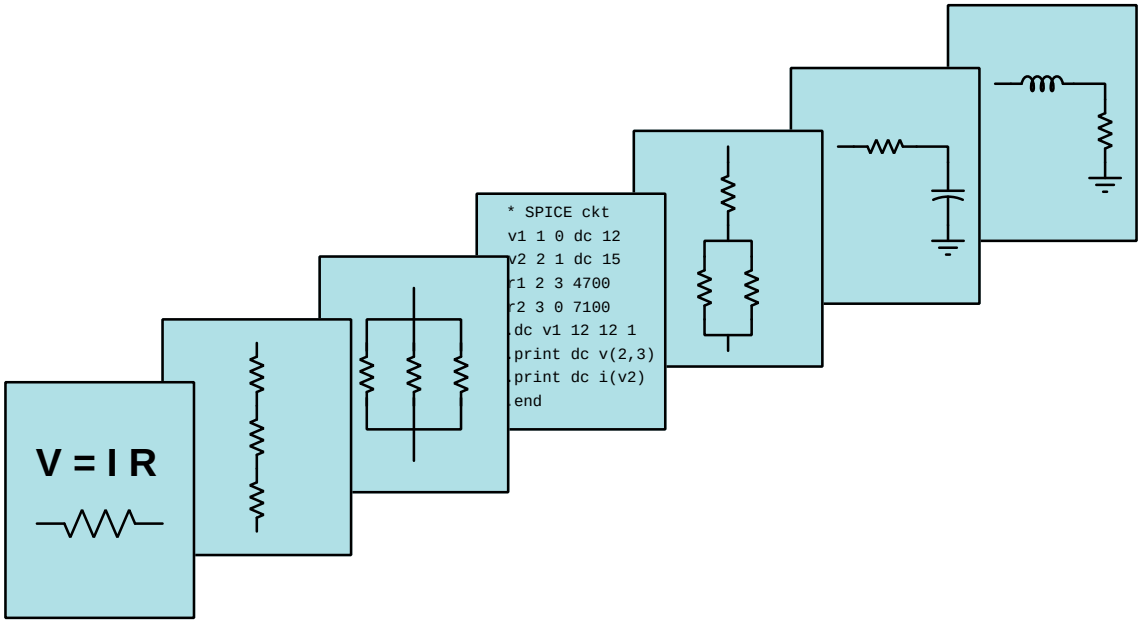
Modular Electronics Learning (ModEL) project



Fundamentals of Calculus

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Last update = 14 February 2025

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Chapter 1

Introduction

Chapter 2

Tutorial

Mathematics is the investigation of an artificial world: a universe populated by abstract entities and rigid rules governing those entities. Mathematicians devoted to the study and advancement of pure mathematics have an extremely well-developed respect for these rules, for the integrity of this artificial world depends on them. In order to preserve the integrity of their artificial world, their collective work must be *rigorous*, never allowing for sloppy handling of the rules or allowing intuitive leaps to be left unproven.

However, many of the tools and techniques developed by mathematicians for their artificial world happen to be extremely useful for understanding the real world in which we live and work, and therein lies a problem. In applying mathematical rules to the study of real-world phenomena, we often take a far more pragmatic approach than any mathematician would feel comfortable with.

The tension between pure mathematicians and those who apply math to real-world problems is not unlike the tension between linguists and those who use language in everyday life. All human languages have rules (though none as rigid as in mathematics!), and linguists are the guardians of those rules, but the vast majority of human beings play fast and loose with the rules as they use language to describe and understand the world around them. Whether or not this "sloppy" adherence to rules is good depends on which camp you are in. To the purist, it is offensive; to the pragmatist, it is convenient.

I like to tell my students that mathematics is very much like a language. The more you understand mathematics, the larger "vocabulary" you will possess to describe principles and phenomena you encounter in the world around you. Proficiency in mathematics also empowers you to grasp relationships between different things, which is a powerful tool in learning new concepts.

This book is not written for (or by!) mathematicians. Rather, it is written for people wishing to make sense of electricity, electronics, and industrial process measurement and control. This Tutorial is devoted to a very pragmatic coverage of certain mathematical concepts, for the express purpose of applying these concepts to real-world systems.

Mathematicians, cover your eyes for the rest of this Tutorial!

2.1 Introduction to calculus

Few areas of mathematics are as powerfully useful in describing and analyzing the physical world as calculus: the mathematical study of *changes*. Calculus also happens to be tremendously confusing to most students first encountering it. A great deal of this confusion stems from mathematicians' insistence on rigor[1] and denial of intuition.

Look around you right now. Do you see any mathematicians? If not, good – you can proceed in safety. If so, find another location to begin reading this Tutorial. I will frequently appeal to practical example and intuition in describing the basic principles of single-variable calculus, for the purpose of expanding your mathematical "vocabulary" to be able to describe and better understand phenomena of change related to mechanical and electrical systems.

Silvanus P. Thompson, in his wonderful book *Calculus Made Simple* originally published in 1910, began his text with a short chapter entitled, "To Deliver You From The Preliminary Terrors[2]." I will follow his lead by similarly introducing you to some of the notations frequently used in calculus, along with very simple (though not mathematically rigorous) definitions.

When we wish to speak of a change in some variable's value (let's say x), it is common to precede the variable with the capital Greek letter "delta" as such:

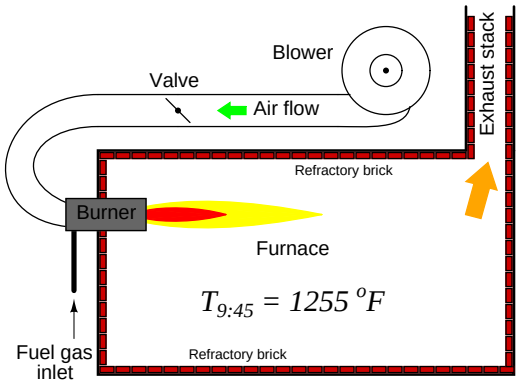
$$\Delta x = \text{"Change in } x\text{"}$$

An alternative interpretation of the "delta" symbol (Δ) is to think of it as denoting a *difference* between two values of the same variable. Thus, Δx could be taken to mean "the difference between two values of x". The precise nature and cause of this difference is not important right now: it may be the difference between the value of x at one point in time versus another point in time, it may be the difference between the value of x at one point in space versus another point in space, or it may simply be the difference between values of x as it relates to some other variable (e.g. y) in a mathematical function. If we have some variable such as x that is known to change value relative to some other variable (e.g. time, space, y), it is nice to be able to express that change using precise mathematical symbols, and this is what the "delta" symbol does for us.

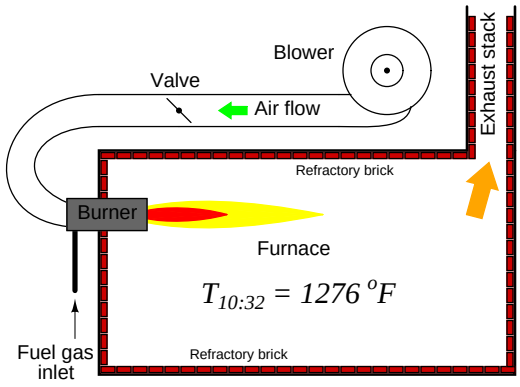
[1] In mathematics, the term *rigor* refers to a meticulous attention to detail and insistence that each and every step within a chain of mathematical reasoning be thoroughly justified by deductive logic, not intuition or analogy.

[2] The book's subtitle happens to be, *Being a very-simplest introduction to those beautiful methods of reckoning which are generally called by the terrifying names of the differential calculus and the integral calculus.* Not only did Thompson recognize the anti-pragmatic tone with which calculus is too often taught, but he also infused no small amount of humor in his work.

For example, if the temperature of a furnace (T) increases over time, we might wish to describe that change in temperature as ΔT:



Temperature of furnace at 9:45 AM = 1255 °F



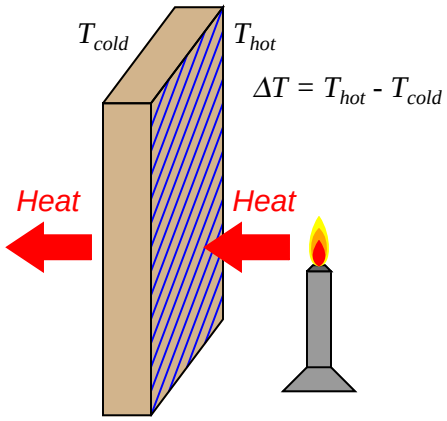
Temperature of furnace at 10:32 AM = 1276 °F

$$\Delta T = T_{10:32} - T_{9:45}$$

$$\Delta T = 1276\ ^{o}F - 1255\ ^{o}F = 21\ ^{o}F$$

The value of ΔT is nothing more than the difference (subtraction) between the recent temperature and the older temperature. A rising temperature over time thus yields a positive value for ΔT, while a falling temperature over time yields a negative value for ΔT.

We could also describe differences between the temperature of two *locations* (rather than a difference of temperature between two *times*) by the notation ΔT, such as this example of heat transfer through a heat-conducting wall where one side of the wall is hotter than the other:



Once again, ΔT is calculated by subtracting one temperature from another. Here, the sign (positive or negative) of ΔT denotes the *direction* of heat flow through the thickness of the wall.

One of the major concerns of calculus is changes or differences between variable values lying *very close to each other along a continuum of values*. In the context of a heating furnace, this could mean increases in temperature over miniscule time periods. In the context of heat flowing through a wall, this could mean differences in temperature sampled between points within the wall immediately adjacent each other. If our desire is to express the change in a variable between neighboring points along a continuum rather than over some discrete period, we may use a different notation than the capital Greek letter delta (Δ); instead, we use a lower-case Roman letter d (or in some cases, the lower-case Greek letter delta: δ).

Thus, a change in furnace temperature from one instant in time to the next could be expressed as dT (or δT), and likewise a difference in temperature between two adjacent positions within the heat-conducting wall could also be expressed as dT (or δT). Just as with the "delta" (Δ) symbol, the changes denoted by either d or δ may apply to space or time.

We even have a unique name for this concept of extremely small differences: while ΔT is called a *difference* in temperature, dT is called a *differential* of temperature. This distinction may not seem useful right now, but it is meaningful when describing *continuous changes* versus changes *sampled at discrete points*.

Another major concern in calculus is how quantities accumulate, especially how differential quantities add up to form a larger whole. A furnace's temperature rise since start-up (ΔT_{total}), for example, could be expressed as the accumulation (sum) of many temperature differences (ΔT) measured periodically. The total furnace temperature rise calculated from a sampling of temperature once every minute from 9:45 to 10:32 AM could be written as:

$$\Delta T_{total} = \Delta T_{9:45} + \Delta T_{9:46} + \cdots \Delta T_{10:32} = \text{Total temperature rise over time, from 9:45 to 10:32}$$

A more sophisticated expression of this series uses the capital Greek letter sigma (meaning "sum of" in mathematics) with notations specifying which temperature differences to sum:

$$\Delta T_{total} = \sum_{n=9:45}^{10:32} \Delta T_n = \text{Total temperature rise over time, from 9:45 to 10:32}$$

However, if our furnace temperature monitor scans at an infinite pace, measuring temperature *differentials* (dT) and summing them in rapid succession, we may express the same accumulated temperature rise as an *infinite* sum of *infinitesimal* (infinitely small) changes, rather than as a finite sum of temperature changes measured once every minute. Just as we introduced a unique mathematical symbol to represent differentials (d) over a continuum instead of differences (Δ) over discrete periods, we will introduce a unique mathematical symbol to represent the summation of differentials ($\int$) instead of the summation of differences ($\sum$):

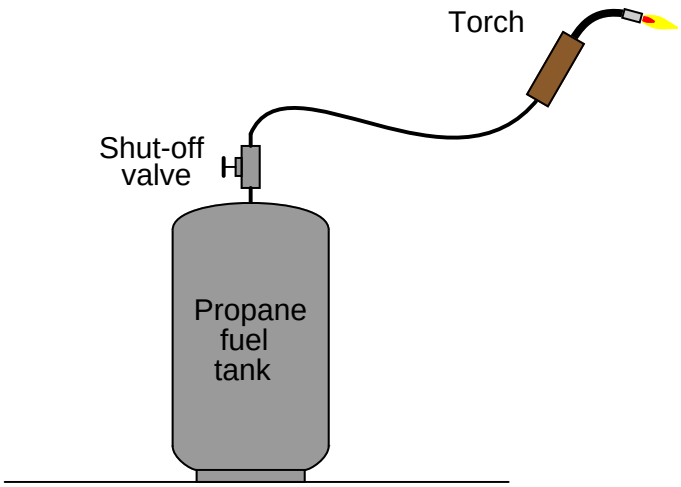
$$\Delta T_{total} = \int_{9:45}^{10:32} dT = \text{Total temperature rise over time, from 9:45 to 10:32}$$

This summation of infinitesimal quantities is called *integration*, and the elongated "S" symbol ($\int$) is the *integral* symbol.

These are the two major ideas and notations of calculus: *differentials* (tiny changes represented by d or δ) and *integrals* (accumulations represented by $\int$). Now that wasn't so frightening, was it?

2.2 The concept of differentiation

Suppose we wished to measure the rate of propane fuel gas flow through a hose to a torch:



Flowmeters appropriate for measuring low flow rates of any gas are typically very expensive, making it impractical to directly measure the flow rate of propane fuel gas consumed by this torch at any given moment. We could, however, *indirectly* measure the flow rate of propane gas by placing the tank on a scale where its mass (m) could be monitored over time. By taking measurements of mass over short time periods (Δt), we could calculate the corresponding differences in mass (Δm), then calculate the ratio of mass lost over time to calculate average mass flow rate ($\overline{W}$):

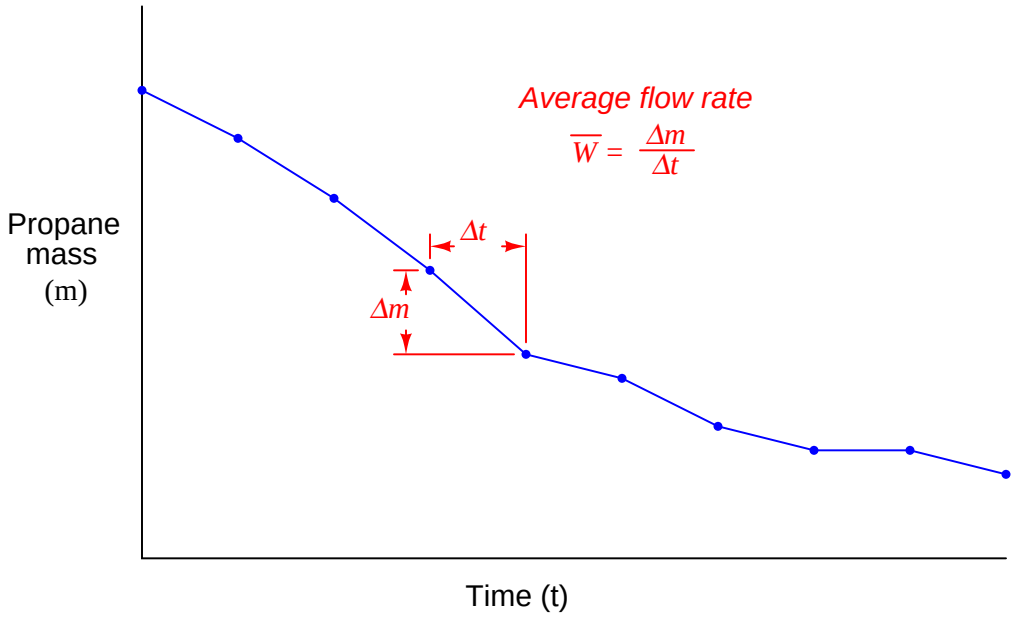
$$\overline{W} = \frac{\Delta m}{\Delta t} = \text{Average mass flow rate}$$

Where,
$\overline{W}$ = Average mass flow rate within each time period (kilograms per minute)
Δm = Measured mass difference over time period (kilograms)
Δt = Time period of mass measurement sample (minutes)

Note that flow rate is a ratio (quotient) of mass change over time change. The units used to express flow even reflect this process of division: kilograms *per* minute.

$$\overline{W} = \frac{[\text{kg}]}{[\text{min}]} = \text{Average mass flow rate} = \left[\frac{\text{kg}}{\text{min}}\right]$$

Graphed as a function over time, the tank's mass will be seen to decrease as time elapses. Each dot represents a mass and time measurement coordinate pair (e.g. 20 kilograms at 7:38, 18.6 kilograms at 7:51, etc.):

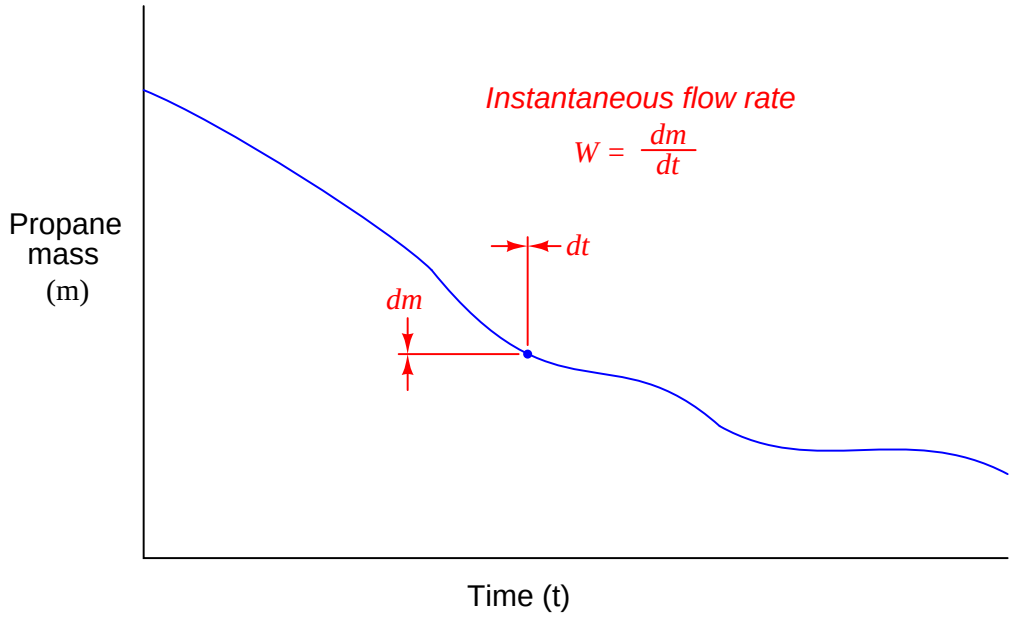


We should recall from basic geometry that the slope of a line or line segment is defined as its *rise* (vertical height) divided by its *run* (horizontal width). Thus, the average mass flow rate calculated within each time period may be represented as the pitch (slope) of the line segments connecting dots, since mass flow rate is defined as a change in mass per (divided by) change in time.

Periods of high propane flow (large flame from the torch) appear as steeply-pitched line segments. Periods of no propane flow reveal themselves as flat portions on the graph (no rise or fall over time).

If the determination of average flow rates between significant gaps in time is good enough for our application, we need not do anything more. However, if we wish to detect mass flow rate at any particular *instant* in time, we need to perform the same measurements of mass loss, time elapse, and division of the two at an infinitely fast rate.

Supposing such a thing were possible, what we would end up with is a smooth graph showing mass consumed over time. Instead of a few line segments roughly approximating a curve, we would have an *infinite* number of infinitely short line segments forming a seamless curve. The flow rate at any particular point in time would be the ratio of the mass and time differentials (the slope of the infinitesimal line segment) at that point:



$$W = \frac{dm}{dt} = \text{Instantaneous mass flow rate}$$

Where,

W = Instantaneous mass flow rate at a given time (kilograms per minute)
dm = Mass differential at a single point in time (kilograms)
dt = Time differential at a single point in time (minutes)

Flow is calculated just the same as before: a quotient of mass and time differences, except here the differences are infinitesimal in magnitude. The unit of flow measurement reflects this process of division, just as before, with mass flow rate expressed in units of kilograms *per* minute. Also, just as before, the rate of flow is graphically represented by the *slope* of the graph: steeply-sloped points on the graph represent moments of high flow rate, while shallow-sloped points on the graph represent moments of low flow rate.

Such a ratio of differential quantities is called a *derivative* in calculus[3]. Derivatives – especially time-based derivatives such as flow rate – find many applications in the physical world. Some of the most common time-based derivative functions include the relationships between *position* (x), *velocity* (v), and *acceleration* (a).

Velocity (v) is the rate at which an object changes position over time. Since position is typically denoted by the variable x and time by the variable t, the derivative of position with respect to time may be written as such:

$$v = \frac{dx}{dt} \qquad [\text{meters/second}] = \frac{[\text{meters}]}{[\text{seconds}]}$$

The metric units of measurement[4] for velocity (meters per second, miles per hour, etc.) betray this process of division: a differential of position (meters) divided by a differential of time (second).

Acceleration (a) is the rate at which an object changes velocity over time. Thus, we may express acceleration as the time-derivative of velocity, just as velocity was expressed as the time-derivative of position:

$$a = \frac{dv}{dt} \qquad [\text{meters/second}^2] = \frac{[\text{meters/second}]}{[\text{seconds}]}$$

We may even express acceleration as a function of position (x), since it is the rate of change of the rate of change in position over time. This is known as a *second derivative*, since it is applying the process of "differentiation" twice:

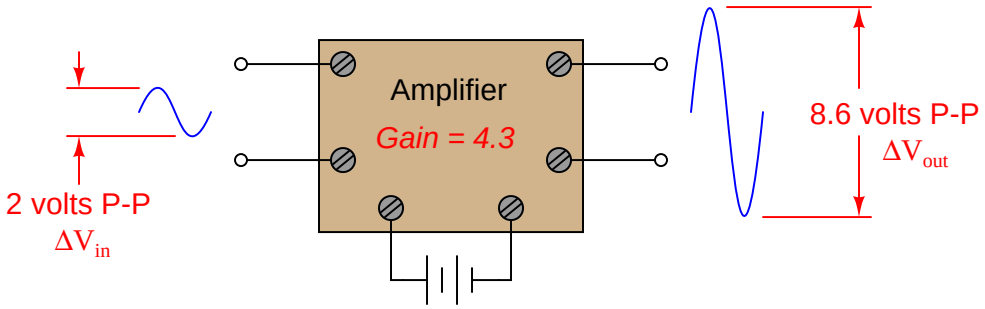
$$a = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \qquad [\text{meters/second}^2] = \frac{[\text{meters}]}{[\text{seconds}^2]}$$

As with velocity, the units of measurement for acceleration (meters per second squared, or alternatively meters per second per second) suggest a compounded quotient.

[3] Isaac Newton referred to derivatives as *fluxions*, and in Silvanus Thompson's day they were known as *differential coefficients*.

[4] British units of measurement for velocity indicate this same process of division: the number of feet traveled in a time period of seconds yields a velocity in feet per second. There is nothing unique about metric units in this regard.

It is also possible to express rates of change between different variables not involving time. A common example in electronics is the concept of *gain*, generally defined as the ratio of output change to input change. An electronic amplifier, for example, with an input signal of 2 volts (peak-to-peak) and an output signal of 8.6 volts (peak-to-peak), would be said to have a gain of 4.3, since the change in output measured in peak-to-peak volts is 4.3 times larger than the corresponding change in input voltage:



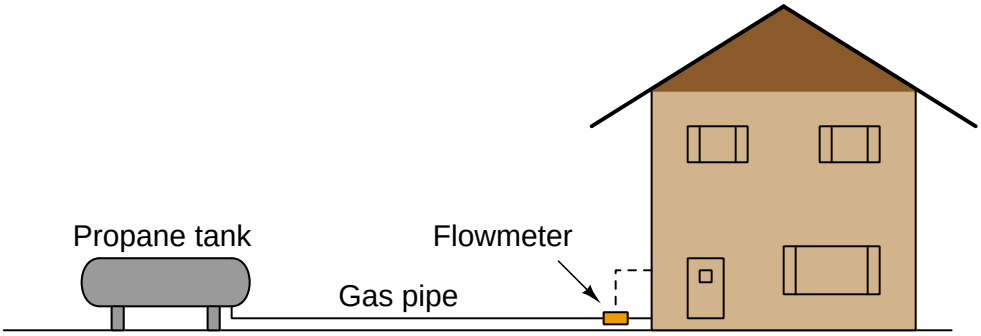
This gain may be expressed as a quotient of differences ($\frac{\Delta V_{out}}{\Delta V_{in}}$), or it may be expressed as a derivative instead:

$$\text{Gain} = \frac{dV_{out}}{dV_{in}}$$

If the amplifier's behavior is perfectly linear, there will be no difference between gain calculated using differences and gain calculated using differentials (the derivative), since the average slope of a straight line is the same as the instantaneous slope at any point along that line. If, however, the amplifier does not behave in a perfectly linear fashion, gain calculated from voltage values sampled over discrete intervals ($\frac{\Delta V_{out}}{\Delta V_{in}}$) will not be the same as gain calculated from infinitesimal changes at different points along the amplifier's operating voltage range.

