocity vector, the orbit and orbit parameters are uniquely determined.
B. THE ORBIT IN SPACE

The orbit in space is defined by an orthogonal set of axes along perigee, the semilatus rectum, and the angular momentum vector. The ordered set of right hand rotations (Euler angles) to achieve this orientation from the earth-centered-inertial (ECI) frame are illustrated in Figure 1 and given by the expression

\[
\begin{bmatrix}
P \\
Q \\
W
\end{bmatrix}
= [R_Z(\Omega)] [R_Y(\delta)] [R_X(\gamma)]
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
= [A]
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\]

where
- \( \Omega \) = longitude of ascending node
- \( \delta \) = angle of incidence
- \( \gamma \) = argument of perigee

Figure 1. The Orbit in Space

The angles \( \Omega, \delta, \) and \( \gamma \) are generally unknowns; therefore, the elements in \([A]\) must be evaluated by some other means. To this end, Equation 3 is expressed as follows:

\[
\begin{bmatrix}
P_x \\
P_y \\
P_z
\end{bmatrix}
= [A]
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

The unit vector along perigee (that is, \( P_x, P_y, P_z \)) can be determined by Equation 141, Page 20 of Battin as follows:

\[
P = \frac{1}{\mu c} \left( (v^2 - \frac{\mu}{r}) \mathbf{r} - (\mathbf{r} \cdot \mathbf{v}) \mathbf{v} \right)
\]

where
- \( v \) = absolute magnitude of velocity
- \( r \) = absolute magnitude of the radius vector
- \( \mathbf{v} \) = inertial velocity
- \( \mathbf{r} \) = radius vector

This relationship is obtained as follows:

\[
\begin{align*}
P &= \frac{1}{\mu c} \left( (v^2 - \frac{\mu}{r}) \mathbf{r} - (\mathbf{r} \cdot \mathbf{v}) \mathbf{v} \right) \\
&= \frac{1}{\mu c} \left( \left( v^2 - \frac{\mu}{r} \right) \mathbf{r} - (\mathbf{r} \cdot \mathbf{v}) \mathbf{v} \right)
\end{align*}
\]

Combining 1, 2, and 3 into Equation 6:

\[
P = \frac{1}{\mu c} \left[ \left( v^2 - \frac{\mu}{r} \right) \mathbf{r} - (\mathbf{r} \cdot \mathbf{v}) \mathbf{v} \right]
\]

By substituting

\[
h = r v \sin \beta, \quad \cos \beta = \frac{\mathbf{r} \cdot \mathbf{v}}{r v}
\]

and adding the factor

\[
\beta = \frac{h^2 v^2}{\mu} - r v^2 \cos^2 \beta \delta
\]

the above equation reduces to

\[
P = \frac{1}{\mu c} \left[ \left( v^2 - \frac{\mu}{r} \right) \mathbf{r} - (\mathbf{r} \cdot \mathbf{v}) \mathbf{v} \right]
\]

but

\[
\mathbf{v} = v \cos \beta \delta \mathbf{v} + v \sin \beta \mathbf{e}_n
\]

therefore

\[
P = \frac{1}{\mu c} \left[ \left( v^2 - \frac{\mu}{r} \right) \mathbf{r} - (\mathbf{r} \cdot \mathbf{v}) \mathbf{v} \right]
\]

Since \( h \) = constant, the unit vector along \( h \) (that is, \( W_x, W_y, W_z \)) is determined given any \( \mathbf{v} \) and corresponding \( \mathbf{v} \) in the orbit. That is

\[
W = \frac{\mathbf{v} \times \mathbf{v}}{h v^2}
\]

Having determined \( \mathbf{P} \) and \( \mathbf{W} \), the remaining unit vector, \( \mathbf{Q} \), is calculated as follows:

\[
\mathbf{Q} = \mathbf{P} \times \mathbf{W}
\]

By using Equations 6, 7, and \( \mathbf{Q} \), the matrix \([B]\) is determined. That is, the orbit is defined in inertial space. From Equations 3 and 4, \([A] = [B]\). By equating elements and by using the constraint \( \mathbf{Q} \cdot \mathbf{r} = 0 \), the Euler angles \( \Omega, \delta, \gamma \) are determined.
C. ORBIT DETERMINATION

The problem of orbit determination can be stated as follows: Given an initial $t_1$ and corresponding $\mathbf{v}_1$, vector expressed in ECEF coordinates, determine the position and velocity in ECEF coordinates at some time $t_2$.