2.3 The concept of integration

Suppose we wished to measure the consumption of propane fuel gas over time for a large propane storage tank supplying a building with heating fuel, because the tank lacked a level indicator to show how much fuel was left at any given time. The flow rate is sufficiently large, and the task sufficiently important, to justify the installation of a mass flowmeter[5], which registers flow rate at an indicator inside the building:



By measuring true mass flow rate, it should be possible to indirectly measure how much propane gas has been consumed at any time following the most recent filling of the tank. For example, if the mass flow rate of propane out of the tank happened to be a constant 5 kilograms per hour for 30 hours straight, it would be a simple matter of multiplication to calculate the consumed mass:

$$\left(\frac{5 \text{ kg}}{\text{hr}}\right)\left(\frac{30 \text{ hrs}}{1}\right) = 150 \text{ kg of propane consumed}$$

Expressing this mathematically as a function of differences in mass and differences in time, we may write the following equation:

$$\Delta m = \overline{W}\,\Delta t$$

Where,
$\overline{W}$ = Average mass flow rate within the time period (kilograms per hour)
Δm = Mass difference over time period (kilograms)
Δt = Time period of flow measurement sample (hours)

It is easy to see how this is just a variation of the quotient-of-differences equation used previously in this Tutorial to define mass flow rate:

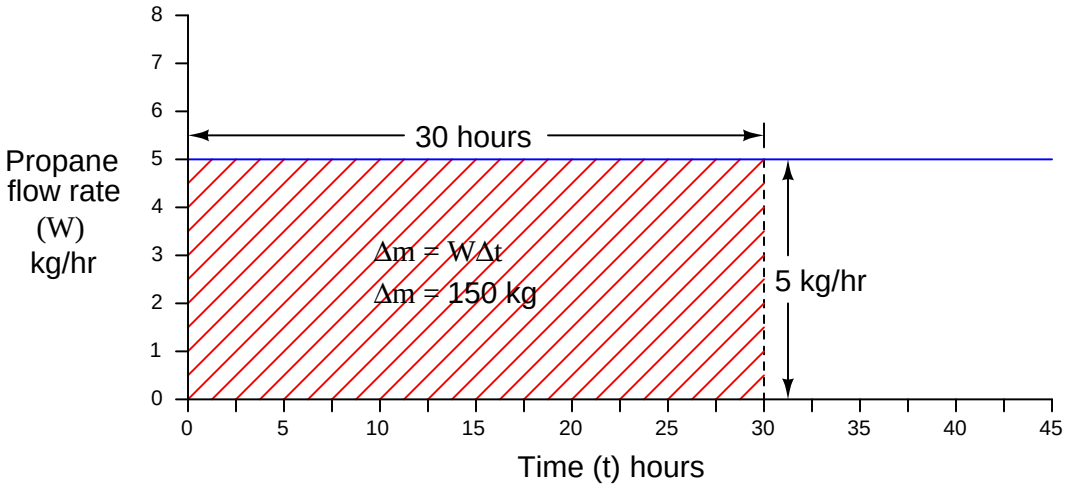
$$\overline{W} = \frac{\Delta m}{\Delta t} = \text{Average mass flow rate}$$

Inferring mass flow rate from changes in mass over time periods is a process of *division*. Inferring changes in mass from flow rate over time periods is a process of *multiplication*. The units of measurement used to express each of the variables makes this quite clear.

[5] Most likely a thermal mass flowmeter or a Coriolis-effect flowmeter.

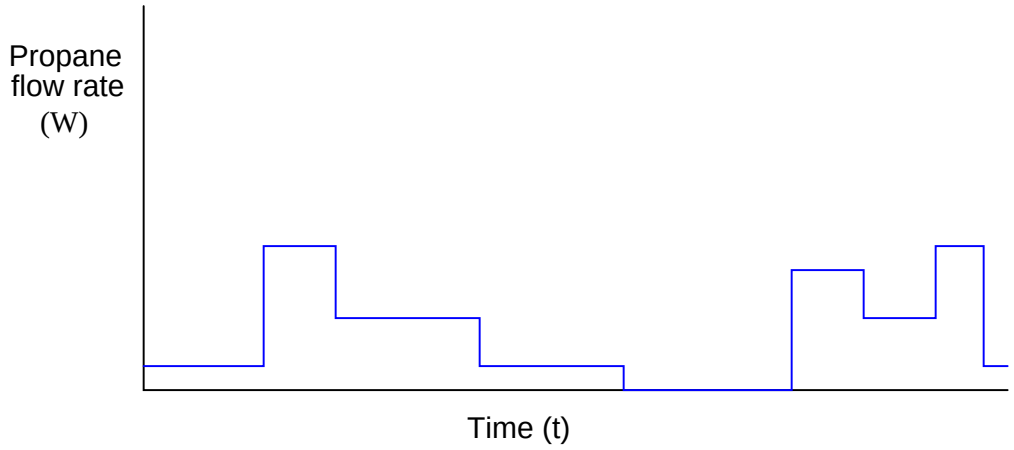
As we learned previously, the process of differentiation is really just a matter of determining the *slope* of a graph. A graph of propane fuel mass (m) plotted over discrete points in time (t) has a slope corresponding to mass flow rate ($W = \frac{\Delta m}{\Delta t}$). Here, we are attempting to do the opposite: the data reported by the sensing instrument is propane mass flow rate (W), and our goal is to determine total mass lost (Δm) as the propane is consumed from the storage tank over a period of time (Δt). This operation is fundamentally distinct from differentiation, which means its graphical interpretation will not be the same. Instead of calculating the slope of the graph, we will have to do something else.

Using the previous example of the propane flowmeter sensing a constant mass flow rate (W) of 5 kilograms of propane per hour for 30 hours (for a total consumption of 150 kilograms), we may plot a trend graph showing flow rate (vertical) as a function of time (horizontal). We know the consumed propane quantity is the simple product (multiplication) of constant flow rate and time, which relates to the geometric *area* enclosed by the graph, since the area of any rectangle is height times width:



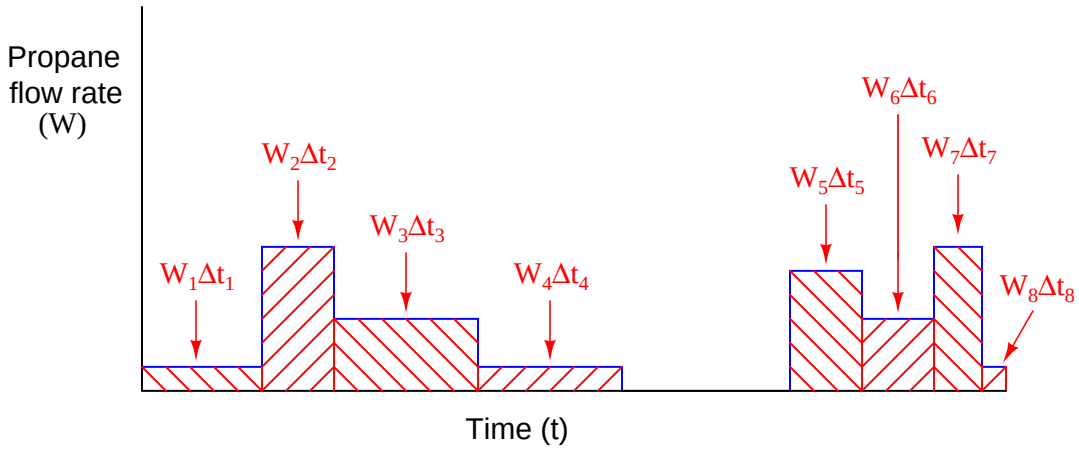
To summarize: the height of this graph represents the rate at which propane exits the storage tank, the width of the graph represents the length of time propane has been consumed from the storage tank, and the geometric area enclosed by these two boundaries represents the total mass of propane consumed during that time. Height in *kilograms per hour* multiplied by width in *hours* equals *kilograms*.

Inferring consumed propane fuel mass over time is easy when the propane flow rate is constant, because that produces a simple rectangular enclosed area on the graph, the area of which is trivial to compute. However, real-world applications are rarely this easy. Consider the following graph, showing periods of increased and decreased flow rate due to different gas-fired appliances turning on and off inside the building:



Here, the propane gas flow rate does not stay constant throughout the entire time interval covered by the graph. This graph is obviously more challenging to analyze than the previous example where the propane flow rate was constant. From that previous example, though, we have learned that the geometric area enclosed by the boundaries of the graph's height (flow rate) and width (time duration) has physical meaning, representing the total quantity of propane passed through the flowmeter. Despite the fact that the graph's enclosed area is more challenging to calculate, the basic principle remains the same as before: *the enclosed area – regardless of how we must compute that area – still represents the amount of propane consumed over the represented time interval.*

In order to accurately calculate the amount of propane mass consumed over time, we must treat each period of constant flow as its own propane quantity, calculating the mass lost during each period, then summing those mass differences to arrive at a total mass loss for the entire time interval covered by the graph. Since we know the difference (loss) in mass over a time period is equal to the average flow rate for that period multiplied by the period's duration ($\Delta m = W\ \Delta t$), we may calculate each period's mass as an *area* underneath the graph line, each rectangular area being equal to height (W) times width (Δt):



Each rectangular area underneath the flow line on the graph ($W_n \Delta t_n$) represents a quantity of propane gas consumed during that time period. To find the total amount of propane consumed in the time represented by the entire graph, we simply sum these mass quantities together:

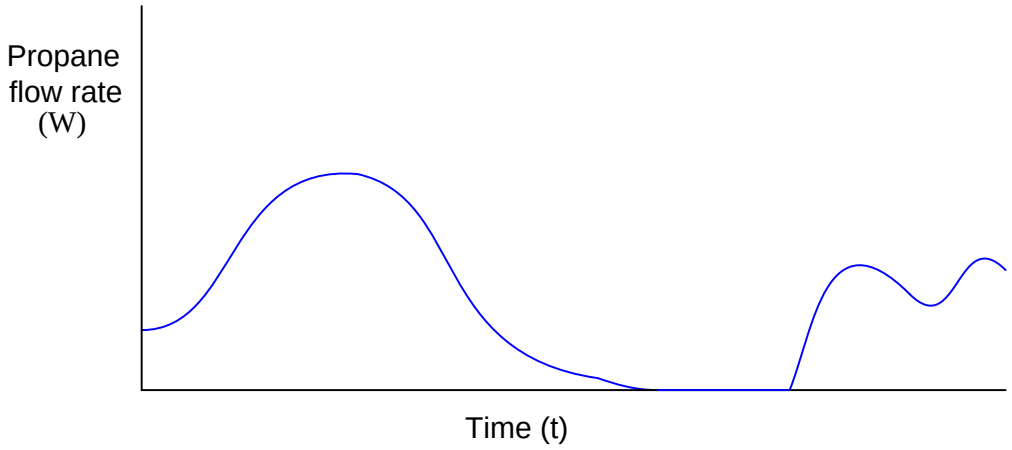
$$\Delta m = (W_1 \Delta t_1) + (W_2 \Delta t_2) + (W_3 \Delta t_3) + (W_4 \Delta t_4) + (W_5 \Delta t_5) + (W_6 \Delta t_6) + (W_7 \Delta t_7) + (W_8 \Delta t_8)$$

A "shorthand" notation for this sum uses the capital Greek letter sigma to represent a series of repeated products (multiplication) of mass flow and time periods for the eight rectangular areas enclosed by the graph:

$$\Delta m = \sum_{n=1}^{8} W_n\ \Delta t_n$$

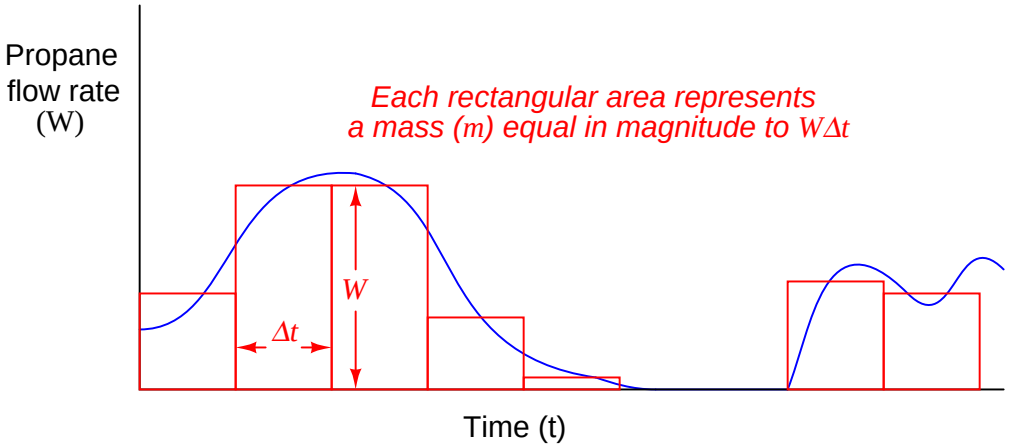
While $W_n \Delta t_n$ represents the area of just one of the rectangular periods, $\sum_{n=1}^{8} W_n \Delta t_n$ represents the total combined areas, which in this example represents the total mass of propane consumed over the eight time periods shown on the graph.

The task of inferring total propane mass consumed over time becomes even more complicated if the flow does not vary in stair-step fashion as it did in the previous example. Suppose the building were equipped with *throttling* gas appliances instead of on/off gas appliances, thus creating a continuously variable flow rate demand over time. A typical flow rate graph might look something like this:

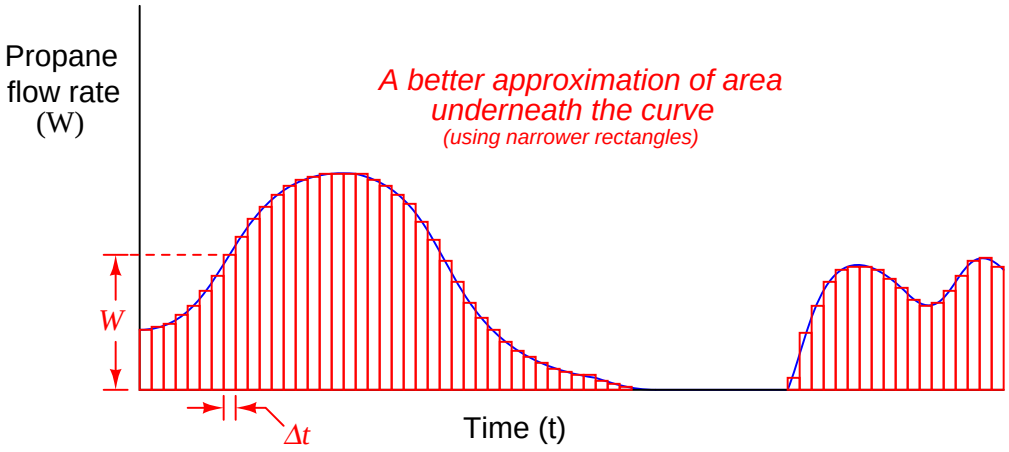


The physics of gas flow and gas mass over time has not changed: *total propane mass consumed over time must still be the area enclosed beneath the flow curve.* The only difference between this example and the two previous examples is the complexity of actually calculating that enclosed area.

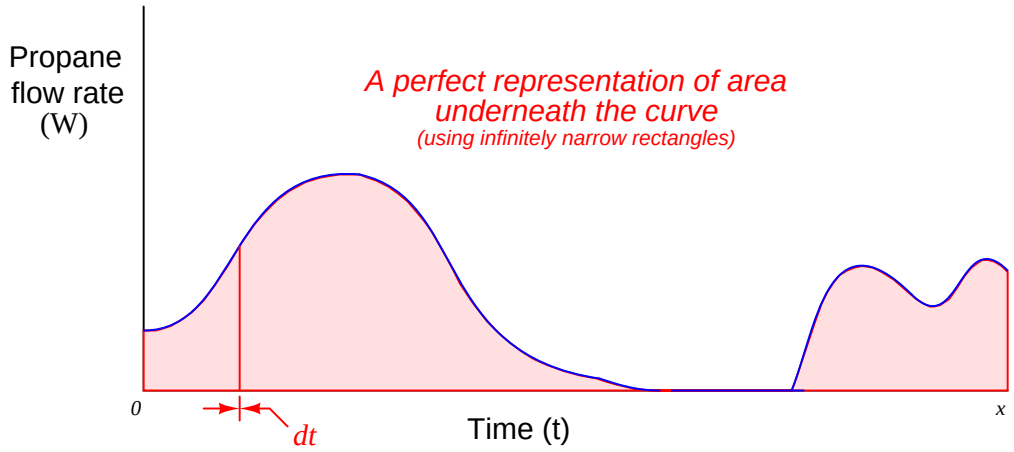
We can, however, *approximate* the area underneath this curve by overlaying a series of rectangles, the area of each rectangle being height (W) times width (Δt):



This strategy of approximating the area underneath a curve improves with the number of rectangles used: each rectangle still has an area $W\Delta t$, but using more rectangles makes each Δt period shorter, which in turn makes it easier to fit the rectangles to the exact curve of the graph. The summation of a series of rectangular areas intended to approximate the area enclosed by a graphed function is commonly referred to as a *Riemann Sum* in honor of the mathematician Bernhard Riemann:



Taking this idea to its ultimate realization, we could imagine a super-computer sampling mass flow rates at an infinite speed, then calculating the rectangular area covered by each flow rate (W) times each infinitesimal increment of time (dt). With time increments of negligible width, the "approximation" of area underneath the graph found by the sum of all these rectangles would be perfect – indeed, it would not be an approximation at all, but rather an exact match:



If we represent infinitesimal time increments using the notation "dt" as opposed to the notation "Δt" used to represent discrete time periods, we must also use different notation to represent the mathematical sum of those quantities. Thus, we will replace the "sigma" symbol ($\sum$) used for summation and replace it with the integral symbol ($\int$), which means a *continuous* summation of infinitesimal quantities:

$$\Delta m = \sum_{n=0}^{x} W \, \Delta t_n \qquad \text{Summing discrete quantities of } W \Delta t$$

$$\Delta m = \int_{0}^{x} W \, dt \qquad \text{Summing continuous quantities of } W \, dt$$

This last equation tells us the total change in mass (Δm) from time 0 to time x is equal to the continuous sum of mass quantities found by multiplying mass flow rate measurements (W) over corresponding increments of time (dt). We refer to this summation of infinitesimal quantities as *integration* in calculus. Graphically, the *integral* of a function is the geometric area enclosed by the function over a specified interval.

An important detail to note is that this process of integration (multiplying flow rates by infinitesimal time increments, then summing those products) only tells us how much propane mass was consumed – it does *not* tell us how much propane remains in the tank, which was the purpose of installing the mass flowmeter and performing all this math! The integral of mass flow and time ($\int W\ dt$) will always be a negative[6] quantity in this example, because a flow of propane gas out of the tank represents a *loss* of propane mass within the tank. In order to calculate the amount of propane mass left in the tank, we would need to know the initial value of propane in the tank before any of it flowed to the building, then we would add this initial mass quantity (m_0) to the negative mass loss calculated by integration.

Thus, we would mathematically express the propane mass inside the tank at time x as such[7]:

$$m_x = \int_0^x W\ dt + m_0$$

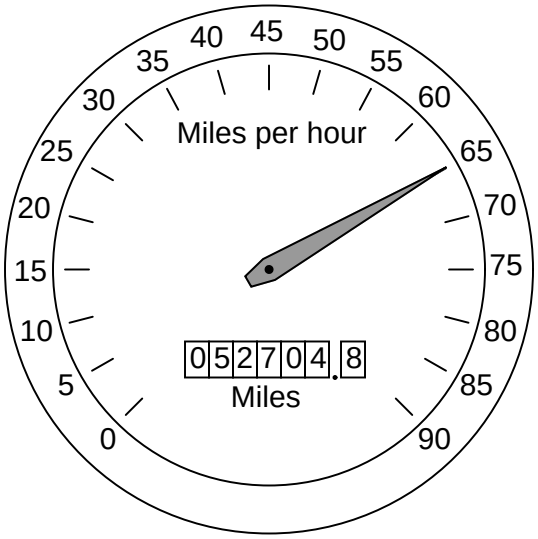
This initial value must always be considered in problems of integration if we attempt to absolutely define some integral quantity. Otherwise, all the integral will yield is a relative quantity (how much something has *changed* over an interval).

[6] Although we will measure time, and differentials of time, as positive quantities, the mass flowmeter should be configured to show a negative flow rate (W) when propane flows from the tank to the building. This way, the *integrand* (the product "inside" the integration symbol; $W\ dt$) will be a negative quantity, and thus the integral over a positive time interval (from 0 to x) will likewise be a negative quantity.

[7] According to calculus convention, the differential dt represents the end of the integrand. It is safe to regard the long "S" symbol and the differential (dx, dt, etc.) as complementary *grouping symbols* declaring the beginning and end of the integrand. This tells us m_0 is not part of the integrand, but rather comes after it. Using parentheses to explicitly declare the boundaries of the integrand, we may re-write the expression as $m_x = (\int_0^x W\ dt) + m_0$

The problem of initial values is very easy to relate to common experience. Consider the *odometer* indication in an automobile. This is an example of an integral function, the distance traveled (x) being the time-integral[8] of speed (or velocity, v):

$$\Delta x = \int v\, dt$$



Although the odometer does accumulate to larger and larger values as you drive the automobile, its indication does not necessarily tell me how many miles *you* have driven it. If, for example, you purchased the automobile with 32411.6 miles on the odometer, its current indication of 52704.8 miles means you have driven it 20293.2 miles. The automobile's *total* distance traveled since manufacture is equal to the distance you have accumulated while driving it ($\int v\, dt$) *plus* the initial mileage accumulated at the time you took ownership of it (x_0):

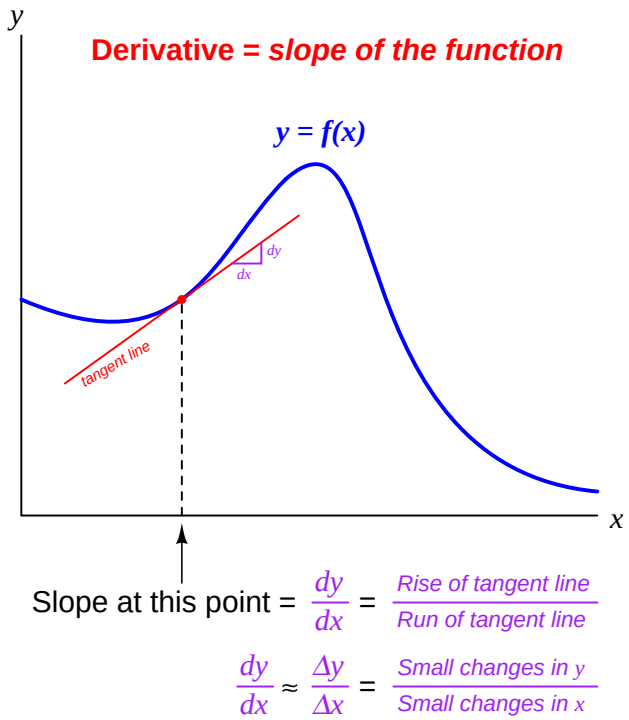
$$x_{total} = \int v\, dt + x_0$$

[8] Recall from the previous section ("The Concept of Differentiation") that velocity could be defined as the time-derivative of position: $v = \frac{dx}{dt}$ All we have done here is algebraically solved for changes in x by first multiplying both sides of the equation by dt to arrive at $dx = v\, dt$. Next, we integrate both sides of the equation in order to "un-do" the differential (d) applied to x: $\int dx = \int v\, dt$. Since accumulations ($\int$) of any differential (dx) yields a discrete change for that variable, we may substitute Δx for $\int dx$ and get our final answer of $\Delta x = \int v\, dt$.

2.4 How derivatives and integrals relate to one another

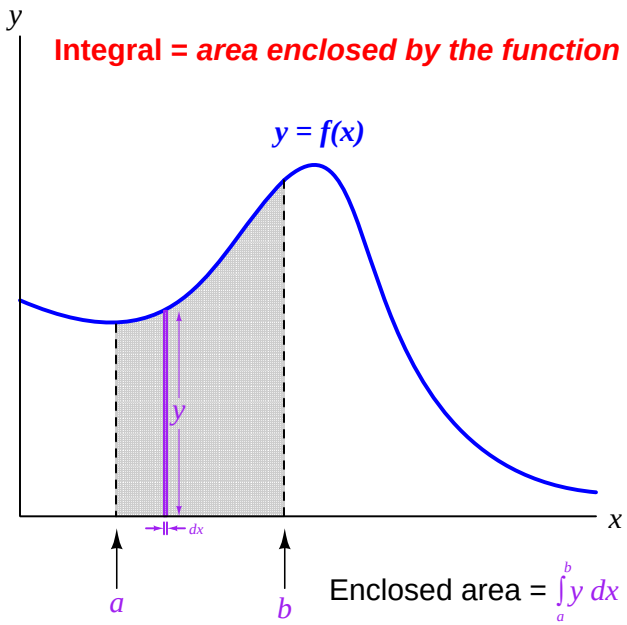
First, let us review some of the properties of *differentials* and *derivatives*, referencing the expression and graph shown below:

- A *differential* is an infinitesimal increment of change (difference) in some continuously-changing variable, represented either by a lower-case Roman letter d or a lower-case Greek letter "delta" (δ). Such a change in time would be represented as dt; a similar change in temperature as dT; a similar change in the variable x as dx.
- A derivative is always a *quotient of differences*: a process of subtraction (to calculate the amount each variable changed) followed by division (to calculate the *rate* of one change to another change).
- The units of measurement for a derivative reflect this final process of division: one unit divided by some other unit (e.g. gallons *per* minute, feet *per* second).
- Geometrically, the derivative of a function is its graphical *slope* (its "rise over run").
- When computing the value of a derivative, we must specify a single point along the function where the slope is to be calculated.
- The *tangent line* matching the slope at that point has a "rise over run" value equal to the derivative of the function at that point.



Next, let us review some of the properties of *integrals*, referencing the expression and graph shown below:

- An integral is always a *sum of products*: a process of multiplication (to calculate the product of two variables) followed by addition (to sum those quantities into a whole).
- The units of measurement for an integral reflect this initial process of multiplication: one unit times some other unit (e.g. kilowatt-hours, foot-pounds, volt-seconds).
- When computing the value of an integral, we must specify both the starting and ending points along the function defining the interval of integration (a and b).
- Geometrically, the integral of a function is the graphical *area* enclosed by the function and the interval boundaries.
- The area enclosed by the function may be thought of as an infinite sum of extremely narrow rectangles, each rectangle having a height equal to one variable (y) and a width equal to the differential of another variable (dx).



Upper boundary of area = y
Lower boundary of area = 0
Starting boundary of area = a
Ending boundary of area = b

Just as division and multiplication are *inverse* mathematical functions (i.e. one "un-does" the other), differentiation and integration are also inverse mathematical functions. The two examples of propane gas flow and mass measurement highlighted previously illustrates this complementary relationship. We may use differentiation with respect to time to convert a mass measurement (m) into a mass flow measurement (W, or $\frac{dm}{dt}$). Conversely, we may use integration with respect to time to convert a mass flow measurement (W, or $\frac{dm}{dt}$) into a measurement of mass gained or lost (Δm).

Likewise, the common examples of position (x), velocity (v), and acceleration (a) used to illustrate the principle of differentiation are also related to one another by the process of integration. Reviewing the derivative relationships:

$$v = \frac{dx}{dt} \qquad \text{Velocity is the derivative of position with respect to time}$$

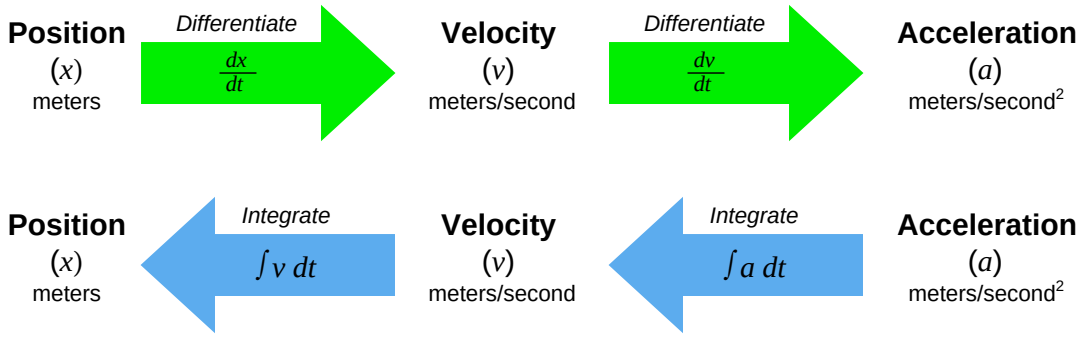
$$a = \frac{dv}{dt} \qquad \text{Acceleration is the derivative of velocity with respect to time}$$

Now, expressing position and velocity as *integrals* of velocity and acceleration, respectively[9]:

$$x = \int v\,dt \qquad \text{Position is the integral of velocity with respect to time}$$

$$v = \int a\,dt \qquad \text{Velocity is the integral of acceleration with respect to time}$$

Differentiation and integration may be thought of as processes transforming these quantities into one another. Note the transformation of units with each operation – differentiation always divides while integration always multiplies:



[9] To be perfectly accurate, we must also include initial values for position and velocity. In other words, $x = \int v dt + x_0$ and $v = \int a\,dt + v_0$

The inverse nature of these two calculus operations is codified in mathematics as the *Fundamental Theorem of Calculus*, shown here:

$$\frac{d}{dx}\left[\int_a^b f(x)\,dx\right] = f(x)$$

What this equation tells us is that the derivative of the integral of any continuous function is that original function. In other words, we can take any mathematical function of a variable that we know to be continuous over a certain range – that function represented by $f(x)$, with the range of integration starting at a and ending at b – and integrate that function over that range, then take the derivative of that result and end up with the original function. By analogy, we can take the *square-root* of any quantity, then *square* the result and end up with the original quantity, because these are inverse functions as well.

A feature of this module which may be helpful to your understanding of derivatives, integrals, and their relation to each other is found in the Animations chapter (animation 4.1 beginning on page 68). This animation takes the form of illustrations arranged in sequence you may "flip" through to view the filling and emptying of a water storage tank, with graphs showing stored volume (V) and volumetric flow rate (Q). Since flow rate is the time-derivative of volume ($Q = \frac{dV}{dt}$) and volume change is the time-integral of volumetric flow rate ($\Delta V = \int Q\,dt$), the animation demonstrates both concepts in action.

2.5 Symbolic versus numerical calculus

Calculus has a reputation for being difficult to learn, and with good reason. The traditional approach begins by manipulating symbols (variables) in equations, learning how different types of mathematical functions become transformed by the calculus operations of differentiation and integration. For example, suppose a first-semester calculus student were given the following function to differentiate. The function is expressed as y in terms of x:

$$y = \frac{3x^2 - 2x + 5}{x^2 - 8}$$

That calculus student would first apply two basic rules of symbolic differentiation (namely, the *Power Rule* and the *Quotient Rule*) followed by algebraic distribution and combination of like terms to arrive at the derivative of y with respect to x (written as $\frac{dy}{dx}$) in terms of x:

$$\frac{dy}{dx} = \frac{(x^2 - 8)(6x - 2) - (3x^2 - 2x + 5)(2x)}{(x^2 - 8)^2}$$

$$\frac{dy}{dx} = \frac{6x^3 - 2x^2 - 48x + 16 - (6x^3 - 4x^2 + 10x)}{x^4 - 16x^2 + 64}$$

$$\frac{dy}{dx} = \frac{2x^2 - 58x + 16}{x^4 - 16x^2 + 64}$$

The resulting derivative expresses the rate-of-change of y with respect to x of the original function for any value of x. In other words, anyone can now plug any arbitrary value of x they wish into the derivative equation, and the result ($\frac{dy}{dx}$) will tell them *how steep the slope is* of the original function at that same x value[10].

Rules such as the Power Rule and even the Quotient Rule are not difficult to memorize, but they are far from intuitive. Although it is possible to formally prove each one of them from more fundamental principles of algebra, doing so is tedious, and so most students simply resign themselves to memorizing all the calculus rules of differentiation and integration. There are many such rules to memorize in symbolic calculus.

Symbolic integration is even more difficult to learn than symbolic differentiation. Most calculus textbooks reserve pages at the very end listing the general rules of differentiation and integration. Whereas a table of derivatives might occupy a single page in a calculus text, tables of integrals may fill five or more pages!

The next logical topic in the sequence of a calculus curriculum is *differential equations*. A "differential equation" is a function relating some variable to one or more of its own derivatives. To use the variables y and x, a differential equation would be one containing both y and at least one derivative of y ($\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, etc.). $\frac{dV}{dt} = -kV$ is an example of a simple differential equation. The various forms and solution techniques for different kinds of differential equations are numerous and complex.

[10] For instance, at $x = 1$, the original function tells us that y will be equal to $-\frac{6}{7}$. If we plug this same value of 1 into x of the derivative function, the result $\frac{dy}{dx} = -\frac{40}{49}$ tells us the original function $y = f(x)$ has a slope of $-\frac{40}{49}$ when $x = 1$.

It has been said that the laws of the universe are written in the language of calculus. This is immediately evident in the study of physics, but it is also true for chemistry, biology, astronomy, and other "hard sciences." Areas of applied science including engineering (chemical, electrical, mechanical, and civil) as well as economics, statistics, and genetics would be impoverished if not for the many practical applications of symbolic calculus. To be able to express a function of real-life quantities as a set of symbols, then apply the rules of calculus to those symbols to transform them into functions relating rates of change and accumulations of those real-life quantities, is an incredibly powerful tool.

Two significant problems exist with symbolic calculus, however. The first problem with symbolic calculus is its complexity, which acts as a barrier to many people trying to learn it. It is quite common for students to drop out of calculus or to change their major of study in college because they find the subject so confusing and/or frustrating. This is a shame, not only because those students end up missing out on the experience of being able to see the world around them in a new way, but also because mastery of calculus is an absolute requirement of entry into many professions. One cannot become a licensed engineer in the United States, for example, without passing a series of calculus courses in an accredited university and demonstrating mastery of those calculus concepts on a challenging exam.

The second significant problem with symbolic calculus is its limitation to a certain class of mathematical functions. In order to be able to symbolically differentiate a function (e.g. $y = f(x)$) to determine its derivative ($\frac{dy}{dx}$), we must first have a function written in mathematical symbols to differentiate. This rather obvious fact becomes a barrier when the data we have from a real-life application defies symbolic expression. It is trivial for a first-semester calculus student to determine the derivative of the function $V = 2t^2 - 4t + 9$, but what if V and t only exist as recorded values in a table, or as a graph appearing on the display of an oscilloscope? Without a mathematical formula showing V as a function of t, none of the rules learned in a calculus course for manipulating those symbols directly apply. The problem is even worse for differential equations, where a great many examples exist that have so far defied solution by the greatest mathematicians.

An alternative approach to calculus exists which is easily understood by anyone with the ability to perform basic arithmetic (addition, subtraction, multiplication, and division) and sketching (drawing lines and points on a graph). Numerical calculus uses simple *arithmetic* to approximate derivatives and integrals on real-world data. The results are not as precise as with symbolic calculus, but the technique works on *any* data as well as most mathematical functions written in symbolic form. Furthermore, the simplicity of these techniques opens a door to those people who might otherwise be scared away by the mathematical rigor of symbolic calculus. Any way we can find to showcase the beauty and practicality of calculus principles to more people is a good thing!

Suppose we needed to calculate the derivative of some real-world function, such as the volume of liquid contained in a storage vessel. The derivative of volume (V) with respect to time (t) is *volumetric flow rate* ($\frac{dV}{dt}$), thus the time-derivative of the vessel's volume function at any specified point in time will be the net flow rate into (or out of) that vessel at that point in time.

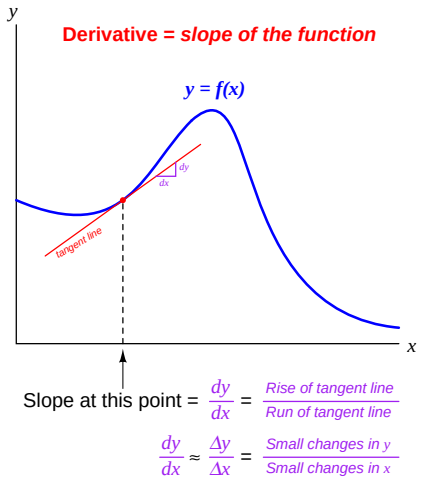
To numerically determine the derivative of volume from tabulated data, we could follow these steps:

- Choose two values of volume both near the point in time we're interesting in calculating flow rate.
- Subtract the two volume values: this will be ΔV.
- Subtract the two time values corresponding to those volume values: this will be Δt.
- Divide ΔV by Δt to approximate $\frac{dV}{dt}$ between those two points in time.

A slightly different approach to numerical differentiation follows these steps:

- Sketch a graph of the volume versus time data for this vessel (if not already provided for you).
- Locate the point in time on this graph you are interested in, and sketch a tangent line to that point (a straight line having the same slope as the graphed data at that point).
- Estimate the rise-over-run slope of this tangent line to approximate $\frac{dV}{dt}$ at this point.

An illustration is a helpful reminder of what differentiation means for any graphed function: the *slope* of that function at a specified point:



Suppose we needed to calculate the integral of some real-world function, such as the flow rate of liquid through a pipe. The integral of volumetric flow (Q) with respect to time (t) is *total volume* (V), thus the time-integral of the flow rate over any specified time interval will be the total volume of liquid that passed by over that time.

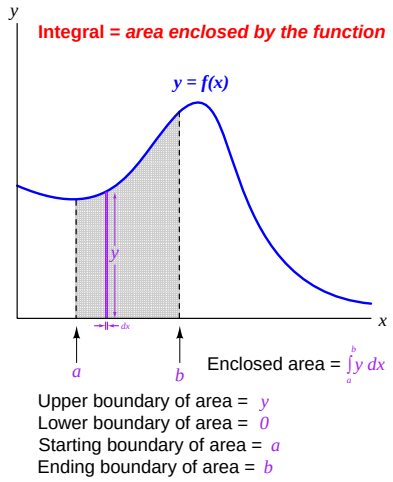
To numerically determine the integral of flow from tabulated data, we could follow these steps:

- Identify the time interval over which we intend to calculate volume, and the duration of each measured data point within that interval.
- Multiply each measured value of flow by the duration of that measurement (the interval between that measurement and the next one) to obtain a volume over each duration.
- Repeat the last step for each and every flow data point up to the end of the interval we're interested in.
- Add all these volume values together – the result will be the approximate liquid volume passed through the pipe over the specified time interval.

A slightly different approach to numerical integration follows these steps:

- Sketch a graph of the flow versus time data for this pipe (if not already provided for you).
- Mark the time interval over which we intend to calculate volume (two straight vertical lines on the graph).
- Use any geometrical means available to estimate the area bounded by the graph and the two vertical time markers – the result will be the approximate liquid volume passed through the pipe over the specified time interval.

An illustration is a helpful reminder of what integration means for any graphed function: the *area* enclosed by that function within a specified set of boundaries:



The next sections of this Tutorial delve into more specific details of numerical differentiation and integration, with realistic examples to illustrate.

2.6 Numerical differentiation

As we have seen, the concept of *differentiation* is finding the rate-of-change of one variable compared to another (related) variable. In this section, we will explore the practical application of this concept to real-world data, where actual numerical values of variables are used to calculate relative rates of change.

In industrial instrumentation, for example, we are often interested in knowing the rate of change of some process variable (pressure, level, temperature, flow, etc.) over time, and so we may use computers to calculate those rates of change, either after the fact (from recorded data) or in real time (as the data is being received by sensors and acquired by the computer). We may be similarly interested in calculating the rate at which one process variable changes with respect to another process variable, both of which measured and recorded as tables of data by instruments.

Numerical (data-based) differentiation is fundamentally a two-step arithmetic process. First, we must use *subtraction* to calculate the change in a variable between two different points. Actually, we perform this step twice to determine the change in *two variables* which we will later compare. Then, we must use *division* to calculate the ratio of the two variables' changes, one to the other (i.e. the "rise-over-run" steepness of the function's graph).

For example, let us consider the application of pressure measurement for a pipeline. One of the diagnostic indicators of a burst pipeline is that the measured pressure rapidly drops. It is not the existence of low pressure in and of itself that suggests a breach, but rather the *rate* at which the pressure falls that reveals a burst pipe. For this reason, pipeline control systems may be equipped with automatic shut-down systems triggered by rate-of-change pressure calculations.

The association of rapid pressure drop with pipeline ruptures is nothing new to pipeline operations. Here is an illustration taken from page 566 of volume 8 of *Cassier's Magazine* published in the year 1895, showing water pressure measurements taken by a paper strip chart recorder on a city water main line. The pressure drop created by a burst in that 36-inch pipe is clearly seen and commented on the recording:

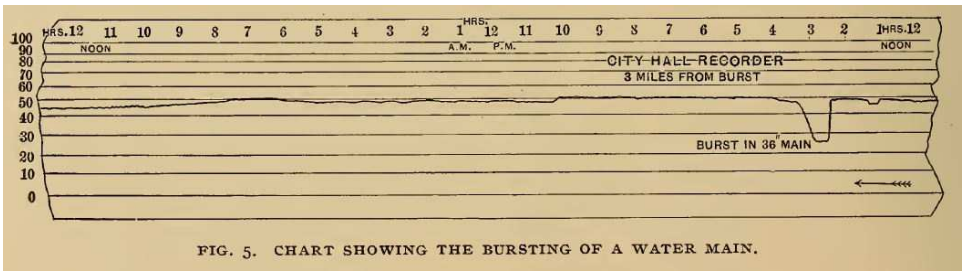
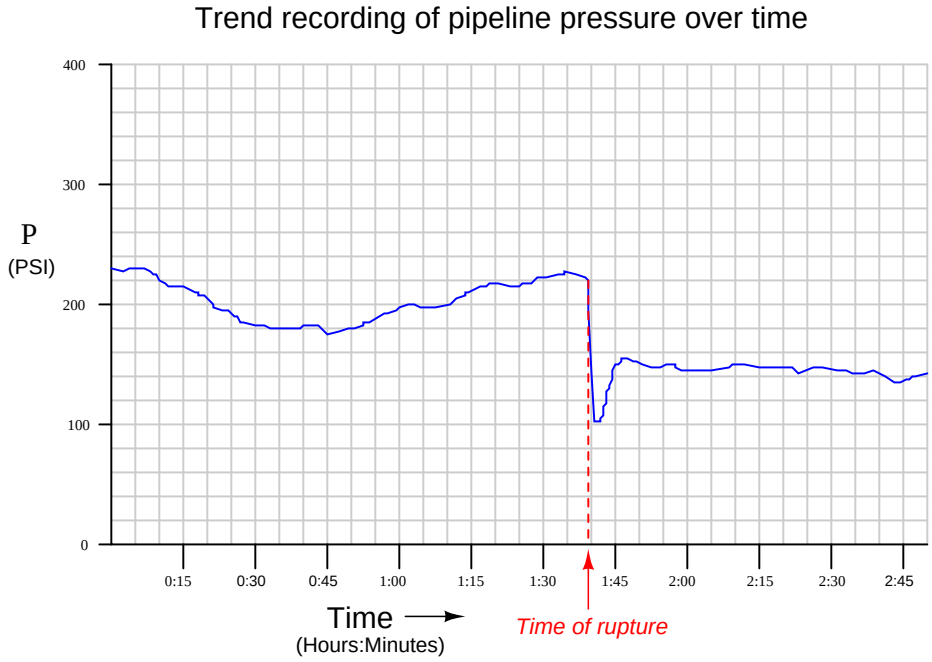


FIG. 5. CHART SHOWING THE BURSTING OF A WATER MAIN.

An example of a modern[11] pressure-trend recording during a pipeline rupture is shown here:

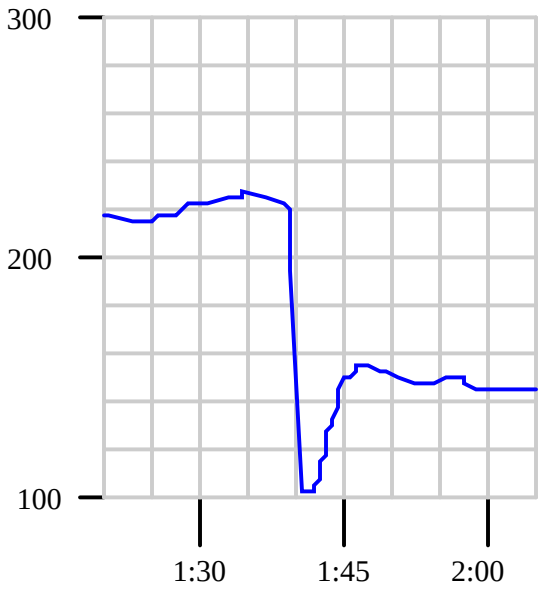


While it may seem surprising that pipeline pressure should recover after the low point immediately following the pipe's rupture, it is important to bear in mind that many pipelines are *pressure-controlled* processes. After a pipeline ruptures, the pumping equipment will attempt to compensate for the reduced pressure automatically, which is why the pressure jumps back up (although not to its previous level) after the initial drop.

This phenomenon helps explain why pressure rate-of-change is a more reliable diagnostic indicator of a ruptured pipe than pressure magnitude alone: any automatic rupture-detection scheme based on a simple comparison of pipeline pressure against a pre-set threshold may fail to reliably detect a rupture if the pressure-regulating equipment is able to quickly restore pipeline pressure following the rupture. A rate-of-change system, on the other hand, will still detect the rupture based on the sharp pressure decrease following the break, even if the pressure quickly recovers.

[11] Unlike the recording shown from *Cassier's Magazine*, which runs chronologically from right to left, modern chart recordings all run from left to right.

A computer tasked with calculating the pressure's rate of change over time ($\frac{dP}{dt}$) would have to continuously sample the pressure value over short time periods, then calculate the quotient of pressure changes over time changes. Given a sample rate of once every 5 minutes, we see how the computer would tabulate the pressure data over time:



Pressure	Time
217.5 PSI	1 hour, 20 minutes
215.0 PSI	1 hour, 25 minutes
222.5 PSI	1 hour, 30 minutes
226.3 PSI	1 hour, 35 minutes
150.0 PSI	1 hour, 40 minutes
150.0 PSI	1 hour, 45 minutes
151.3 PSI	1 hour, 50 minutes
148.8 PSI	1 hour, 55 minutes
145.0 PSI	2 hours, 0 minutes
145.0 PSI	2 hours, 5 minutes

To calculate the rate of pressure change over time in each of these periods, the computer would subtract the two adjacent pressure values, subtract the two corresponding adjacent time values, and then divide those two differences to arrive at a figure in units of PSI per minute. Taking the first two data coordinates in the table as an example:

$$\frac{\Delta P}{\Delta t} = \frac{215.0 \text{ PSI} - 217.5 \text{ PSI}}{1{:}25 - 1{:}20} = \frac{-2.5 \text{ PSI}}{5 \text{ min}} = -0.5 \frac{\text{PSI}}{\text{min}}$$

The sample period where the computer would detect the pipeline rupture lies between 1:35 and 1:40. Calculating this rate of pressure change:

$$\frac{\Delta P}{\Delta t} = \frac{150.0 \text{ PSI} - 226.3 \text{ PSI}}{1{:}40 - 1{:}35} = \frac{-76.3 \text{ PSI}}{5 \text{ min}} = -15.26 \frac{\text{PSI}}{\text{min}}$$

Clearly, a pressure drop rate of −15.26 PSI per minute is far greater than a typical drop of −0.5 PSI per minute, thus signaling a pipeline rupture.

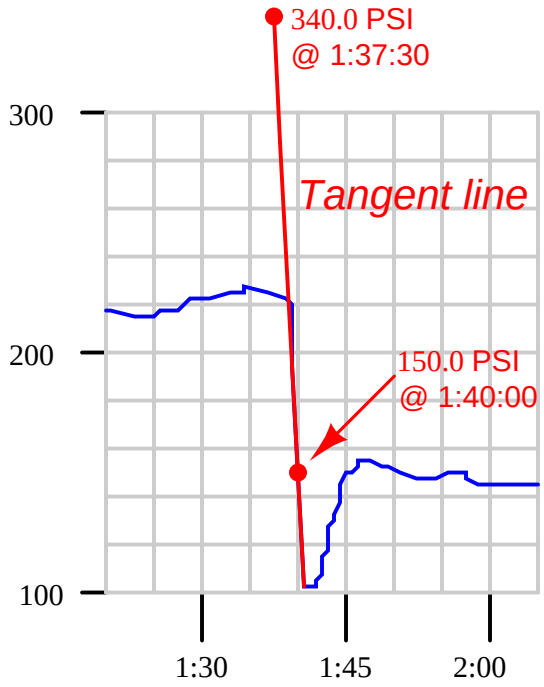
As you can see, the pipeline monitoring computer is not technically calculating *derivatives* ($\frac{dP}{dt}$), but rather *difference quotients* ($\frac{\Delta P}{\Delta t}$). Being a digital device, the best it can ever do is perform calculations at discrete points in real time. It is evident that calculating rates of change over 5-minute period misses much detail[12]. The actual rate of change at the steepest point of the pressure drop *far* exceeds −15.26 PSI per minute.

It is possible for us to calculate the instantaneous rate-of-change of pressure ($\frac{dP}{dt}$) at the moment of the rupture by examining the graph and sketching a straight line called a *tangent line* matching the slope where the graph is steepest. Our goal is to calculate the exact slope of that single (steepest) point on that graph, rather than an estimate of slope between two points as the computer did. In essence, the computer "drew" short line segments between pairs of points and calculated the slopes (rise-over-run) of those line segments. The slope of each line segment[13] is a difference quotient: $\frac{\Delta P}{\Delta t}$. The slope of a tangent line matching the slope at a single point on the function graph, however, is a derivative: $\frac{dP}{dt}$.

[12] Not only does a 5-minute rate calculation period miss a lot of detail, but it also results in a time delay of (up to) 5 minutes detecting a pipeline rupture.

[13] The technical term for a line passing through a pair of points on a curve is called a *secant line*.

First we sketch a tangent line (by hand) matching the steepest portion of the pressure trend graph. Then, we calculate the slope of a tangent line by marking convenient points[14] where the line intersects major division marks on the graph's graduated scale, then calculating rise over run:



$$\frac{dP}{dt} = \frac{150.0 \text{ PSI} - 340.0 \text{ PSI}}{1{:}40{:}00 - 1{:}37{:}30} = \frac{-190.0 \text{ PSI}}{2.5 \text{ min}} = -76.0 \frac{\text{PSI}}{\text{min}}$$

This distinction between calculating difference quotients ($\frac{\Delta P}{\Delta t}$) and calculating true derivative values ($\frac{dP}{dt}$) becomes less and less significant as the calculation period shortens. If the computer could sample and calculate at infinite speed, it *would* generate true derivative values instead of approximate derivative values.

[14] Please note that the pipeline pressure is *not* actually 340.0 PSI at a time of 1:37:30. This is simply a coordinate convenient to mark because it how it lines up with the divisions on the trend display. We choose coordinate points on the tangent line easy to visually discern, then calculate the tangent line's slope using those coordinates.

An algorithm applicable to calculating rates of change in a digital computer is shown here, using a notation called *pseudocode*[15]. For more information on pseudocode, refer to section 3.1 beginning on page 55. Each line of text in this listing represents a command for the digital computer to follow, one by one, in order from top to bottom. The `LOOP` and `ENDLOOP` markers represent the boundaries of a program *loop*, where the same set of encapsulated commands are executed over and over again in cyclic fashion:

Pseudocode listing

```
LOOP
  SET x = analog_input_N    // Update x with the latest measured input
  SET t = system_time       // Sample the system clock

  SET delta_x = x - last_x  // Calculate change in x
  SET delta_t = t - last_t  // Calculate change in t (time)

  SET rate = (delta_x / delta_t)  // Calculate ratio of changes

  SET last_x = x        // Update last_x value for next program cycle
  SET last_t = t        // Update last_t value for next program cycle
ENDLOOP
```

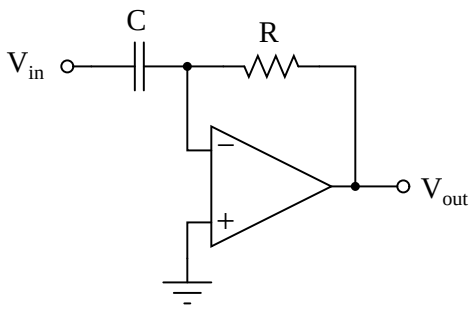
Each `SET` command tells the computer to assign a numerical value to the variable on the left-hand side of the equals sign (=), according to the value of the variable or expression on the right-hand side of the equals sign. Text following the double-dash marks (//) are *comments*, included only to help human readers interpret the code, not for the computer's benefit.

This computer program uses two variables to "remember" the values of the input (x) and time (t) from the previous scan, named `last_x` and `last_t`, respectively. These values are subtracted from the current values for x and t to yield differences (`delta_x` and `delta_t`, respectively), which are subsequently divided to yield a difference quotient. This quotient (`rate`) may be sampled in some other portion of the computer's program to trigger an alarm, a shutdown action, or simply display and/or record the rate value for a human operator's benefit.

The time period (Δt) for this program's difference quotient calculation is simply how often this algorithm "loops," or repeats itself. For a modern digital microprocessor, this could be many thousands of times per second.

[15] "Pseudocode" is a name given to any imaginary computer language used for the purpose of illustrating some procedure or concept without having to make reference to any particular (real) computer programming language. I could have just as well shown you the same algorithm using BASIC, C, or Java code, but pseudocode does just as well without the burden of introducing unfamiliar syntax to the reader.