From what has been presented so far, the approach would be to determine if the conic is a hyperbola or an ellipse. Then use the corresponding form of Kepler's equation to solve for the true anomaly and through the orbit-in-space-transformation determine the position and velocity at $t_2$.

This method has the undesirable feature of first determining if the conic is a hyperbola or an ellipse and requires two sets of equations.

A more unified approach as presented by Battin is the universal conic equations which are given by

$$
\begin{align*}
\hat{e}(t) &= \left[1 - \frac{x^2}{r^2} \frac{C(X^2 \alpha_1)}{r^2} \right] \hat{e}_1 + \left[1 - \frac{x^2}{r^2} \frac{S(X^2 \alpha_1)}{r^2} \right] \hat{v}_1 \\
\hat{v}(t) &= \left[1 - \frac{x^2}{r^2} \frac{X^2 x(X^2 \alpha_1)}{r^2} \right] \hat{v}_1 + \left[1 - \frac{x^2}{r^2} \frac{X^2 x(X^2 \alpha_1)}{r^2} \right] \hat{v}_1
\end{align*}
$$

where

$$
\begin{align*}
C(X^2 \alpha_1) &= \frac{1}{2} - \frac{x^2}{r^2} \frac{C(X^2 \alpha_1)}{r^2} \\
S(X^2 \alpha_1) &= \frac{1}{2} - \frac{x^2}{r^2} \frac{S(X^2 \alpha_1)}{r^2} \\
X &= \frac{\mathbf{X}_1}{\mathbf{v}_1} \quad \text{for ellipse} \\
X &= \frac{\mathbf{X}_1}{\mathbf{v}_1} \quad \text{for hyperbola}
\end{align*}
$$

The parameter $X$ which is required for Equations 9 and 10 is determined as follows:

An initial guess for $X$ is given by

$$
X = X_0 + S (1 - F_0) S (1 - F_2) S \left[\frac{1}{2} \left(1 - \frac{x^2}{r^2} \alpha_1 \right) S \right] \quad (11)
$$

where

$$
X_0 = 0 \quad \text{for the first iteration} \quad \text{and} \quad X_0 = \text{for subsequent iterations}
$$

$$
F_0 = \frac{F_1 - F_2}{2} \quad \text{and} \quad S = \frac{F_1 - F_2}{2} (t_2 - t_1)
$$

Having an initial guess for $X$, an improved value is obtained by a Newton-Raphson iteration scheme as follows:

$$
X_{n+1} = X_n - \frac{F(X_n)}{F'(X_n)} \quad (12)
$$

where

$$
F(X_n) = \frac{1}{2} \frac{X_n^2}{r^2} C(X^2 \alpha_1) + \left(1 - \frac{x^2}{r^2} \right) X_n^2 S(X^2 \alpha_1) + \frac{r_1}{r} X_n
$$

$$
F'(X_n) = \frac{1}{2} \frac{X_n^2}{r^2} C(X^2 \alpha_1) + \left(1 - \frac{x^2}{r^2} \right) X_n^2 S(X^2 \alpha_1) + \frac{r_1}{r} r_1
$$

The iteration scheme of Equations 11 and 12 converges quickly if $X_0 = 0$ is sufficiently small. The good initial guess provided by Equation 11 is due to the iteration number three in most cases. The value for $X_0$, if then substituted into Equations 9 and 10, directly specifying $r_1$ and $v_1$, in ECI normal. Following the numerical conic method just described, the position and velocity vectors referred to ECI space can be determined for any time during the flight.