If a nearly-instantaneous calculation is required for a rate-of-change variable, we may turn to an older technology using *analog*[16] electronic circuitry. Such a *differentiator* circuit uses the natural behavior of a capacitor to generate an output voltage proportional to the instantaneous rate-of-change of the input voltage:



$$V_{out} = -RC\frac{dV_{in}}{dt}$$

The negative feedback of the operational amplifier forms a *virtual ground* at the node where the capacitor, resistor, and inverting input connect. This means the capacitor "sees" the full input voltage (V_{in}) at all times. Current through a capacitor is a direct function of the voltage's time-derivative:

$$I = C\frac{dV}{dt}$$

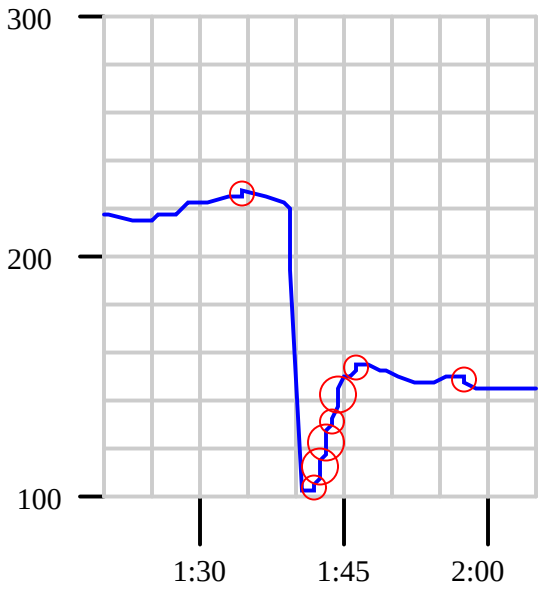
This current finds its way through the feedback resistor, developing a voltage drop that becomes the output signal (V_{out}). Thus, the output voltage of this analog differentiator circuit is directly proportional to the time-derivative of the input voltage (i.e. the input voltage's rate-of-change).

It is indeed impressive that such a simple circuit, possessing far fewer components than a microprocessor, is actually able to do a *better* job at calculating the real-time derivative of a changing signal than modern digital technology. The only real limitations to this device are accuracy (tolerances of the components used) and the bandwidth of the operational amplifier.

It would be a mistake, though, to think that an analog differentiator circuit is better suited to industrial applications of rate calculation than a digital computer, even if it does a superior job differentiating live signals. A very good argument for favoring difference quotients over actual derivatives is the presence of *noise* in the measured signal. A true differentiator, calculating the actual time-derivative of a live signal, will pick up on *any* rise or fall of the signal over time, no matter how brief. This is a serious problem when differentiating real-world signals, because noise (small amounts of "jittering" in the signal caused by any number of phenomena) will be interpreted by a perfect differentiator as very large rates of change over time.

[16] An *analog* system is one where some continuously-variable quantity is analogously represented by another continuously-variable quantity. In this case, we are using variable DC voltages to represent varying numerical quantities (e.g. pipeline pressure).

A close look at the previous pipeline pressure trend illustrates this problem. Note the areas circled (in red) on the graph, representing relatively small increases and decreases in signal occurring over very short periods of time:

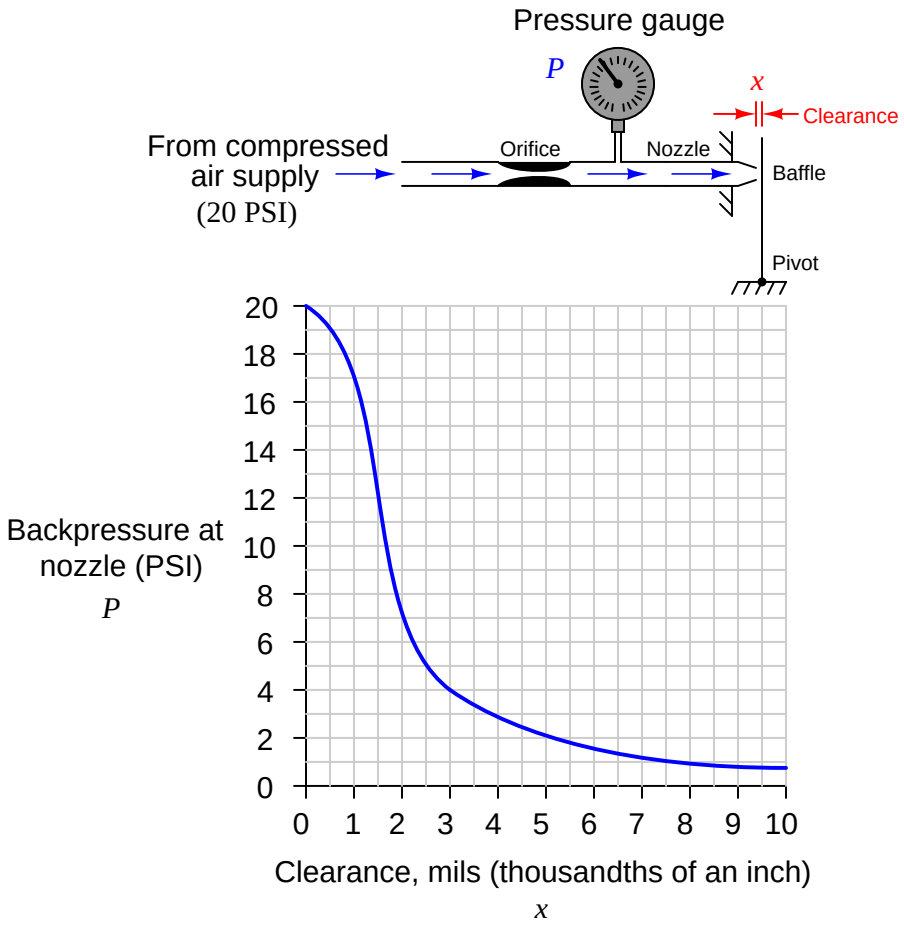


Although each "step" in pressure at these circled locations is small in amplitude, each one occurs over an extremely brief time increment. Thus, each of these steps has a nearly *infinite* rate of change (i.e. a vertical slope). Any rate-of-change sensing system able to apply true differentiation to the pressure signal would falsely declare an alarm (high rate-of-change) condition every time it encountered one of these "steps" in the signal. This means that even under perfectly normal operating conditions the rate-detection system would periodically declare an alarm (or perhaps shut the pipeline down!) given the inevitable presence of small noise-induced[17] "jitters" in the signal.

The best solution to this problem is to use a digital computer to calculate rates of change, setting the calculation period time slow enough that these small "jitters" will be averaged to very low values, yet fast enough that any serious pressure rate-of-change will be detected if it occurs. Back in the days when analog electronic circuits were the *only* practical option for calculating rates of signal change, the solution to this problem was to place a low-pass filter before the differentiator circuit to block such noise from ever reaching the differentiator.

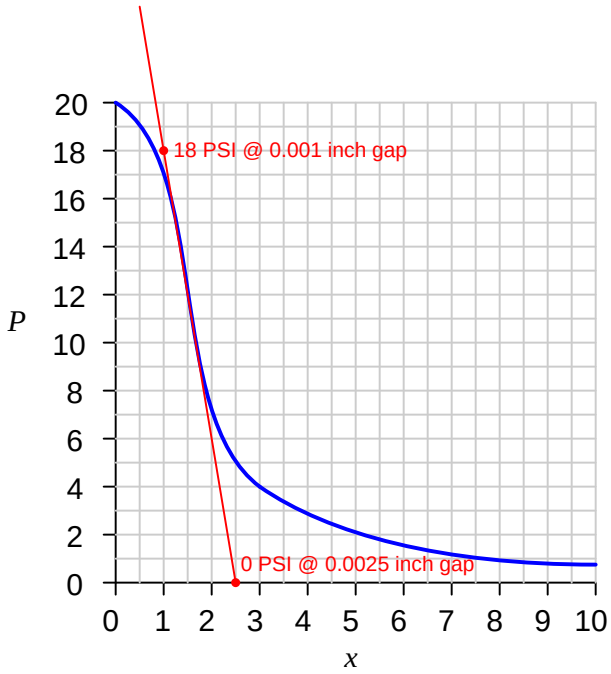
[17] Another source of trouble for differentiation of live signals is when the signal originates from a digital sensor. Digital devices, by their very nature, break analog signals into a series of discrete amplitude steps. As a digital process transmitter encounters a steadily increasing or decreasing process variable, its output rises or falls in discrete "jumps" rather than continuously as a fully analog transmitter would. Now, each of these jumps is quite small, but since each one occurs almost instantly it still translates into an extremely large rate-of-change when detected by a differentiator sampling over small time increments or sampling continuously (as in the case of an analog differentiator circuit). This means the problem of false rates-of-change exists *even in perfectly noiseless systems*, when the detection device (and/or the information channel to the monitoring system) is digital rather than analog.

Differentiation with respect to *time* has many applications, but there are many other applications of differentiation that are not time-based. For example, we may use differentiation to express the *sensitivity* of a non-linear mechanism in terms of the rate-of-change of output over input. One such application is the sensitivity of a legacy mechanism called a *baffle/nozzle* assembly used in pneumatic (compressed-air-powered) instruments to convert a small physical motion (x) into an air pressure signal (P). This very simple mechanism uses a flat piece of sheet metal (the baffle) to restrict air flow out of a small nozzle, causing a variable "backpressure" at the nozzle to develop as the baffle-to-nozzle clearance changes:



The graph expressing the relationship between P and x is clearly non-linear, having different slopes ($\frac{dP}{dx}$) at different points along its range. When used as part of the feedback mechanism for a self-balancing instrument, the purpose of the baffle/nozzle assembly is to detect baffle motion as sensitively as possible: that is, to generate the greatest change in pressure (ΔP) for the least change in motion (Δx). This means the designer of the pneumatic instrument should design it in such a way that the normal baffle/nozzle clearance gap rests at a point of maximum slope (maximum $\frac{dP}{dx}$) on the graph.

Sketching a tangent line near the point of maximum slope (maximum "steepness" on the graph) allows us to approximate the rate of change at that point:



Choosing convenient points[18] on this tangent line aligning with major divisions on the graph's scales, we find two coordinates we may use to calculate the derivative of the curve at its steepest point:

$$\frac{dP}{dx} = \frac{0 \text{ PSI} - 18 \text{ PSI}}{0.0025 \text{ inch} - 0.001 \text{ inch}} = \frac{-18 \text{ PSI}}{0.0015 \text{ inch}} = -12000 \text{ PSI per inch}$$

The phenomenally large value of −12000 PSI per inch is a *rate* of pressure change to clearance (baffle-nozzle gap) change. Do not mistakenly think that this value suggests the mechanism could ever develop a pressure of 12000 PSI – it is simply describing the extreme sensitivity of the mechanism in terms of PSI change per unit change of baffle motion. By analogy, just because an automobile travels at a *speed* of 70 miles per hour does not mean it must travel 70 miles in *distance*!

It should be clear from an examination of the graph that this high sensitivity extends approximately between the pressure values of 9 and 14 PSI. Outside of those pressure values, the graph's slope begins to decrease. While still sensitive, the baffle/nozzle mechanism will not be as sensitive to baffle motion outside those pressure values as it is within.

[18] Once gain, we are looking for points where the tangent line happens to intersect with major divisions on the graph's scale. This makes it relatively easy to calculate the line's slope, since the pressure and distance values for those coordinates are easy to read.

2.7 Numerical integration

As we have seen, the concept of *integration* is finding the accumulation of one variable multiplied by another (related) variable. In this section, we will explore the practical application of this concept to real-world data, where actual numerical values of variables are used to calculate accumulated sums.

In industrial instrumentation, for example, we are often interested in calculating the accumulation of some process fluid based on a measured flow *rate* of that fluid. The rate is, of course, expressed in either mass or volume units per unit time (e.g. gallons per minute), but the total accumulated quantity will be expressed plainly in either mass or volume units (e.g. gallons). We may use computers to calculate those accumulated quantities, either after the fact (from recorded data) or in real time.

Numerical (data-based) integration is fundamentally a two-step arithmetic process. First, we must use *multiplication* to calculate the product of a variable and a small increment of another variable (a change in the second variable between two different points). Then, we must use *addition* to calculate the accumulated sum of the products.

To illustrate, we will first focus on the integration of a flow measurement signal with respect to time. The flow rate of any fluid is always expressed in units of volume or mass *per unit time*. Common volumetric flow units are gallons *per minute*, liters *per second*, cubic feet *per day*, etc. Common mass flow units are pounds *per hour*, kilograms *per minute*, slugs *per second*, etc. If we desire to calculate the volume or mass of fluid passed through a pipe – representing fluid added to or removed from a system – over some interval of time, we may do so by integrating flow rate with respect to time:

$$\Delta V = \int_a^b Q \, dt$$

$$\Delta m = \int_a^b W \, dt$$

Where,
ΔV = Volume of fluid added or removed
Q = Volumetric flow rate of fluid
Δm = Mass of fluid added or removed
W = Mass flow rate of fluid
a = Starting point of integration interval
b = Ending point of integration interval
t = Time

As always, integration is fundamentally a matter of *multiplying* one variable by small increments of another variable. If a flow rate is integrated with respect to time, the result is that the unit for time becomes eliminated. Gallons per minute, for example, becomes gallons after integration; kilograms per second becomes kilograms; etc.

The elimination of time units is also evident if we re-write the integrands in the previous equations to show volumetric and mass flow rates (Q and W, respectively) as the rates of change they are ($Q = \frac{dV}{dt}$ and $W = \frac{dm}{dt}$):

$$\Delta V = \int_a^b \frac{dV}{dt}\, dt$$

$$\Delta m = \int_a^b \frac{dm}{dt}\, dt$$

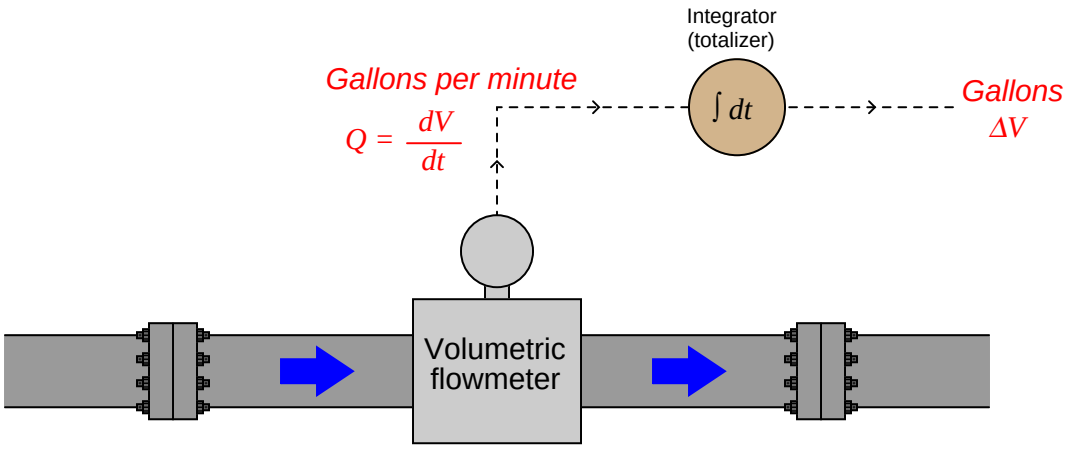
It should be clear that the time differentials (dt) cancel in each integrand, leaving:

$$\Delta V = \int_a^b dV$$

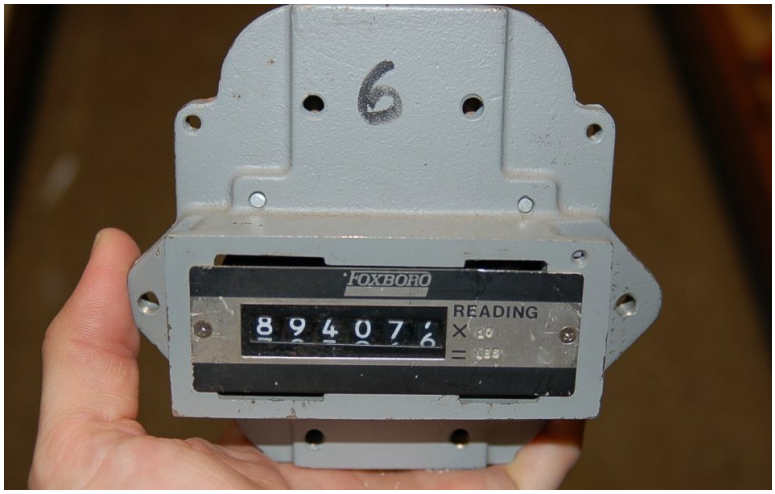
$$\Delta m = \int_a^b dm$$

Since we know the integral symbol ($\int$) simply means the "continuous sum of" whatever follows it, we may conclude in each case that the continuous sum of infinitesimal increments of a variable is simply a larger change of that same variable. The continuous summation of dV is simply the total change in V over the interval beginning at time a and ending at time b; likewise, the continuous summation of dm is simply the total change in m over the interval beginning at time a and ending at time b.

A flowmeter measuring the flow rate of a fluid outputs a signal representing either volume or mass units passing by per unit time. Integrating that signal with respect to time yields a value representing the total volume or mass passed through the pipe over a specific interval. A physical device designed to perform this task of integrating a signal with respect to time is called an *integrator* or a *totalizer*:



An example of a flow integrator, or flow totalizer, made for pneumatic instrument systems is the legacy Foxboro model 14. A view of this instrument's front face shows an odometer-style display, in this particular case showing the total number of pounds (lbs) of fluid passed through the pipe, with a multiplying factor of 10:



The fact that this instrument's display resembles the odometer of an automobile is no coincidence. Odometers are really just another form of mechanical integrator, "totalizing" the distance traveled by a vehicle. If the speedometer of a vehicle registers speed (v) in units of miles per hour, then the odometer will accumulate a distance (Δx) in units of miles, since distance (miles) is the time-integral of speed (miles per hour):

$$\Delta x = \int_a^b v \, dt \qquad \text{. . . or . . .} \qquad \Delta x = \int_a^b \frac{dx}{dt} \, dt$$

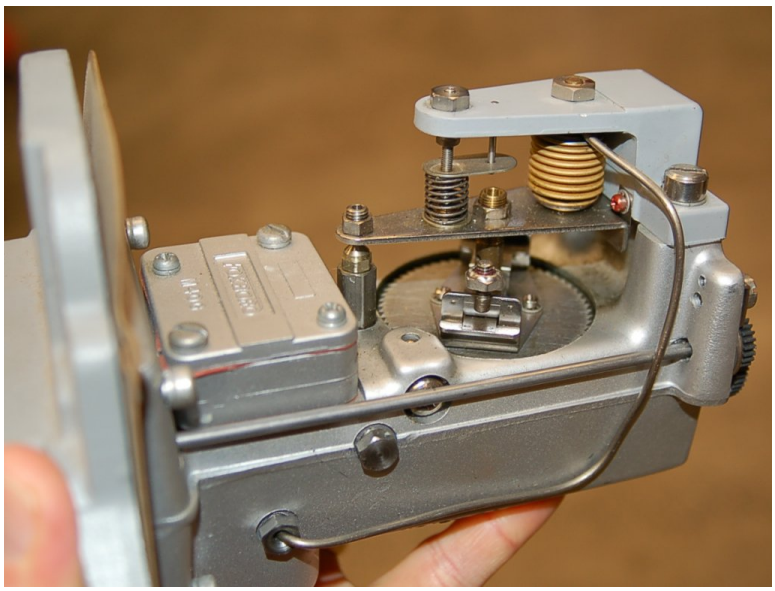
$$[\text{miles}] = \int_a^b \left(\left[\frac{\text{miles}}{\text{hour}} \right] \, [\text{hours}] \right)$$

In this particular case, where the flowmeter measures pounds per hour, and the integrator registers accumulated mass in pounds, the integration of units is as follows:

$$\Delta m = \int_a^b W \, dt \qquad \text{. . . or . . .} \qquad \Delta m = \int_a^b \frac{dm}{dt} \, dt$$

$$[\text{pounds}] = \int_a^b \left(\left[\frac{\text{pounds}}{\text{hour}} \right] \, [\text{hours}] \right)$$

Like many pneumatic instruments of its era, the Foxboro model 14 totalizer operated on rather ingenious mechanical principles. In this design, a turbine wheel was driven by a jet of compressed air from a nozzle, that wheel's speed proportional to the sensed fluid flow rate reported by a pneumatic flowmeter. As fluid flow rate increased through the pipe, the wheel was made to spin faster. This spinning wheel drove a gear-reduction mechanism to slowly turn the odometer-style numerals, registering total fluid quantity passed through the flowmeter:



As pneumatic signal pressure (3-15 PSI) from a pneumatic flowmeter entered the brass bellows of this instrument, pressing down on a lever, forcing a baffle toward a nozzle. This created an increased nozzle backpressure, which was then amplified by a small pneumatic amplifier mechanism and sent through a nozzle to spin the turbine wheel to drive the integrating "odometer" display. Mounted on the turbine wheel was a set of fly-weights, which under the influence of centrifugal force would press upward on the lever to re-establish a condition of force-balance to maintain a (relatively) constant baffle-nozzle gap. Thus, the force-balance mechanism worked to establish an accurate and repeatable relationship[19] between instrument signal pressure and integration rate.

[19] The true ingenuity of the Foxboro model 14 totalizer's design is more evident when we consider that fact that centrifugal force varies with the *square* of angular velocity. This had the effect of naturally performing the *square-root* characterization required of most pneumatic flowmeter instruments due to the quadratic nature of most primary flow-sensing elements (e.g. orifice plate, venturi tubes, pitot tubes, etc.). For example, a doubling of fluid flow rate naturally results in the pressure differential across the flow element quadrupling, and therefore the pneumatic flowmeter's air pressure signal would quadruple in represented value. This would only result in a doubling of the totalizer's turbine wheel, though, because that is how much of a speed increase would be necessary to generate a centrifugal force four times stronger to balance the flowmeter's quadrupled air pressure signal.

A very different style of integrator appears here, an electronic unit used in a ball mill operation for totalizing the amount of crushed limestone applied toward the manufacture of concrete. Limestone is fed into the ball mill on a device called a *weighfeeder*, continuously sensing the mass of crushed limestone as it passes over a conveyor belt. The controller maintains a limestone "flow rate" at a setpoint specified in tons per hour (mass flow of solid material). The red LED digital display shows the total number of tons passed through the mill:

The units involved in the integration of limestone "flow" into the ball mill are slightly different from the example shown with the Foxboro model 14 totalizer, but the concept is the same:

$$\Delta m = \int_a^b W \, dt$$

$$[\text{tons}] = \int_a^b \left(\left[\frac{\text{tons}}{\text{hour}} \right] \, [\text{hours}] \right)$$

As with all cases of numerical integration, an essential piece of information to know when "totalizing" any rate is the initial quantity at the start of the totalization interval. This is the *constant of integration* mentioned previously. For flow totalization, this constant would be the initial volume of fluid recorded at the starting time. For an automobile's odometer, this constant is the initial "mileage" accumulated prior to driving on a trip[20].

[20] Vehicles equipped with a *trip odometer* allow the driver to reset this integration constant to zero at will, thus

An algorithm applicable to integrating real signals with respect to time in a digital computer is shown here, once again using "pseudocode" as the computer language. Each line of text in this listing represents a command for the digital computer to follow, one by one, in order from top to bottom. The `LOOP` and `ENDLOOP` markers represent the boundaries of a program *loop*, where the same set of encapsulated commands are executed over and over again in cyclic fashion:

Pseudocode listing

```
LOOP
  SET x = analog_input_N   // Update x with the latest measured input
  SET t = system_time      // Sample the system clock

  SET delta_t = t - last_t  // Calculate change in t (time)

  SET product = x * delta_t    // Calculate product (integrand)
  SET total = total + product  // Update the running total

  SET last_t = t        // Update last_t value for next program cycle
ENDLOOP
```

This computer program uses a variable to "remember" the value of time (t) from the previous scan, named `last_t`. This value is subtracted from the current value for t to yield a difference (`delta_t`), which is subsequently multiplied by the input value x to form a product. This product is then added to an accumulating total (named `total`), representing the integrated value. This "total" value may be sampled in some other portion of the computer's program to trigger an alarm, a shutdown action, or simply display and/or record the totalized value for a human operator's benefit.

The time period (Δt) for this program's difference quotient calculation is simply how often this algorithm "loops," or repeats itself. For a modern digital microprocessor, this could be upwards of many thousands of times per second. Unlike differentiation, where an excessive sampling rate may cause trouble by interpreting noise as extremely high rates of change, there is no danger of excessive sampling when performing numerical integration. The computer may integrate as fast as it can with no ill effect.

One of the fundamental characteristics of integration is that it *ignores* noise, which is a very good quality for industrial signal processing. Small "jittering" in the signal tends to be random, which means for every "up" spike of noise, one may expect a comparable "down" spike (or collection of "down" spikes having comparable weight) at some later time. Thus, noise tends to cancel itself out when integrated over time.

As with differentiation, applications exist for integration that are not time-based. One such application is the calculation of mechanical *work*, defined as the product of force and displacement (distance moved). In mechanical systems where there is no energy dissipated due to friction, work results in a change in the energy possessed by an object.

allowing the tracking of mileage for individual trips instead of over the life of the automobile.

For example, if we use a hoist to lift a mass weighing 700 pounds straight up against gravity a distance of 3 feet, we will have done 2100 foot-pounds of work. The work done on the mass increases its potential energy (ΔE) by 2100 foot-pounds:

$$\Delta E = Fx$$

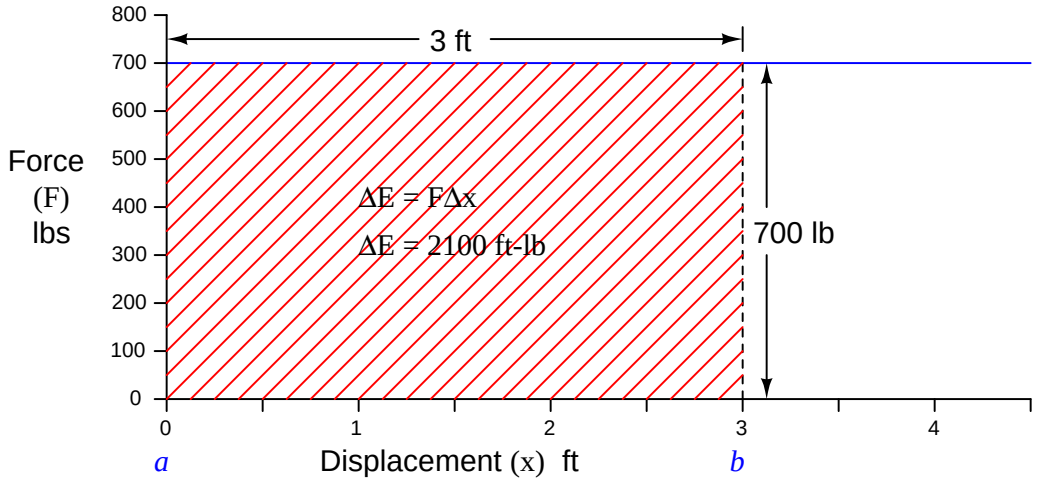
Where,

ΔE = Change in potential energy resulting from work, in Joules (metric) or foot-pounds (British)
F = Force doing the work, in Newtons (metric) or pounds (British)
x = Displacement over which the work was done, in meters (metric) or feet (British)

We may also express this change in potential energy as an integral of force (F) multiplied by infinitesimal increments in displacement (dx) over some interval (from a to b), since we know integration is nothing more than a sophisticated way to multiply quantities:

$$\Delta E = \int_a^b F\, dx$$

Like any other integral, the energy change effected by lifting this mass a vertical distance may be represented graphically as the *area* enclosed by the graph. In this case, the area is very simple to calculate, being a simple rectangle (height times width):



Lifting the mass vertically constitutes a *positive* change in potential energy for this object, because each displacement differential (dx) is a positive quantity as we move from a height of 0 feet to a height of 3 feet:

$$2100 \text{ ft-lbs} = \int_{0ft}^{3ft} (700 \text{ lbs})\, dx$$

A natural question to ask at this point is, *what would the resulting change in energy be if we lowered the mass from its height of 3 feet back down to 0 feet?*. Doing so would cover the exact same distance (3 feet) while exerting the exact same amount of suspending force (700 lbs), and so we can safely conclude the work will have an absolute magnitude of 2100 ft-lbs. However, if we lower the mass, each displacement differential (dx) will be a negative quantity[21] as we move from a greater height to a lesser height. This makes the work – and the resulting energy change – a negative quantity as well:

$$-2100 \text{ ft-lbs} = \int_{3ft}^{0ft} (700 \text{ lbs})\, dx$$

This means if we raise the mass to a height of 3 feet, then lower it back to its original starting height of 0 feet, the total change in potential energy will be zero:

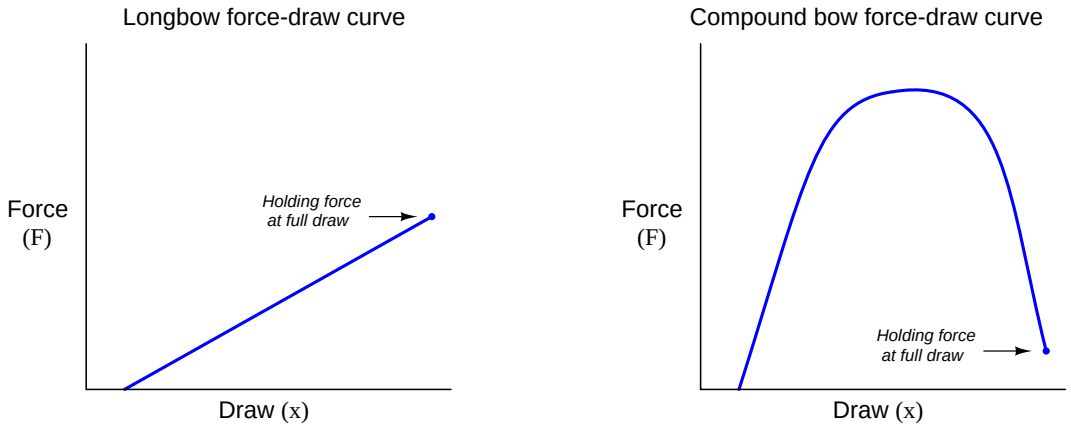
$$0 \text{ ft-lbs} = \int_{0ft}^{3ft} (700 \text{ lbs})\, dx + \int_{3ft}^{0ft} (700 \text{ lbs})\, dx$$

This is true for any integral having an interval of zero (same starting and ending values), regardless of the integrand's value at any point in time:

$$0 \text{ ft-lbs} = \int_{a}^{a} F\, dx$$

[21] As we lower the mass to ground level, height (x) goes from being a positive value to zero. This means each differential (infinitesimal change in value) for x will be negative, thus causing the integrand $F\, dx$ to have a negative value and thus causing the integrated total (work) to be negative as well.

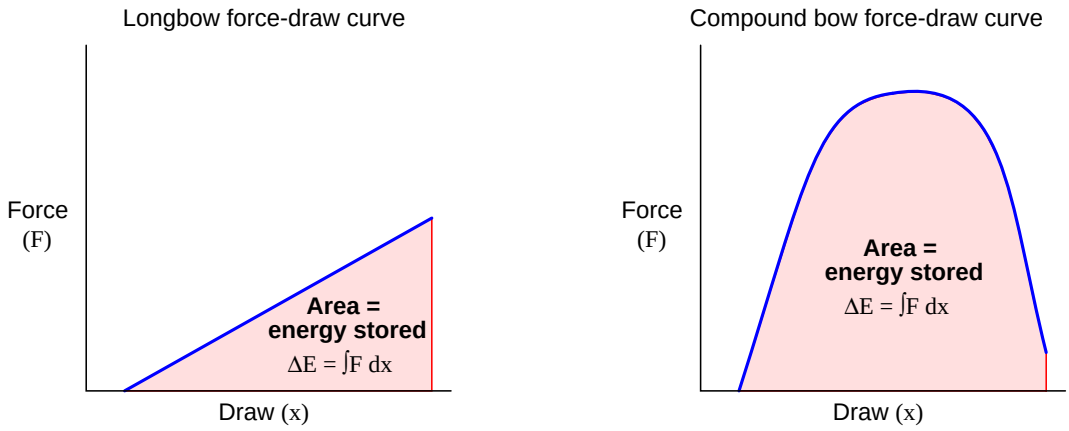
The integration of force and displacement to calculate potential energy change really shows its utility when the force changes as a function of displacement. A practical example of this sort of calculation is the determination of energy stored in an archer's bow when drawn to a certain displacement. The so-called *force-draw curve* of a longbow is nearly ideal for a theoretical spring, with force increasing linearly as the string is drawn back by the archer. The force-draw curve for a compound bow[22] is quite nonlinear, with a much lesser holding force required to maintain the bow at full draw:



The force required to draw a compound bow rises sharply during the first few inches of draw, peaks during the region where the archer's arms are ideally angled for maximum pulling strength, then "lets off" toward the end where the archer's drawing arm is weakest in the "holding" position. The result is a bow that requires substantial force to draw, but is relatively easy to hold in fully-drawn position.

[22] While a longbow is really nothing more than a long and flexible stick with a straight string drawn across it, a compound bow is a sophisticated machine with multiple passes of string and cam-shaped pulleys providing the nonlinear force-draw relationship.

While the compound bow may be easier to hold at full draw than the longbow, for any given holding force the compound bow stores *much* more energy than the longbow, owing to the far greater *area* (force-displacement integral) enclosed by the curve:



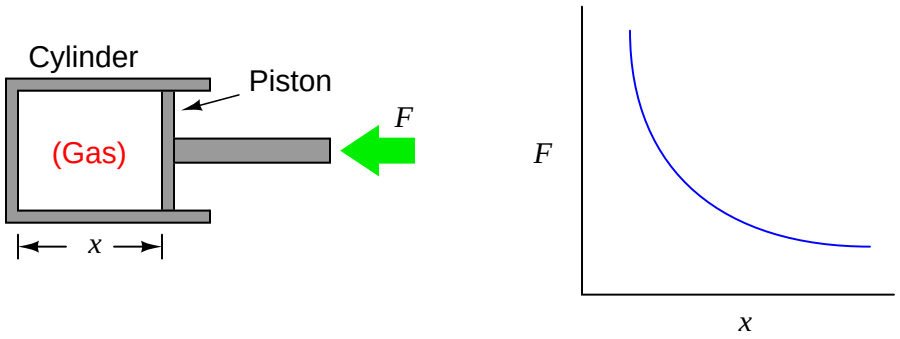
This is why a compound bow is so much more powerful than a longbow or a "recurve" bow with the same holding force: the energy represented by the greater area underneath the force-draw curve equates to greater energy imparted to the arrow when released, and therefore greater kinetic energy in the arrow during flight.

Like any other form of mechanical work, the energy invested into the bow by the archer is readily calculated and expressed in units of force $\times$ displacement, typically newton-meters (Joules) in metric units and foot-pounds in British units. This stands to reason, since we know integration is fundamentally a matter of *multiplying* quantities together, in this case force (pull) and displacement (draw).

To actually calculate the amount of energy stored in a fully-drawn bow, we could measure both force and displacement with sensors as the archer draws the bow, with a computer numerically integrating force over increments of draw in real time. Another method would be to simply graph force versus draw as we have done here, then use geometric methods[23] to approximate the area underneath the curve.

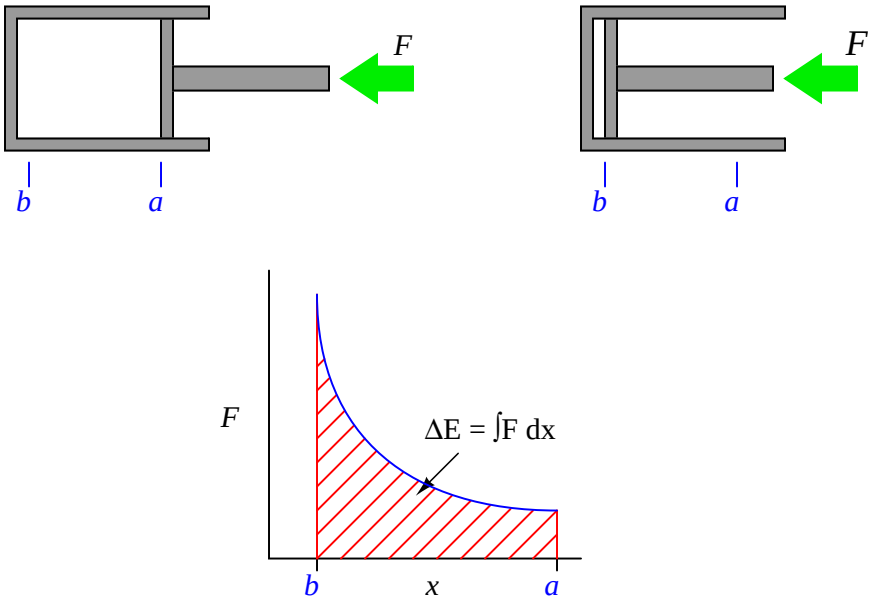
[23] One simple way to do this is to cover the entire integration area using nothing but rectangles and triangles, then measuring all the sketched shapes to totalize their areas.

A more sophisticated example of numerical integration used to calculate work is that of a heat engine, where a piston compresses an enclosed gas:



As the piston is pushed farther into the cylinder, the gas becomes compressed, exerting more force on the piston. This requires an ever-increasing application of force to continue the piston's motion. Unlike the example where a mass of constant weight was lifted against the pull of gravity, here the force is a dynamically changing variable instead of a constant. The graph shows this relationship between piston displacement and piston force.

If we push the piston into the cylinder, the force increases as the displacement decreases. The change in energy is described by the integral of force with respect to displacement, graphically equivalent to the area underneath the force curve:



$$\Delta E = \int_a^b F\, dx$$

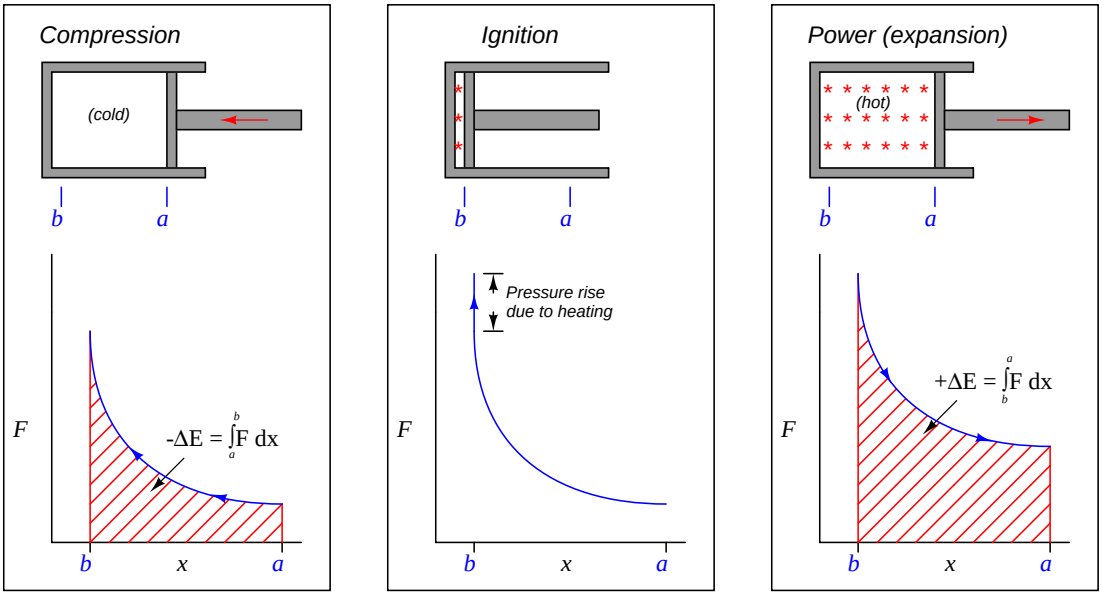
If we slowly allow the piston to return to its original position (letting the pressure of the enclosed gas push it back out), the piston's force decreases as displacement increases. The force/displacement relationship is the same as before, the only difference being the direction of travel is opposite. This means the change in energy is happening over the same interval, in reverse direction (from b to a now instead of from a to b). Expressed as an integral:

$$\Delta E = \int_b^a F\, dx$$

As we have already learned, a reversal of direction means the sign of the integral will be opposite. If pushing the piston farther inside the cylinder represented work being done on the enclosed gas by the applied force, now the gas will be doing work on the source of the applied force as the piston returns to its extended position.

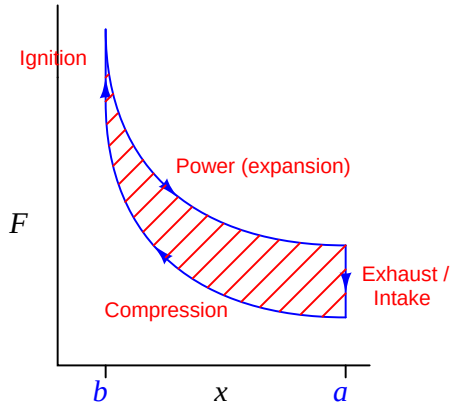
This means we will have done zero net work by pushing the piston into the cylinder and then letting it spring back out to its original position, just as we performed zero net work by lifting a mass 3 feet in the air and then letting it back down.

In order that this piston/cylinder mechanism might function as an engine, we must have some way of making the energy change greater in one direction than the other. This is done by *heating* the enclosed gas at the point of greatest compression. In a spark-ignition engine, the gas is actually a mixture of air and fuel, ignited by an electric spark. In a compression-ignition (diesel) engine, the gas is pure air, with fuel injected at the last moment to initiate combustion. The addition of heat (from combustion) will cause the gas pressure to rise, exerting more force on the piston than what it took to compress the gas when cold. This increased force will result in a *greater* energy change with the piston moving out of the cylinder than with the piston moving in:



Representing the work done by the hot gas as the area enclosed by the curve makes this clear: more mechanical energy is being released as the piston travels from b to a during the "power stroke" than the amount of energy invested in compressing the gas as the piston traveled from a to b during the "compression stroke." Thus, an internal combustion engine produces mechanical power by repeatedly compressing a cold gas, heating that gas to a greater temperature, and then expanding that hot gas to extract energy from it.

At the conclusion of the power stroke, a valve opens to exhaust the hot gas and another valve opens to introduce cold gas. This places the piston and cylinder in the original condition, ready for another set of compression, ignition, and power strokes. This cycle is sometimes represented as a closed "loop" on the force/displacement graph, like this:



Net energy output by the engine at the conclusion of each cycle is equivalent to the area enclosed by the loop. This is the difference in areas (integrals) between the "compression" and "power" strokes. Any design change to the engine resulting in a greater "loop" area (i.e. less energy required to compress the gas, and/or more energy extracted from its expansion) results in a more powerful engine. This is why heat engines output the most power when the difference in temperatures (cold gas versus heated gas) is greatest: a greater temperature shift results in the two curves being farther apart vertically, thus increasing the area enclosed by the "loop."

Chapter 3

Derivations and Technical References

This chapter is where you will find mathematical derivations too detailed to include in the tutorial, and/or tables and other technical reference material.

3.1 Introduction to pseudocode

Pseudocode is a form of text-based programming intended only for human reading, yet similar enough in syntax and structure to real computer programming languages for a human programmer to be able to easily translate to a high-level programming language such as C++, Python, etc. Since pseudocode is not a formal computer language, we may use it to very efficiently describe certain algorithms (procedures) without having to abide by strict "grammatical" rules as we would if writing in a formal programming language. There is no agreed-upon standard for pseudocode, but here I will outline my own conventions.

3.1.1 Program loops

Each line of text in the following listing represents a command for the digital computer to follow, one by one, in order from top to bottom. The `LOOP` and `ENDLOOP` markers represent the boundaries of a program *loop*, where the same set of encapsulated commands are executed over and over again in cyclic fashion:

Pseudocode listing [1]

```
LOOP
  PRINT "Hello World!"  // This line prints text to the screen
  OUTPUT audible beep on the speaker  // This line beeps the speaker
ENDLOOP
```

In this particular case, the result of this program's execution is a continuous printing of the words "Hello World!" to the computer's display with a single "beep" tone accompanying each printed line. The words following a double-slash (//) are called *comments*, and exist only to provide explanatory text for the human reader, not the computer. Admittedly, this example program would be both impractical and annoying to actually run in a computer, but it does serve to illustrate the basic concept of a program "loop" shown in pseudocode.

[1] I have used a typesetting convention to help make my pseudocode easier for human beings to read: all formal commands appear in bold-faced blue type, while all comments appear in italicized red type. All other text appears as normal-faced black type. One should remember that the computer running any program cares not for how the text is typeset: all it cares is that the commands are properly used (i.e. no "grammatical" or "syntactical" errors).

3.1.2 Assigning values

For another example of pseudocode, consider the following program. This code causes a variable (x) in the computer's memory to alternate between two values of 0 and 2 indefinitely:

Pseudocode listing

```
DECLARE x to be an integer variable
SET x = 2  // Initializing the value of x

LOOP
  // This SET command alternates the value of x with each pass
  SET x = 2 - x
ENDLOOP
```

The first instruction in this listing declares the type of variable x will be. In this case, x will be an *integer* variable, which means it may only represent whole-number quantities and their negative counterparts – no other values (e.g. fractions, decimals) are possible. If we wished to limit the scope of x even further to represent just 0 or 1 (i.e. a single bit), we would have to declare it as a *Boolean* variable. If we required x to be able to represent fractional values as well, we would have to declare it as a *floating-point* variable. Variable declarations are important in computer programming because it instructs the computer how much space in its random-access memory to allocate to each variable, which necessarily limits the range of numbers each variable may represent.

The next instruction initializes x to a value of two. Like the declaration, this instruction need only happen once at the beginning of the program's execution, and never again so long as the program continues to run. The single `SET` statement located between the `LOOP` and `ENDLOOP` markers, however, repeatedly executes as fast as the computer's processor allows, causing x to rapidly alternate between the values of two and zero.

It should be noted that the "equals" sign (=) in computer programming often has a different meaning from that commonly implied in ordinary mathematics. When used in conjunction with the `SET` command, an "equals" sign *assigns* the value of the right-hand quantity to the left-hand variable. For example, the command `SET x = 2 - x` tells the computer to first calculate the quantity $2 - x$ and then set the variable x to this new value. It definitely does *not* mean to imply x is actually equal in value to $2 - x$, which would be a mathematical contradiction. Thus, you should interpret the `SET` command to mean "set equal to . . ."

3.1.3 Testing values (conditional statements)

If we mean to simply test for an equality between two quantities, we may use the same symbol (=) in the context of a different command, such as "IF":

Pseudocode listing

```
DECLARE x to be an integer variable

LOOP

  // (other code manipulating the value of x goes here)

  IF x = 5 THEN
    PRINT "The value of the number is 5"
    OUTPUT audible beep on the speaker
  ENDIF
ENDLOOP
```

This program repeatedly tests whether or not the variable x is equal to 5, printing a line of text and producing a "beep" on the computer's speaker if that test evaluates as true. Here, the context of the IF command tells us the equals sign is a test for equality rather than a command to assign a new value to x. If the condition is met ($x = 5$) then all commands contained within the IF/ENDIF set are executed.

Some programming languages draw a more explicit distinction between the operations of equality test versus assignment by using different symbol combinations. In C and C++, for example, a single equals sign (=) represents assignment while a double set of equals signs (==) represents a test for equality. In Structured Text (ST) PLC programming, a single equals sign (=) represents a test for equality, while a colon plus equals sign (:=) represents assignment. The combination of an exclamation point and an equals sign (!=) represents "not equal to," used as a test condition to check for *inequality* between two quantities.

3.1.4 Branching and functions

A very important feature of any programming language is the ability for the path of execution to change (i.e. the program "flow" to *branch* in another direction) rather than take the exact same path every time. We saw shades of this with the `IF` statement in our previous example program: the computer would print some text and output a beep sound if the variable x happened to be equal to 5, but would completely skip the `PRINT` and `OUTPUT` commands if x happened to be any other value.

An elegant way to modularize a program into separate pieces involves writing portions of the program as separate *functions* which may be "called" as needed by the main program. Let us examine how to apply this concept to the following conditional program:

Pseudocode listing

```
DECLARE x to be an integer variable

LOOP

  // (other code manipulating the value of x goes here)

  IF x = 5 THEN
    PRINT "The value of the number is 5"
    OUTPUT audible beep on the speaker
  ELSEIF x = 7 THEN
    PRINT "The value of the number is 7"
    OUTPUT audible beep on the speaker
  ELSEIF x = 11 THEN
    PRINT "The value of the number is 11"
    OUTPUT audible beep on the speaker
  ENDIF
ENDLOOP
```

This program takes action (printing and outputting beeps) if ever the variable x equals either 5, 7, or 11, but not for any other values of x. The actions taken with each condition are quite similar: print the numerical value of x and output a single beep. In fact, one might argue this code is ugly because we have to keep repeating one of the commands verbatim: the `OUTPUT` command for each condition where we wish to computer to output a beep sound.

We may streamline this program by placing the `PRINT` and `OUTPUT` commands into their own separate "function" written outside the main loop, and then *call* that function whenever we need it. The boundaries of this function's code are marked by the `BEGIN` and `END` labels shown near the bottom of the listing:

Pseudocode listing

```
DECLARE n to be an integer variable
DECLARE x to be an integer variable
DECLARE PrintAndBeep to be a function

LOOP

  // (other code manipulating the value of x goes here)

  IF x = 5 OR x = 7 OR x = 11 THEN
    CALL PrintAndBeep(x)
  ENDIF
ENDLOOP


BEGIN PrintAndBeep (n)
  PRINT "The value of the number is" (n) "!"
  OUTPUT audible beep on the speaker
  RETURN
END PrintAndBeep
```

The main program loop is much shorter than before because the repetitive tasks of printing the value of x and outputting beep sounds has been moved to a separate function. In older computer languages, this was known as a *subroutine*, the concept being that flow through the main program (the "routine") would branch to a separate sub-program (a "subroutine") to do some specialized task and then return back to the main program when the sub-program was done with its task.

Note that the program execution flow never reaches the PrintAndBeep function unless x happens to equal 5, 7, or 11. If the value of x never matches any of those specific conditions, the program simply keeps looping between the `LOOP` and `ENDLOOP` markers.

Note also how the value of x gets *passed* on to the PrintAndBeep function, then read inside that function under another variable name, n. This was not strictly necessary for the purpose of printing the value of x, since x is the only variable in the main program. However, the use of a separate ("local") variable within the PrintAndBeep function enables us at some later date to use that function to act on other variables within the main program while avoiding conflict. Take this program for example:

Pseudocode listing

```
DECLARE n to be an integer variable
DECLARE x to be an integer variable
DECLARE y to be an integer variable
DECLARE PrintAndBeep to be a function

LOOP

  // (other code manipulating the value of x and y goes here)

  IF x = 5 OR x = 7 OR x = 11 THEN
    CALL PrintAndBeep(x)
  ENDIF
  IF y = 0 OR y = 2 THEN
    CALL PrintAndBeep(y)
  ENDIF
ENDLOOP


BEGIN PrintAndBeep (n)
  PRINT "The value of the number is" (n) "!"
  OUTPUT audible beep on the speaker
  RETURN
END PrintAndBeep
```

Here, the PrintAndBeep function gets used to print certain values of x, then re-used to print certain values of y. If we had used x within the PrintAndBeep function instead of its own variable (n), the function would only be useful for printing the value of x. Being able to pass values to functions makes those functions more useful.

A final note on branching and functions: most computer languages *allow a function to call itself* if necessary! This concept is known as *recursion* in computer science.

3.2 Solving RC and LR circuit differential equations using C++

A *differential equation* is a mathematical equation containing both a variable and at least one of its derivatives (i.e. its rate-of-change). Differential equations are extremely useful for describing a wide range of physical phenomena including the energization and de-energization of energy-storing components such as capacitors and inductors. While differential equations require calculus techniques to solve analytically, their solutions may be approximated quite closely using repeated calculations of simple arithmetic. Computers excel at doing such tasks, and so here we will view multiple programs written in C++ to solve for voltages and currents in resistor-capacitor and resistor-inductor networks.

Each of these programs outputs data in comma-separated variable (CSV) text format, suitable for plotting using mathematical visualization software such as `gnuplot` or spreadsheets such as Microsoft `Excel`.

3.2.1 Differential equation solver for an RC circuit and voltage source

```
#include <iostream>
#include <cmath>
using namespace std;

/* Schematic:
  +-----R-----+
  |           |
  Vs          C   Ic = C dVc/dt
  |           |
  +-----------+
*/

int main (void)
{
  double Ic, C = 33e-6, R = 10e3, Vr, Vs = 0.0, Vc = 10.0, dVc, t, dt = 0.0001;

  cout << "Time , Vc" << endl;

  for (t = 0 ; t < 2.0 ; t = t + dt)
  {
    Vr = Vs - Vc;         // Kirchhoff's Voltage Law
    Ic = Vr / R;          // Ohm's Law
    dVc = dt * (Ic / C);  // I = C dV/dt "Ohm's Law" for capacitors
    Vc = Vc + dVc;        // Integrating capacitor voltage from small changes
    cout << t << " , " << Vc << endl;
  }

  return 0;
}
```

With the initialized values shown in the `double` line of this program, it will compute capacitor voltages at 0.1 millisecond intervals assuming an initial capacitor voltage of 10 Volts and discharging toward zero volts (i.e. the voltage source value `Vs` is set to 0.0). Of course, any combination of `Vs` and initial `Vc` values will work perfectly well, as this program will show the gradual progression from initial to final voltage over time.