The universal conic equations employ the transformation of trajectory segments under a wide variety of conditions. Depending on the computational method, the set of universal equations will solve:

1. Initial Problem; given $x, y, z$, and $v_x$, solve for $v_y$ and $v_z$.
2. Initial Problem; given $s$, solve for $v_x$, $v_y$, and $v_z$.
3. Initial Problem; given $x, y, z$, and $v$, solve for $s$.
4. Initial Problem; given $s$, solve for $v_x$, $v_y$, and $v_z$.

where:

- $x, y, z, v_x, v_y$: initial position, velocity
- $s$: arc length
- $v$: magnitude of velocity
- $\alpha$: true anomaly
- $\mu$: gravitation constant
D. DISTURBANCE ACCELERATIONS

So far the planet in question has been assumed to be spherical which is not the case for either the earth or the moon. Nor have the effects of the sun and the moon on the earth or vice versa been considered. Therefore, Equation 1 must be modified as follows:

\[ \frac{d^2 \ddot{r}}{dt^2} = \frac{\mu}{r^3} \ddot{r} - \ddot{a}_d. \]  \hspace{1cm} (27)

where

\[ \ddot{a}_d \] is the disturbance acceleration due to

1. oblateness of the earth or the nonspherical shape of the moon depending on which reference body is used,
2. the effects of the sun,
3. the effects of the secondary body on the primary body; that is, effects of moon on earth if earth is primary body and vice versa.

Analytical expressions for \( \ddot{a}_d \) are given in Ref. 517, Section 5. Since \( \ddot{a}_d \) is small in comparison to \( \ddot{a}_\text{gr} \), the two-body orbit given by Equation 1 is used as a reference or osculating orbit which is perturbed by \( \ddot{a}_d \). The actual position and velocity vectors are, therefore, given by:

\[ \ddot{r} = \ddot{r}_0 + \ddot{\delta}, \]
\[ \ddot{v} = \ddot{v}_0 + \ddot{\sigma}. \] \hspace{1cm} (24)

Differentiating Equation 24 and substituting into Equation 23 gives the following expression for the differential acceleration

\[ \frac{d^2 \ddot{\delta}}{dt^2} = \frac{\mu}{r^3} \left[ \left( 1 - \frac{\ddot{r}_0^2}{r^2} \right) \ddot{r} - \ddot{\delta} \right] + \ddot{a}_d. \] \hspace{1cm} (25)

subject to the initial conditions

\[ \ddot{\delta} (t_0) = 0, \quad \ddot{\sigma} (t_0) = 0. \]

This method (Equation 25) is known as Encke's method of differential accelerations.

Since the coefficient of \( \ddot{r} \) in Equation 25 requires the subtraction of nearly equal quantities, prohibitive errors are introduced by solving Equation 25 in its present form. This difficulty can be overcome by making the substitution

\[ \left( 1 - \frac{\ddot{r}_0^2}{r^2} \right) \ddot{r} = \left( 1 + \frac{\ddot{r}_0^2}{1 + \rho} \right) \left( \ddot{r}_0 + \ddot{r} \right) \cdot \ddot{\delta} \] where \( \rho = \ddot{r}_0/r \)

Equation 25, therefore, becomes

\[ \frac{d^2 \ddot{\delta}}{dt^2} = \frac{\mu}{r^3} \left[ \left( 1 + \frac{\ddot{r}_0^2}{1 + \rho} \right) \left( \ddot{r}_0 + \ddot{r} \right) - \ddot{\delta} \right] + \ddot{a}_d. \] \hspace{1cm} (26)

Equation 26 can be solved by any number of numerical integration schemes. The method used in the Apollo Guidance Computer (AGC) is the Nystrom method.

In order to maintain the efficiency of Encke's method of differential accelerations, \( \ddot{\delta} (t) \) must remain small. Therefore, a new osculating orbit must be defined by the total position and velocity vectors \( \ddot{r} (t) \) and \( \ddot{v} (t) \) when \( \ddot{\delta} (t) \) reaches a predetermined limit. The process of selecting a new conic orbit from which to calculate deviations is called rectification.