3.2.2 Differential equation solver for an RC circuit and current source

```
#include <iostream>
#include <cmath>
using namespace std;

/* Schematic:
  +-----+-----+
  |     |     |
  Is    R     C   I = C dVc/dt
  |     |     |
  +-----+-----+
*/

int main (void)
{
  double Ic, C = 33e-6, R = 10e3, Is = 1e-3, Ir, Vc = 0.0, dVc, t, dt = 0.0001;

  cout << "Time , Vc" << endl;

  for (t = 0 ; t < 2.0 ; t = t + dt)
  {
    Ir = Vc / R;          // Ohm's Law
    Ic = Is - Ir;         // Kirchhoff's Current Law
    dVc = dt * (Ic / C);  // I = C dV/dt "Ohm's Law" for capacitors
    Vc = Vc + dVc;        // Integrating capacitor voltage from small changes
    cout << t << " , " << Vc << endl;
  }

  return 0;
}
```

With the initialized values shown in the `double` line of this program, it will compute capacitor voltages at 0.1 millisecond intervals assuming an initial capacitor voltage of 0 Volts and charging toward 10 Volts (i.e. the source current of 1 milliAmpere passing entirely through the 10 kΩ resistor once the capacitor reaches full voltage). Of course, any combination of source current (`Is`) and initial capacitor voltage `Vc` values will work perfectly well, as this program will show the gradual progression from initial to final voltage over time.

3.2.3 Differential equation solver for an LR circuit and voltage source

```
#include <iostream>
#include <cmath>
using namespace std;

/* Schematic:
  +-----R-----+
  |           |
  Vs          L   Vl = L dIl/dt
  |           |
  +-----------+
*/

int main (void)
{
  double Vl, L = 50.0, R = 100.0, Vs = 100.0, Vr, Il = 0.0, dIl, t, dt = 0.0001;

  cout << "Time , Il" << endl;

  for (t = 0 ; t < 2.0 ; t = t + dt)
  {
    Vr = Il * R;          // Ohm's Law
    Vl = Vs - Vr;         // Kirchhoff's Voltage Law
    dIl = dt * (Vl / L);  // V = L dI/dt "Ohm's Law" for capacitors
    Il = Il + dIl;        // Integrating inductor current from small changes
    cout << t << " , " << Il << endl;
  }

  return 0;
}
```

With the initialized values shown in the `double` line of this program, it will compute inductor currents at 0.1 millisecond intervals assuming zero initial inductor current and charging toward 1 Ampere (i.e. the voltage source value `Vs` of 100 Volts divided by 100 Ohms). Of course, any combination of `Vs` and initial `Il` values will work perfectly well, as this program will show the gradual progression from initial to final current over time.

3.2.4 Differential equation solver for an LR circuit and current source

```
#include <iostream>
#include <cmath>
using namespace std;

/* Schematic:
  +-----+-----+
  |     |     |
  Is    R     L   Vl = L dIl/dt
  |     |     |
  +-----+-----+
*/

int main (void)
{
  double Vl, L = 50.0, R = 100.0, Is = 100e-3, Ir, Il = 0.0, dIl, t, dt = 0.0001;

  cout << "Time , Il" << endl;

  for (t = 0 ; t < 2.0 ; t = t + dt)
  {
    Ir = Is - Il;         // Kirchhoff's Current Law
    Vl = Ir * R;          // Ohm's Law
    dIl = dt * (Vl / L);  // V = L dI/dt "Ohm's Law" for capacitors
    Il = Il + dIl;        // Integrating inductor current from small changes
    cout << t << " , " << Il << endl;
  }

  return 0;
}
```

With the initialized values shown in the `double` line of this program, it will compute inductor currents at 0.1 millisecond intervals assuming zero initial inductor current and charging toward 100 milliAmperes. Of course, any combination of `Is` and initial `Il` values will work perfectly well, as this program will show the gradual progression from initial to final current over time.

Chapter 4

Animations

Some concepts are much easier to grasp when seen in *action*. A simple yet effective form of animation suitable to an electronic document such as this is a "flip-book" animation where a set of pages in the document show successive frames of a simple animation. Such "flip-book" animations are designed to be viewed by paging forward (and/or back) with the document-reading software application, watching it frame-by-frame. Unlike video which may be difficult to pause at certain moments, "flip-book" animations lend themselves very well to individual frame viewing.

4.1 Differentiation and integration animated

The following animation shows the calculus concepts of differentiation and integration (with respect to time) applied to the filling and draining of a water tank.

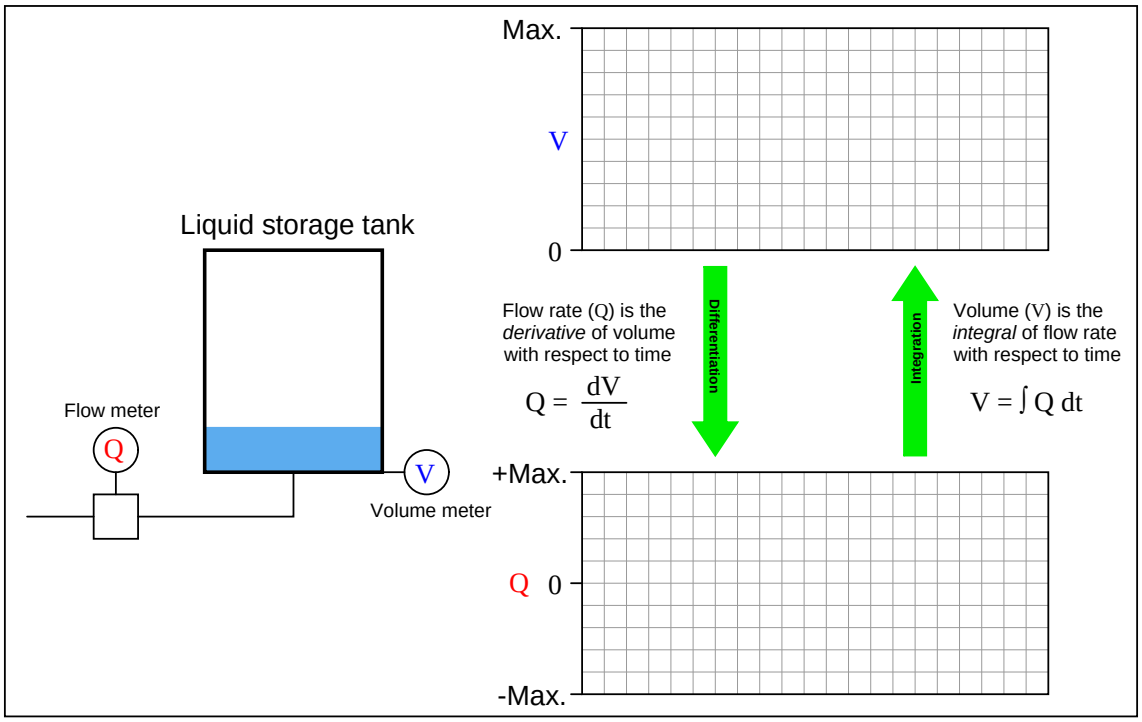
The animation shows two graphs relating to the water storage tank: one showing the volume of stored water in the tank (V) and the other showing volumetric flow rate in and out of the tank (Q). We know from calculus that volumetric flow rate is the *time-derivative* of volume:

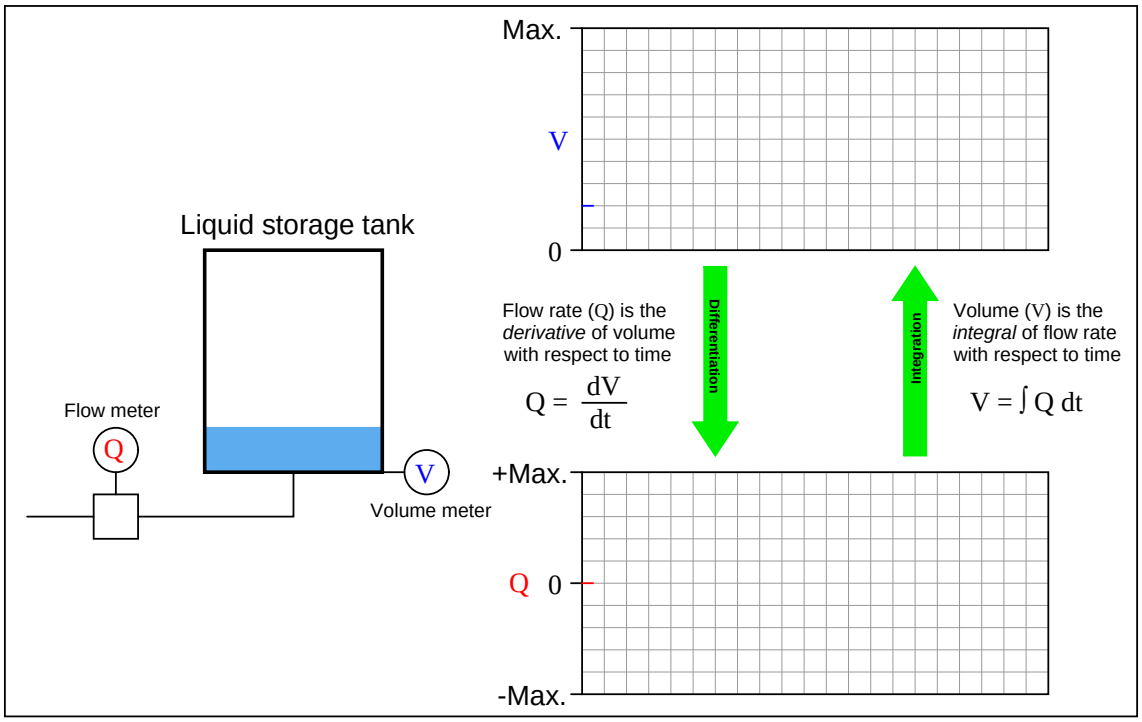
$$Q = \frac{dV}{dt}$$

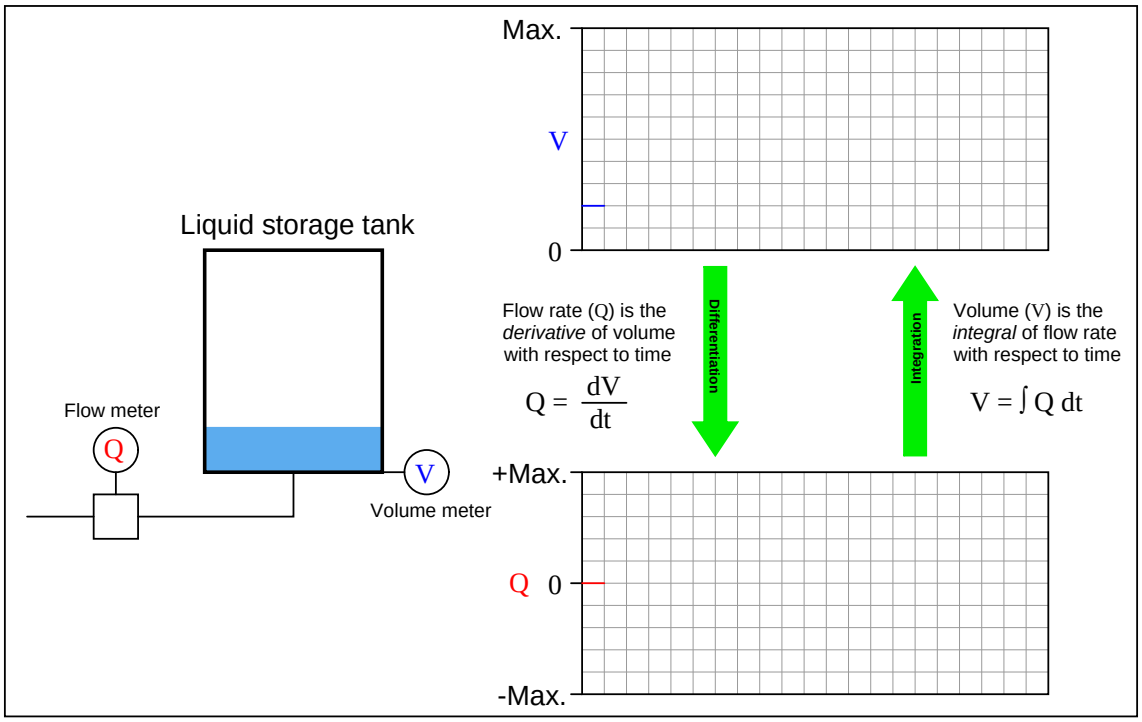
We also know that change in volume is the *time-integral* of volumetric flow rate:

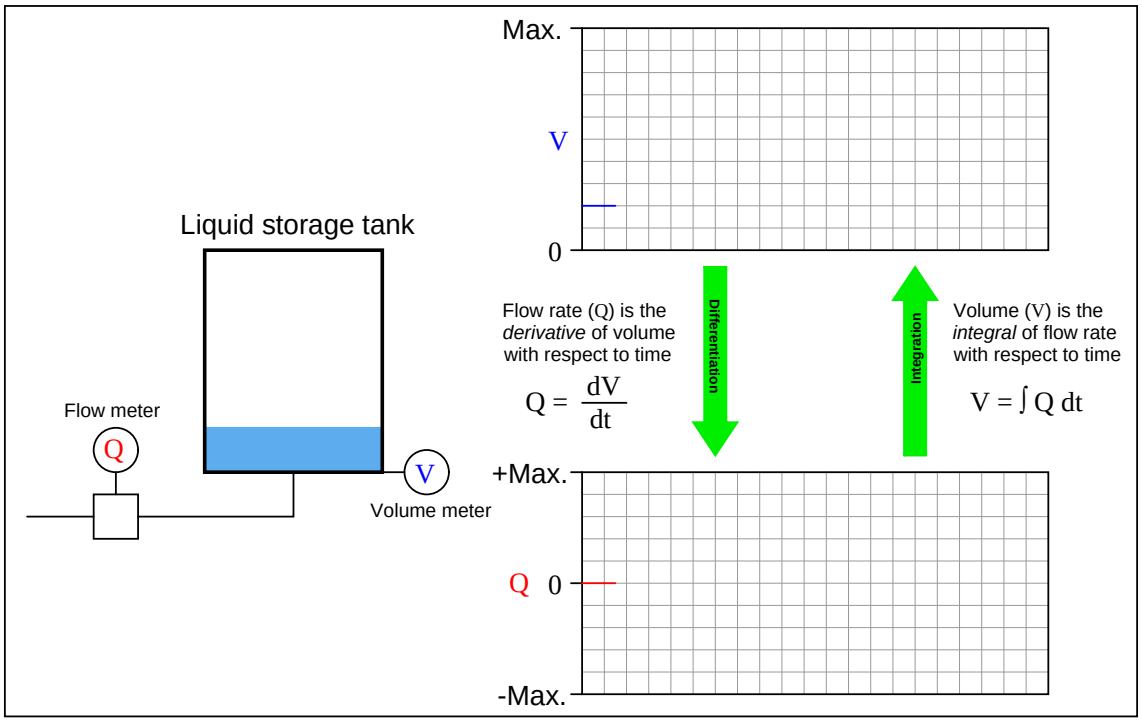
$$\Delta V = \int_{t_0}^{t_1} Q \, dt$$

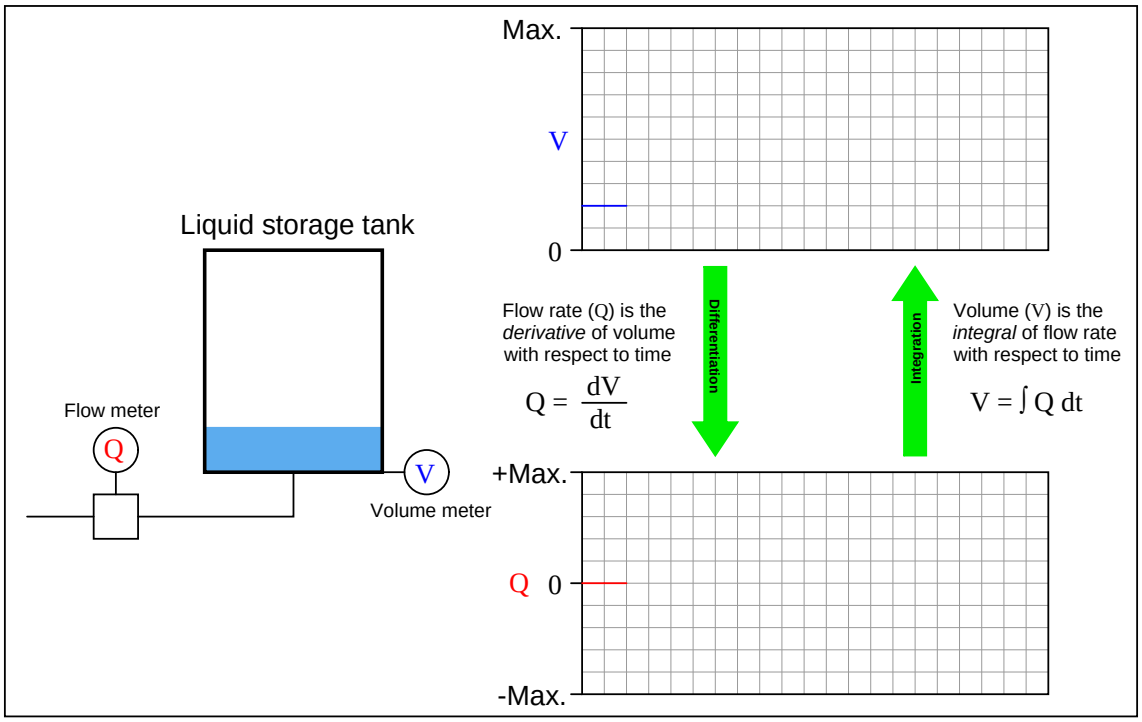
Thus, the example of a water storage tank filling and draining serves to neatly illustrate both concepts in relation to each other.

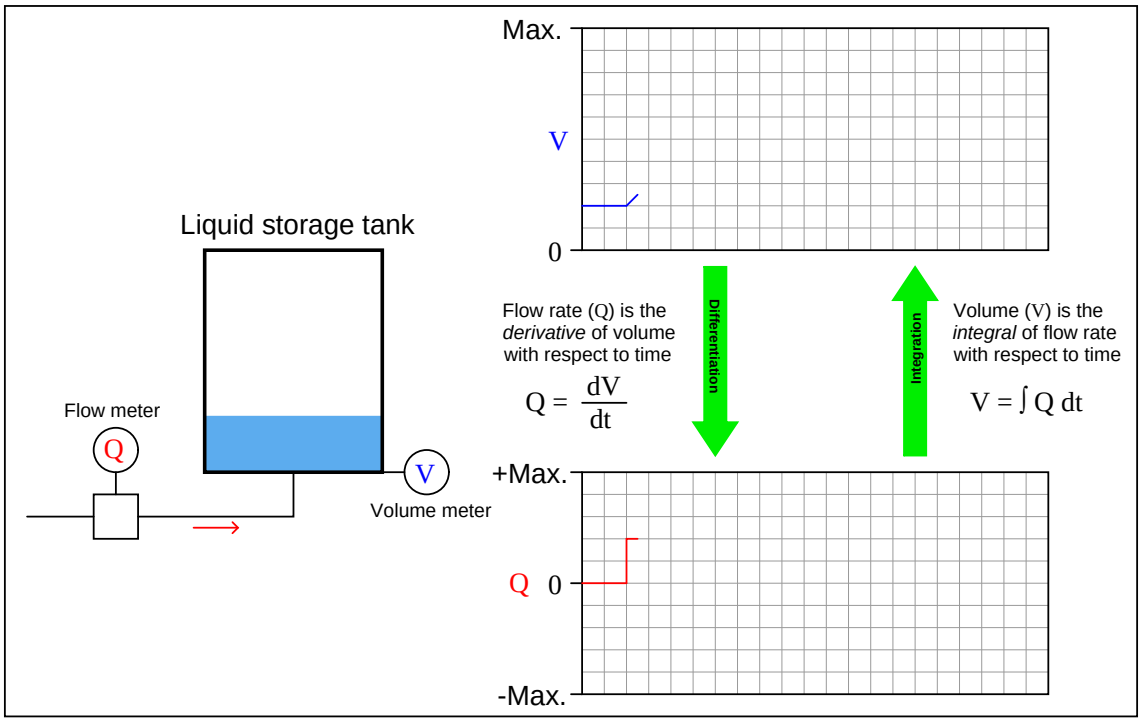

Max.
V
0
Liquid storage tank
Flow meter
Q
V
Volume meter
Flow rate (Q) is the *derivative* of volume with respect to time
Q = dV/dt
Differentiation
Integration
Volume (V) is the *integral* of flow rate with respect to time
V = ∫ Q dt
+Max.
Q
0
-Max.


Liquid storage tank
Flow meter
Q
V
Volume meter
Max.
V
0
Flow rate (Q) is the *derivative* of volume with respect to time
$Q = \frac{dV}{dt}$
Differentiation
Integration
Volume (V) is the *integral* of flow rate with respect to time
$V = \int Q\, dt$
+Max.
Q 0
-Max.


Liquid storage tank
Flow meter
Q
V
Volume meter
Max.
V
0
Flow rate (Q) is the *derivative* of volume with respect to time
Q = dV/dt
Differentiation
Integration
Volume (V) is the *integral* of flow rate with respect to time
V = ∫ Q dt
+Max.
Q 0
-Max.


Max.
V
0
Liquid storage tank
Flow meter
Q
V
Volume meter
Flow rate (Q) is the *derivative* of volume with respect to time
Q = dV/dt
Differentiation
Integration
Volume (V) is the *integral* of flow rate with respect to time
V = ∫ Q dt
+Max.
Q 0
-Max.


Max.
V
0
Liquid storage tank
Flow meter
Q
V
Volume meter
Flow rate (Q) is the *derivative* of volume with respect to time
Q = dV/dt
Differentiation
Integration
Volume (V) is the *integral* of flow rate with respect to time
V = ∫ Q dt
+Max.
Q
0
-Max.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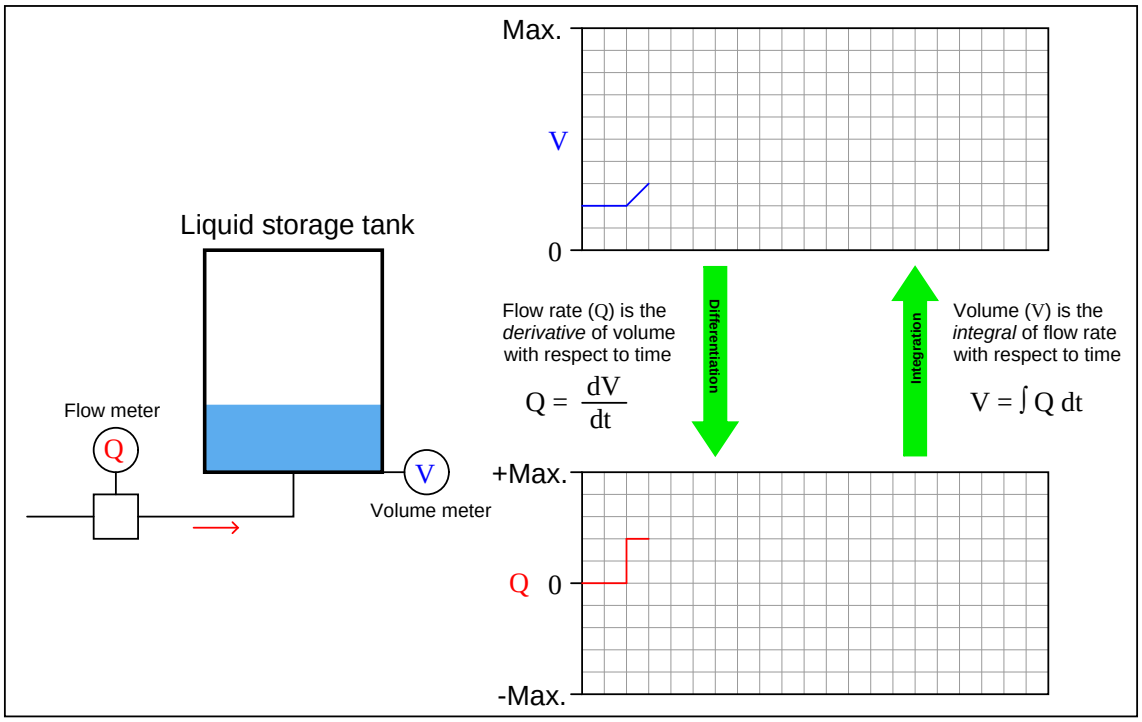
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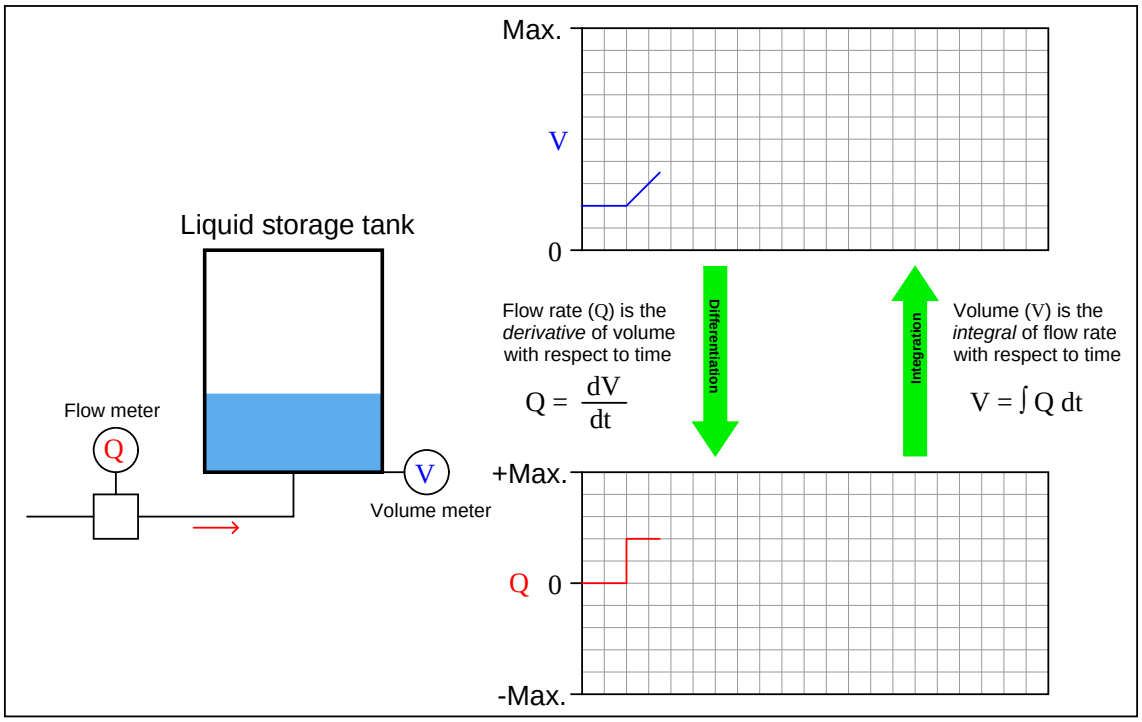
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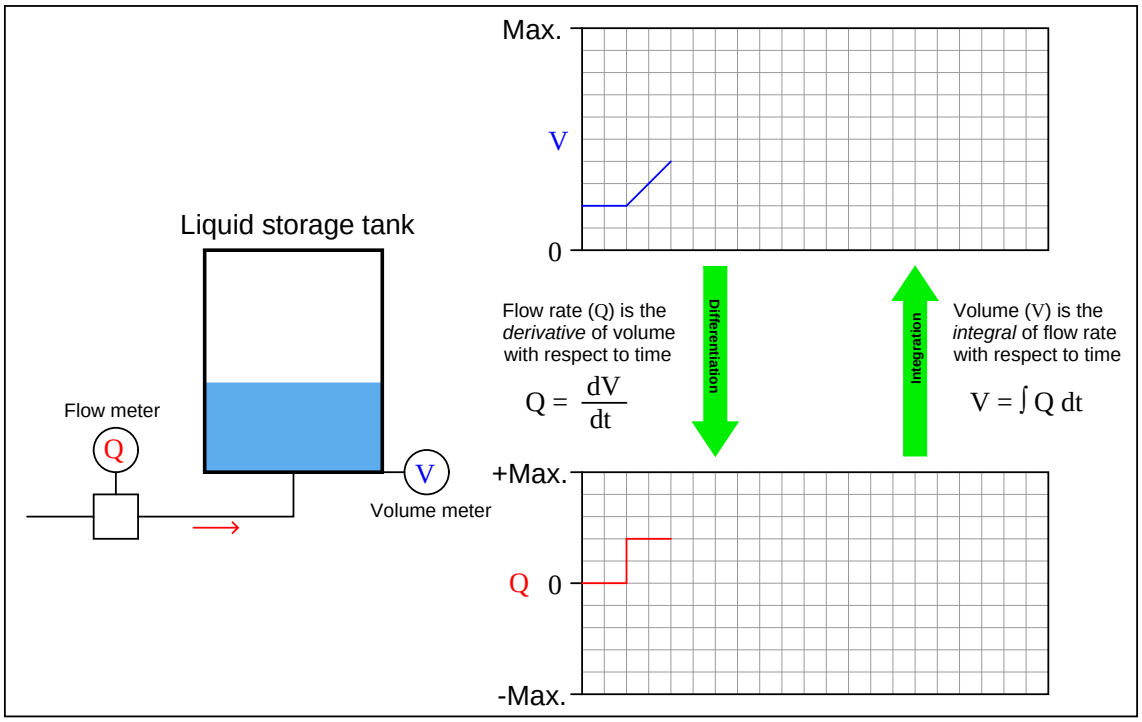
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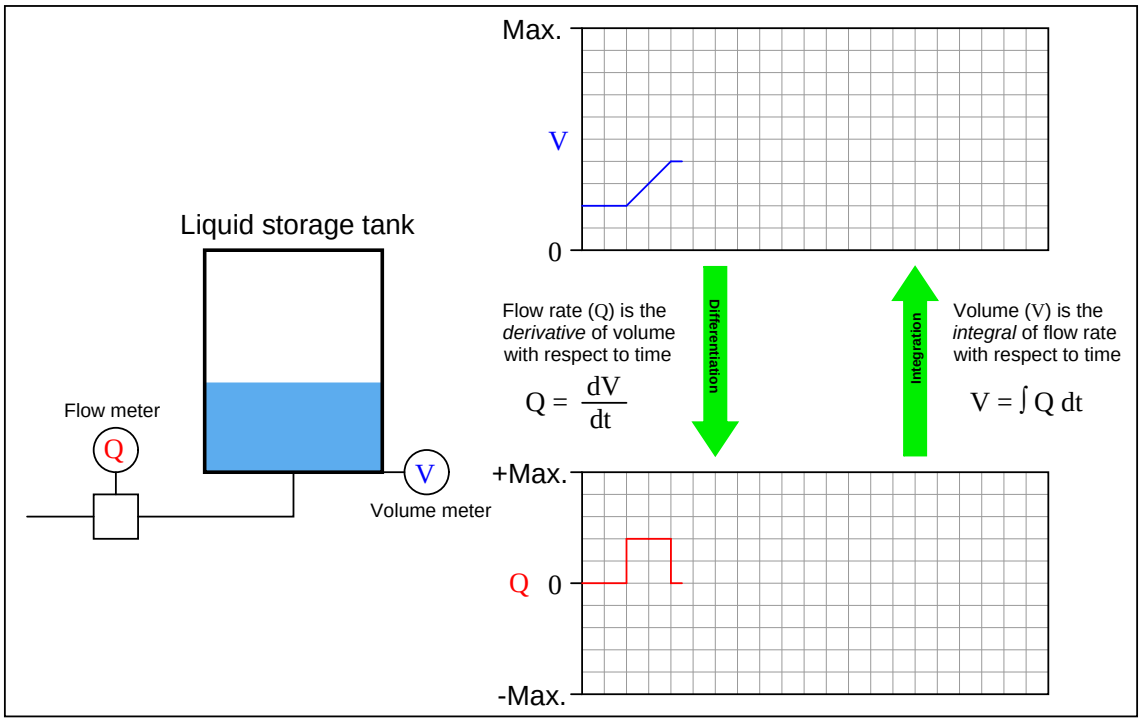
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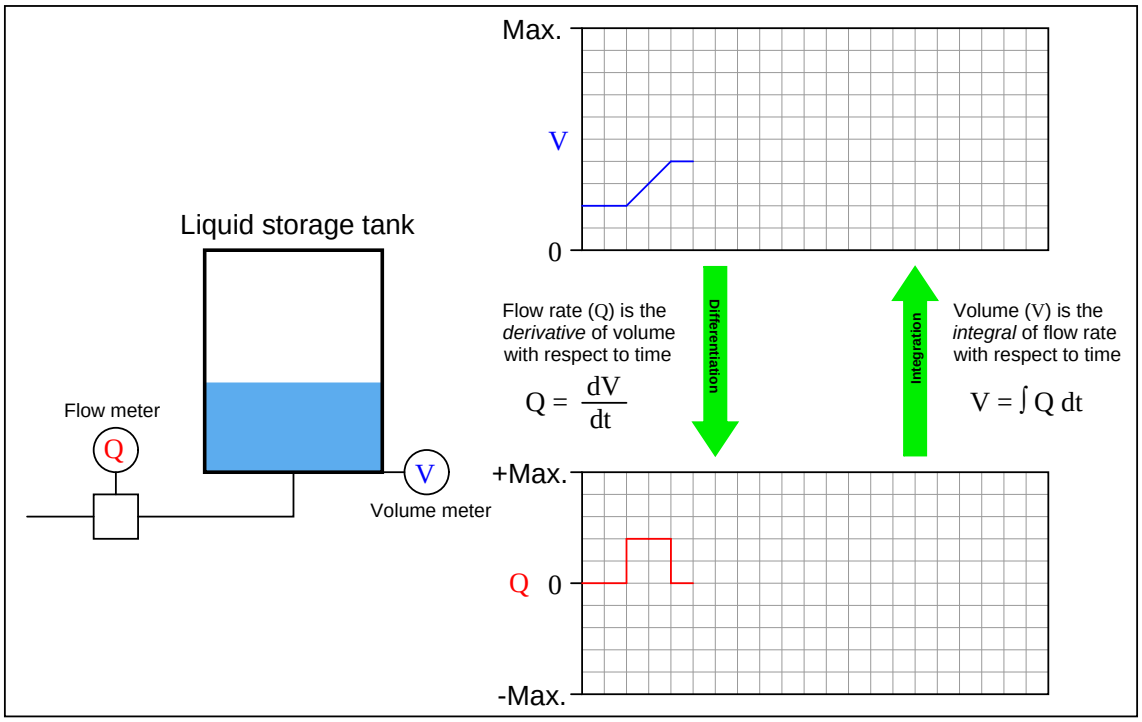
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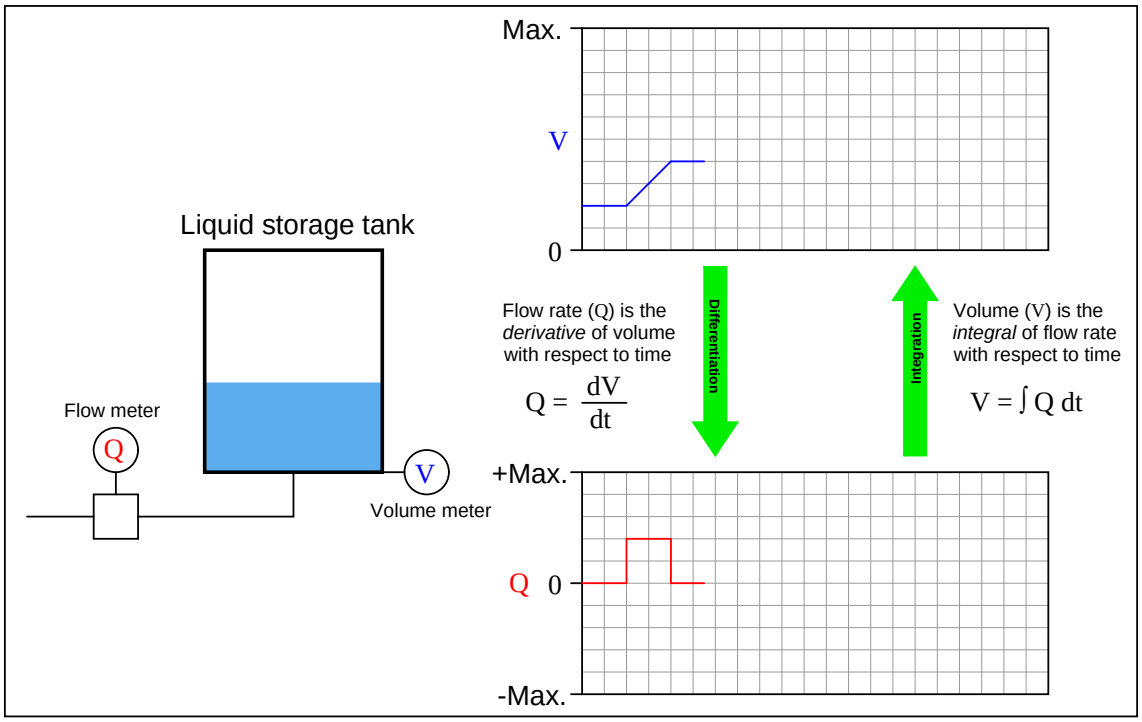
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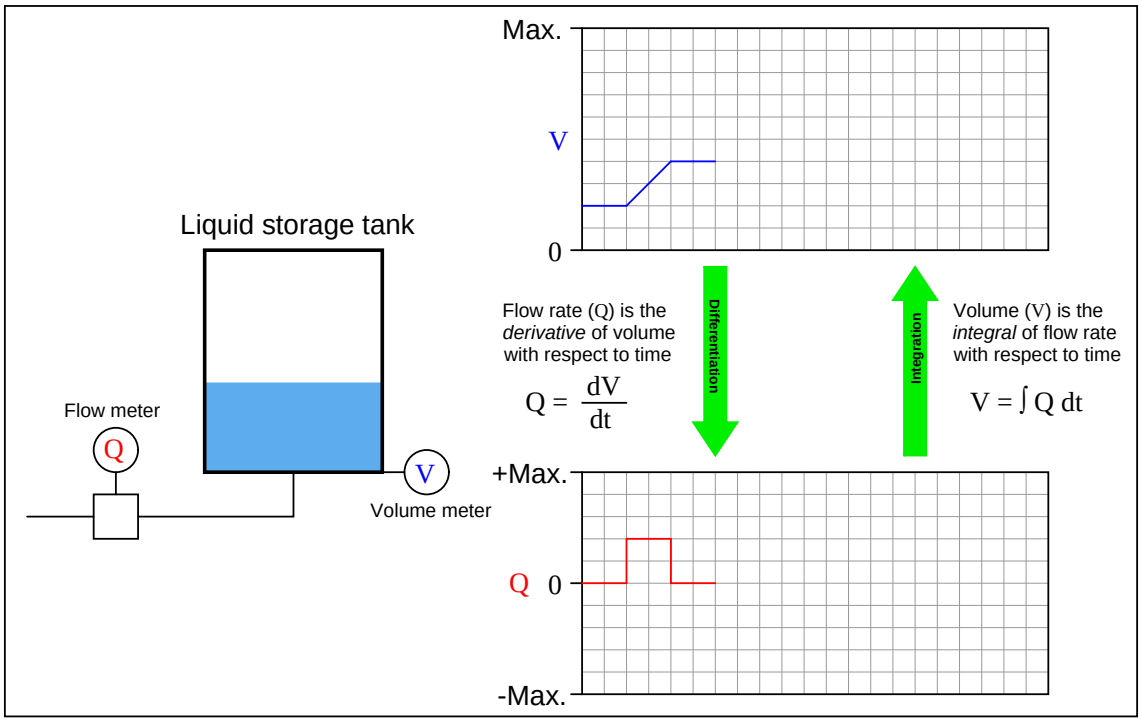
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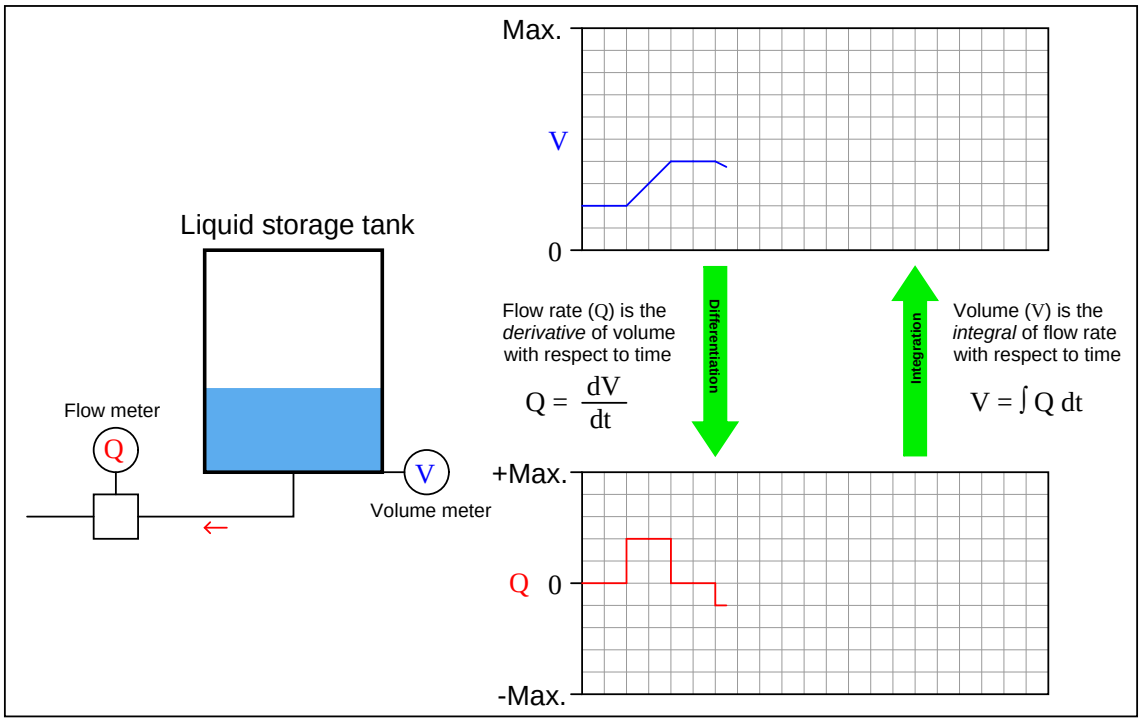
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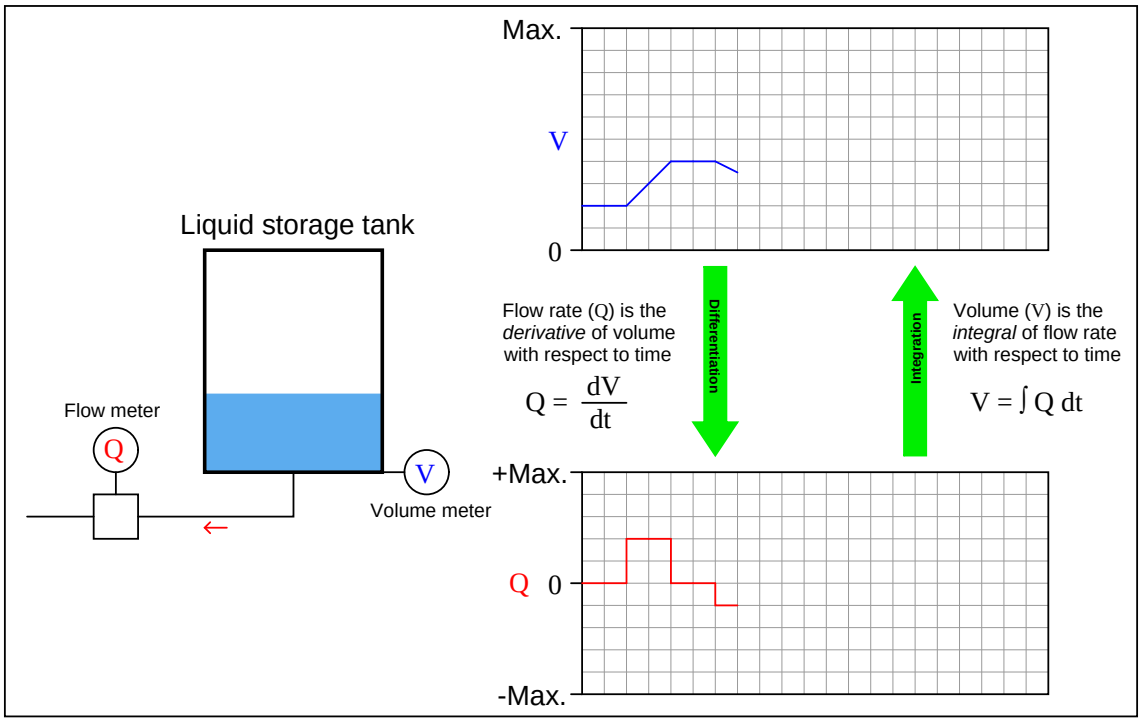
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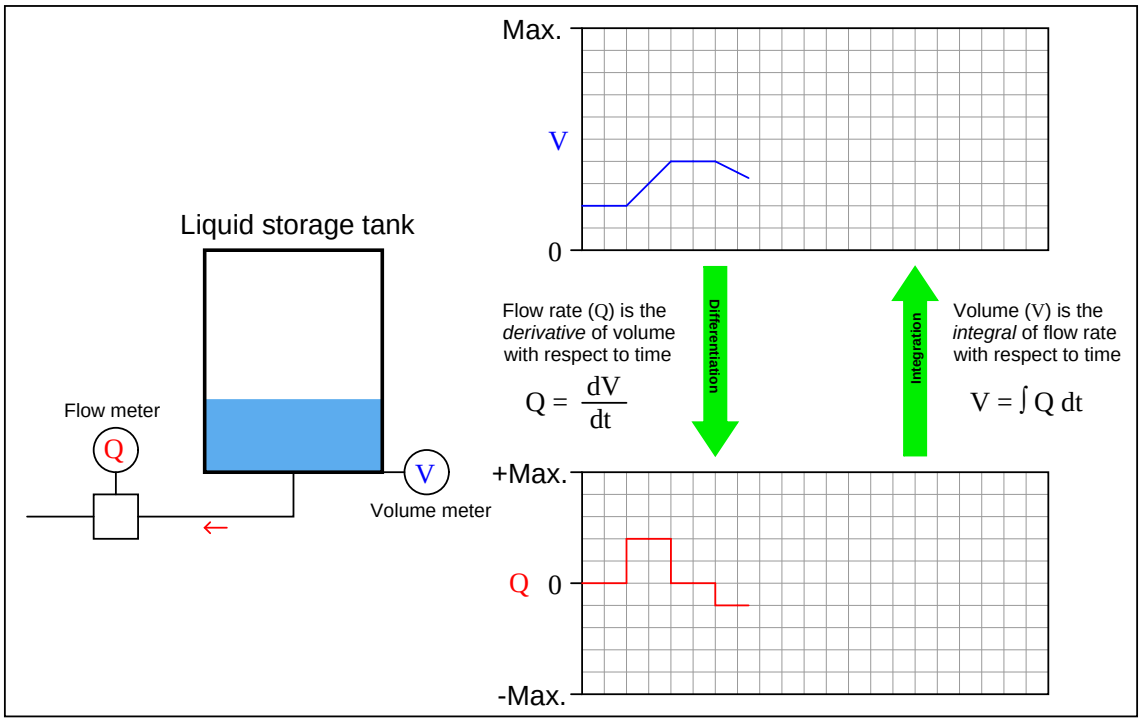