To sum up, the position and velocity, during freefall, at time \( t_1 \), given the position and velocity at time \( t_0 \), are computed as follows:

1. Position and velocity in the osculating orbit at time \( t_1 \) are calculated according to Equations 9 and 10.
2. Deviations are then obtained by numerical integration of Equation 26.
3. A new conic from which to calculate deviations is defined each time the deviations \( \ddot{\delta} (t) \) reach a predetermined limit.

E. DISTURBING FUNCTIONS AND THEIR APPLICABILITY

We define the following disturbance accelerations as applicable to Apollo:

\[ \ddot{a}_{\text{gr}} = \text{acceleration due to the nonspherical gravitational perturbations of the earth} \]
\[ \ddot{a}_{\text{gkm}} = \text{acceleration due to the nonspherical gravitational perturbations of the moon} \]
\[ \ddot{a}_{\text{gy}} = \text{acceleration due to the secondary body on the primary body; that is, moon is secondary body when earth is used as reference and vice versa} \]
\[ \ddot{a}_{\text{gs}} = \text{acceleration due to the sun} \]

analytical expressions for which are given in Ref. 517, Section 5.

The applicable disturbance accelerations and their region of applicability for Apollo are given in Figure 3.
The TPI targeting for the Apollo 16 mission is accomplished using the equations and techniques stated below utilizing the following information:

**Targeting Procedure:**
1. Compute the elevation angle of the LM-CSM LOS above the LM horizontal plane at the time of TPI.

\[ E_A = \cos^{-1} \left( \frac{\mathbf{V}_L \cdot \mathbf{V}_P \text{ SGN} (\mathbf{U}_P \cdot \mathbf{U} \times \mathbf{R}_{LM})}{\mathbf{V}_L \cdot \mathbf{V}_P} \right) \]

where:
- \( \mathbf{U}_L = \text{unit} (\mathbf{R}_{CM} \cdot \mathbf{B}_{LM}) \)
- \( \mathbf{U} = \text{unit} (\mathbf{R}_{LM} \times \mathbf{V}_{LM}) \)
- \( \mathbf{V}_P = \text{unit} \left( \frac{\mathbf{V}_L \cdot \mathbf{B}_{LM} \cdot \mathbf{B}_{LM}}{|\mathbf{B}_{LM}|} \right) \)

If \( E_A < 0^\circ \) define \( E_A = 360^\circ - E_A \)
2. Compute the time \( t_2 \) required for the CSM to travel through the central angle \( \psi_{CM} \) using the Time-Theta method described on page SC-51.

3. Update the state vector of the CSM to the time of rendezvous \( (\Delta T = t_2) \) using Kepler's method as described on page SC-51 obtaining \( (\mathbf{r}_{CMF}, \mathbf{v}_{CMF}) \).

4. Compute the central angle \( \psi_{LM} \) through which the LM must traverse between its position at TPI and the CSM's position at rendezvous.

\[
\psi_{LM} = \text{SGN} \left( \mathbf{R}_{LM} \times \mathbf{R}_{CMF} \cdot \mathbf{U} \right) \cos^{-1} \left( \mathbf{R}_{LM} \cdot \mathbf{R}_{CMF} \right)
\]

if \( \psi_{LM} < 0^\circ \) define \( \psi_{LM} = \psi_{LM} + 2\pi \)

5. Using the Time-Theta method, determine the flight path angle \( \gamma \) required of the LM at the end of the TPI burn such that the time required by the LM to traverse the central angle of \( \psi_{LM} \) is the same as \( t_2 \).

6. Calculate the velocity to be gained.

\[
\Delta \mathbf{v}_g = \mathbf{v}'_{LM} - \mathbf{v}_{LM}
\]

where:

\[
\mathbf{v}'_{LM} = \sqrt{\frac{\mu}{R_{LM}}} \left( \cot \gamma \cdot \text{unit } \mathbf{R}_{LM} + \mathbf{v}_N \times \text{unit } \mathbf{R}_{LM} \right)
\]

\( \mu \) = semilatus rectum of the TPI orbit

\[
\mathbf{v}_N = \text{unit } \mathbf{R}_{CMF} \times \text{unit } \mathbf{R}_{LM}
\]

The velocity to be gained is burned out using the APS and Lambert Aimpoint Guidance.