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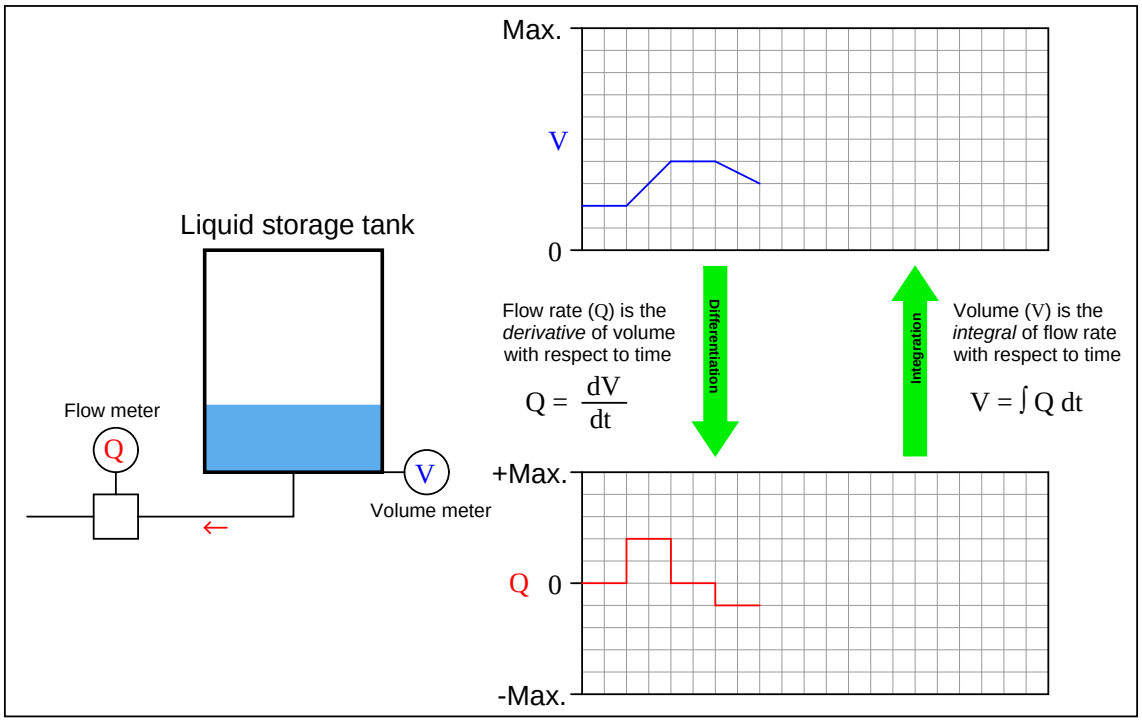
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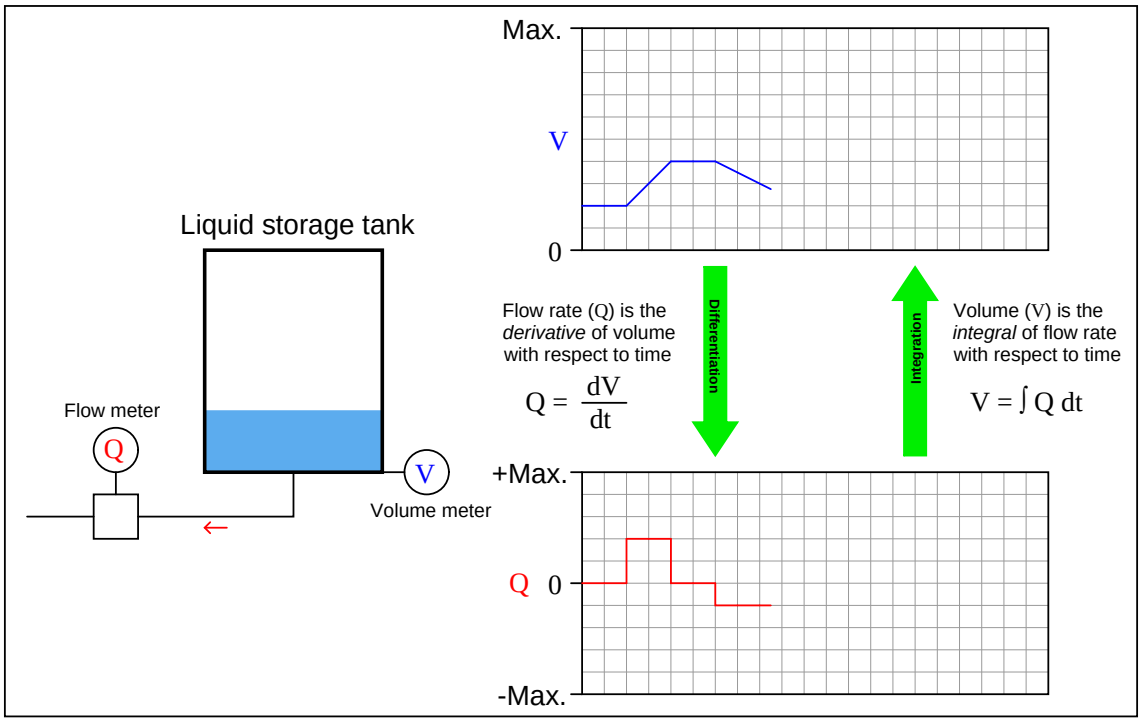
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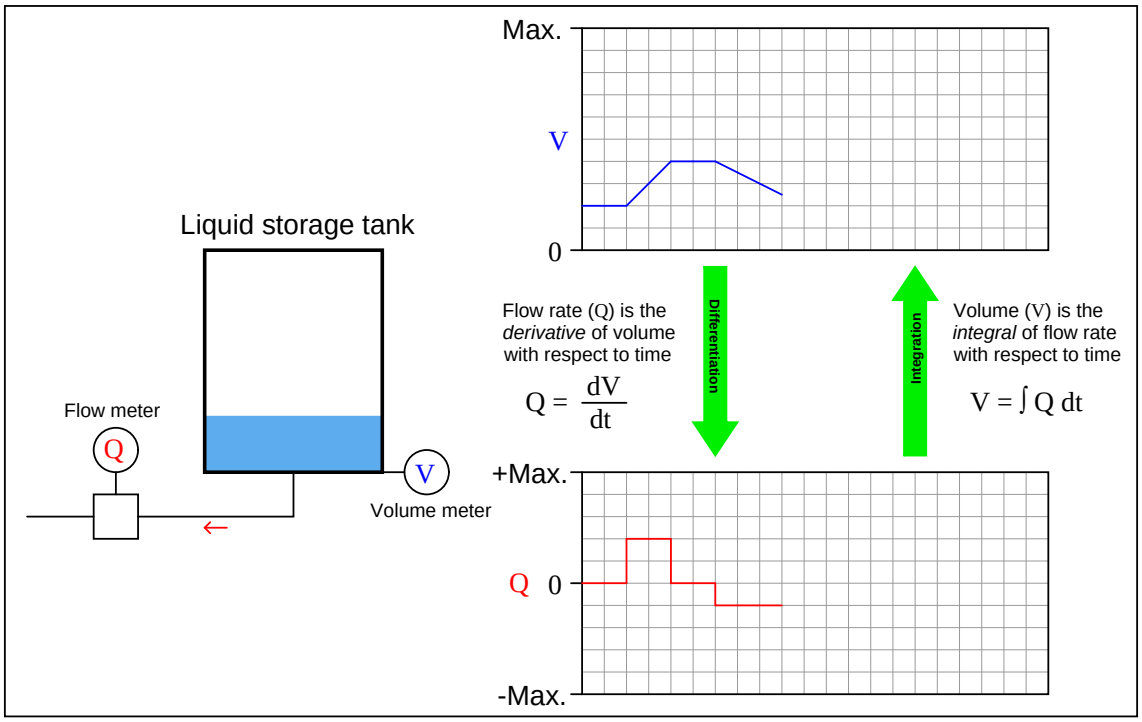
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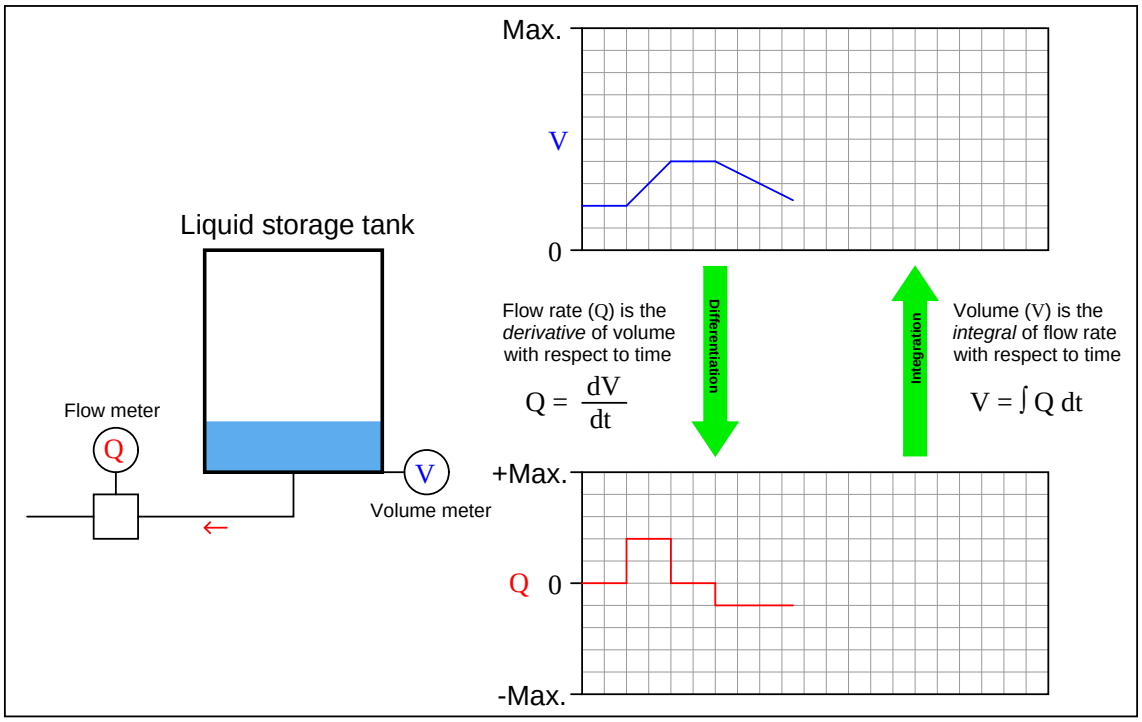
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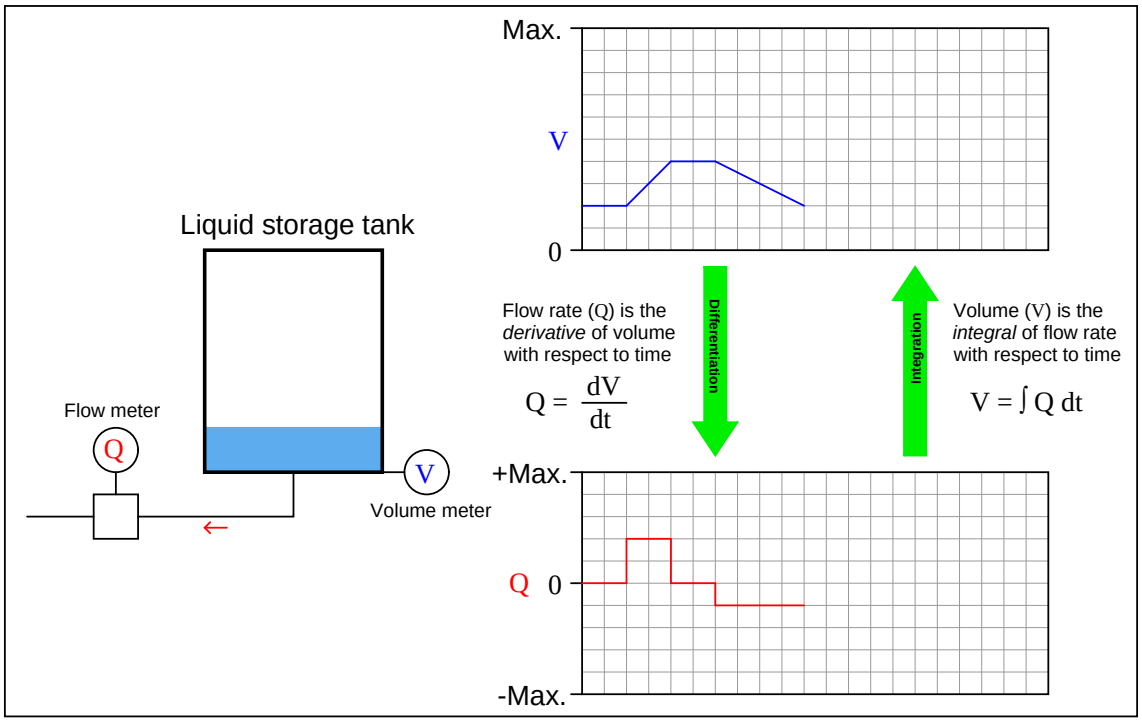
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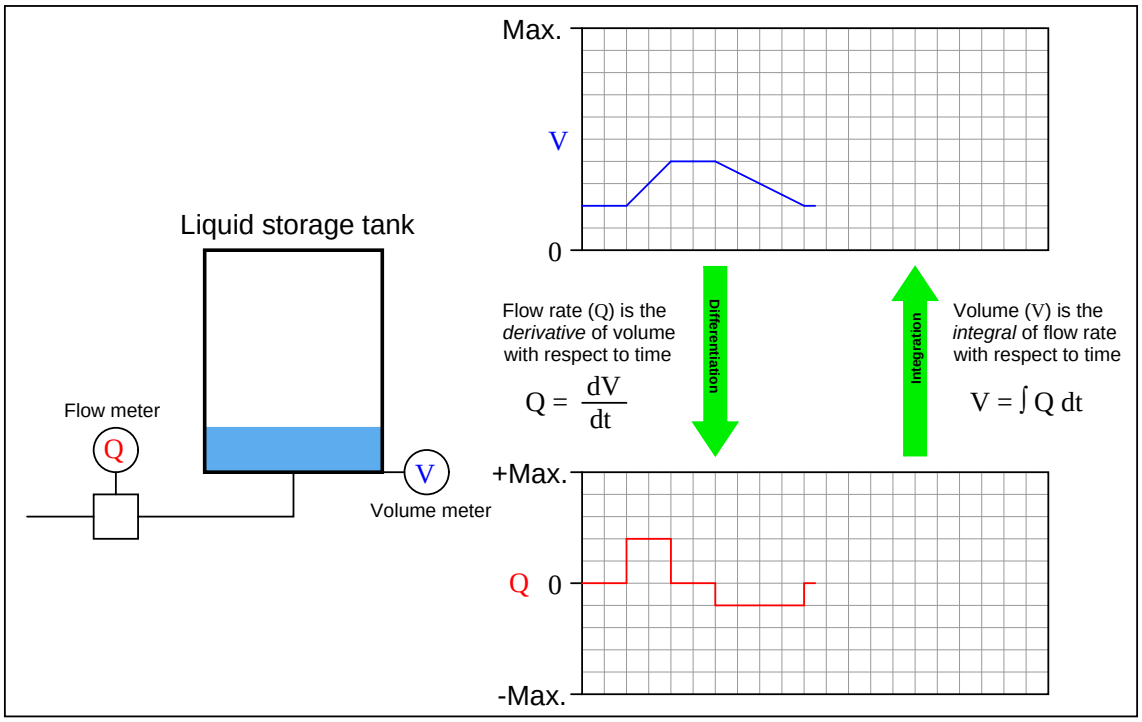
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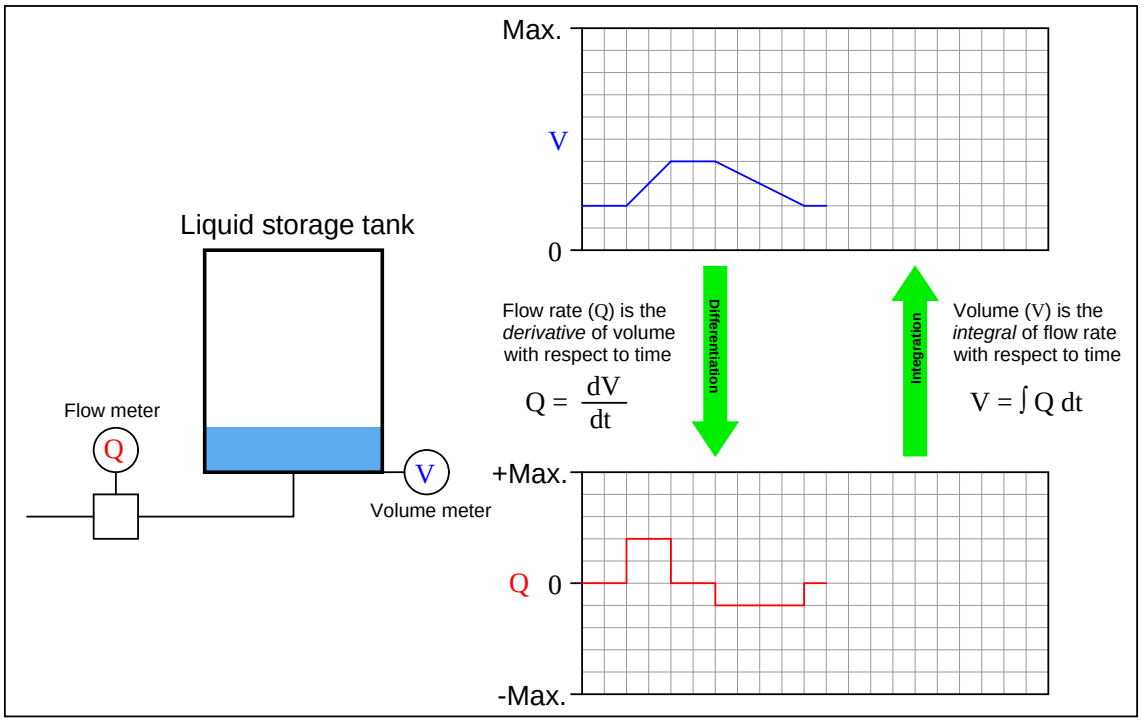
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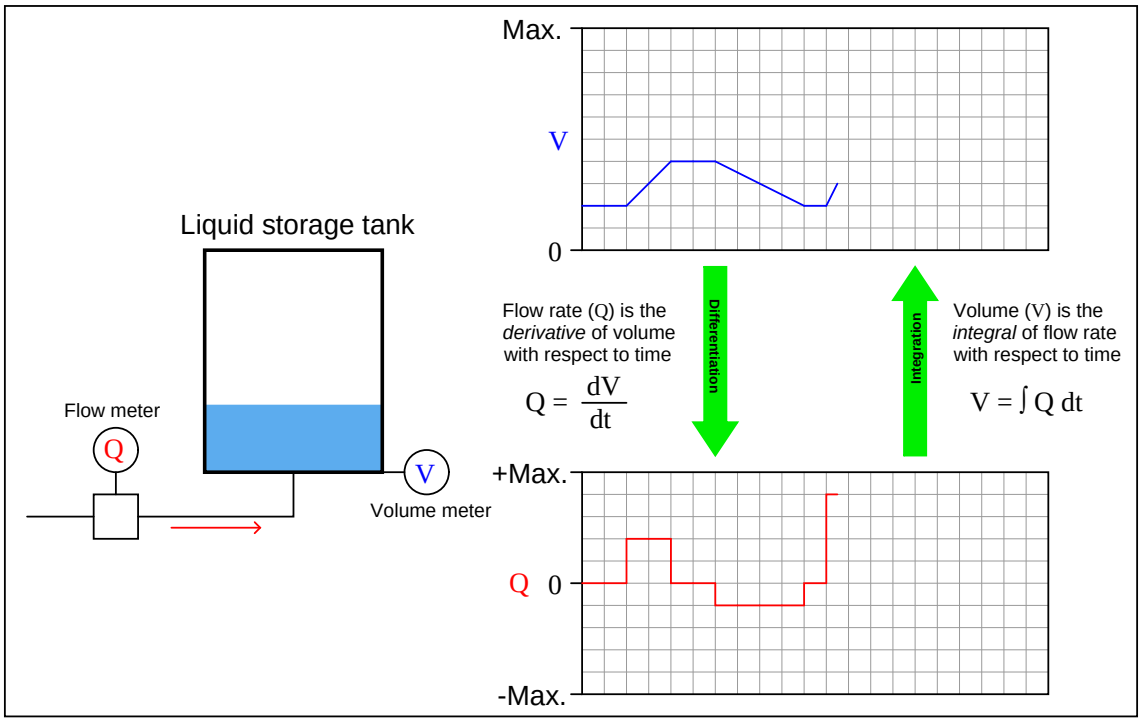
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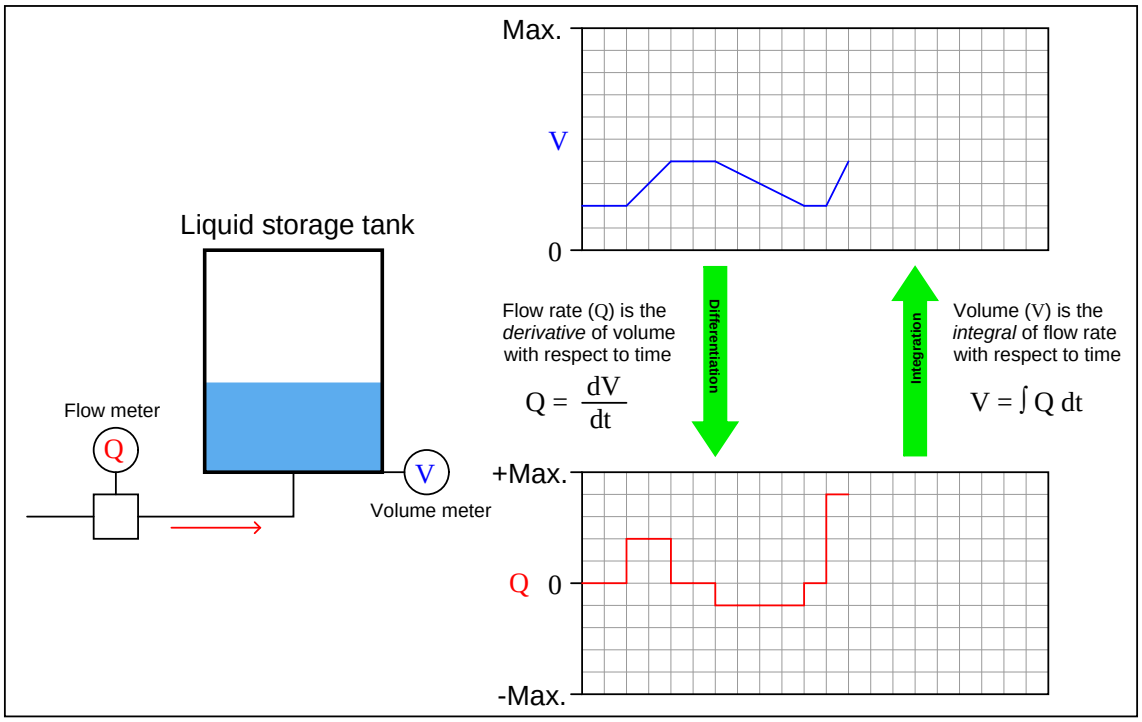
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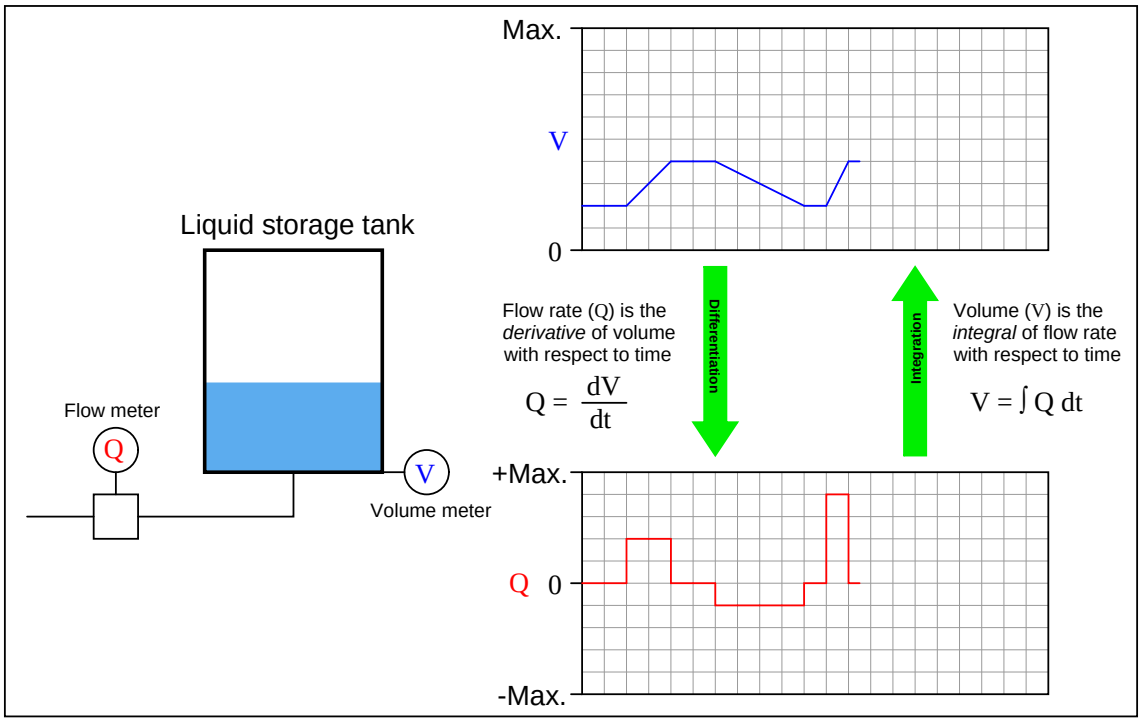
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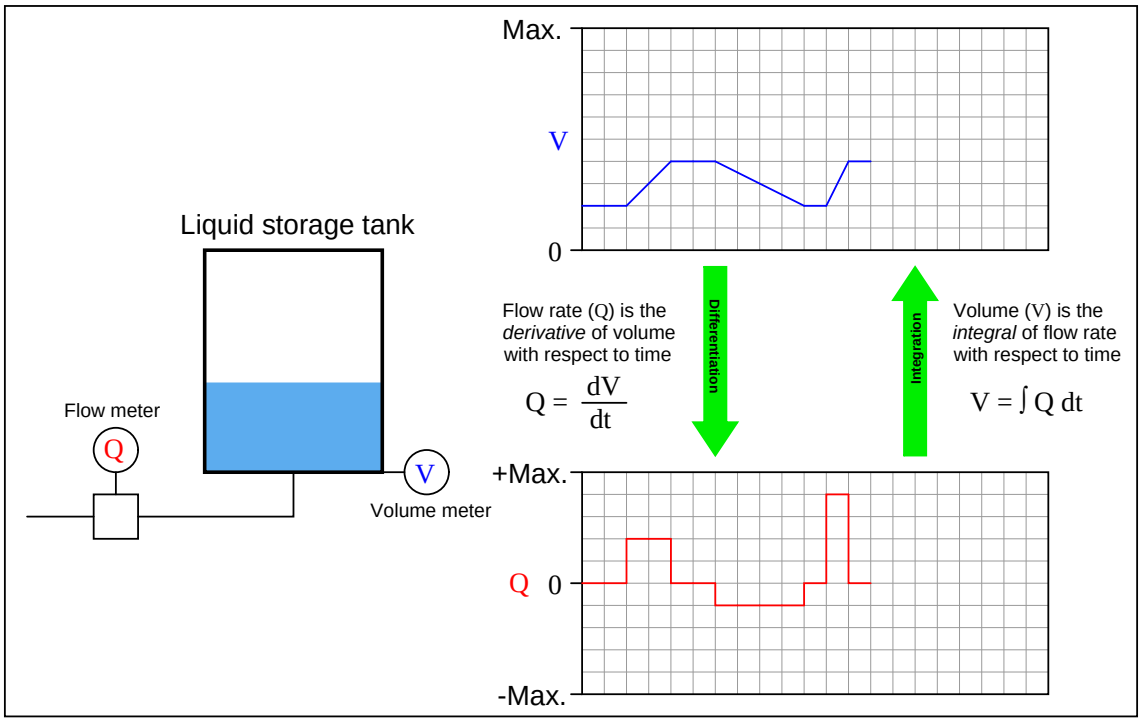
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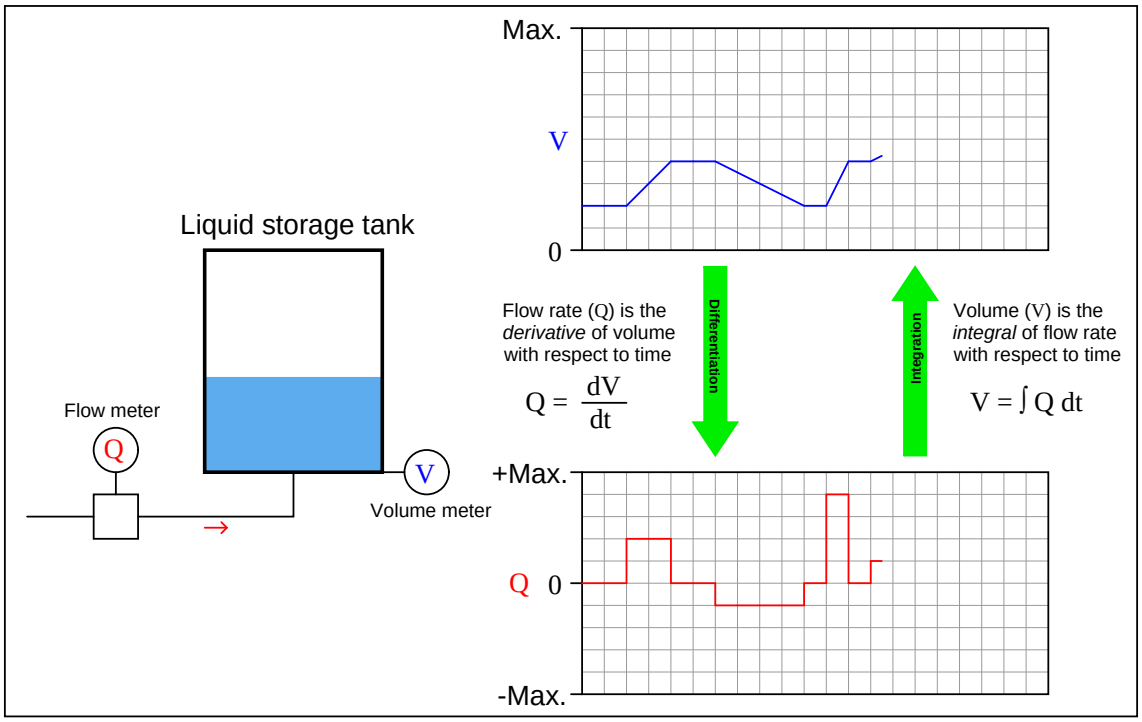
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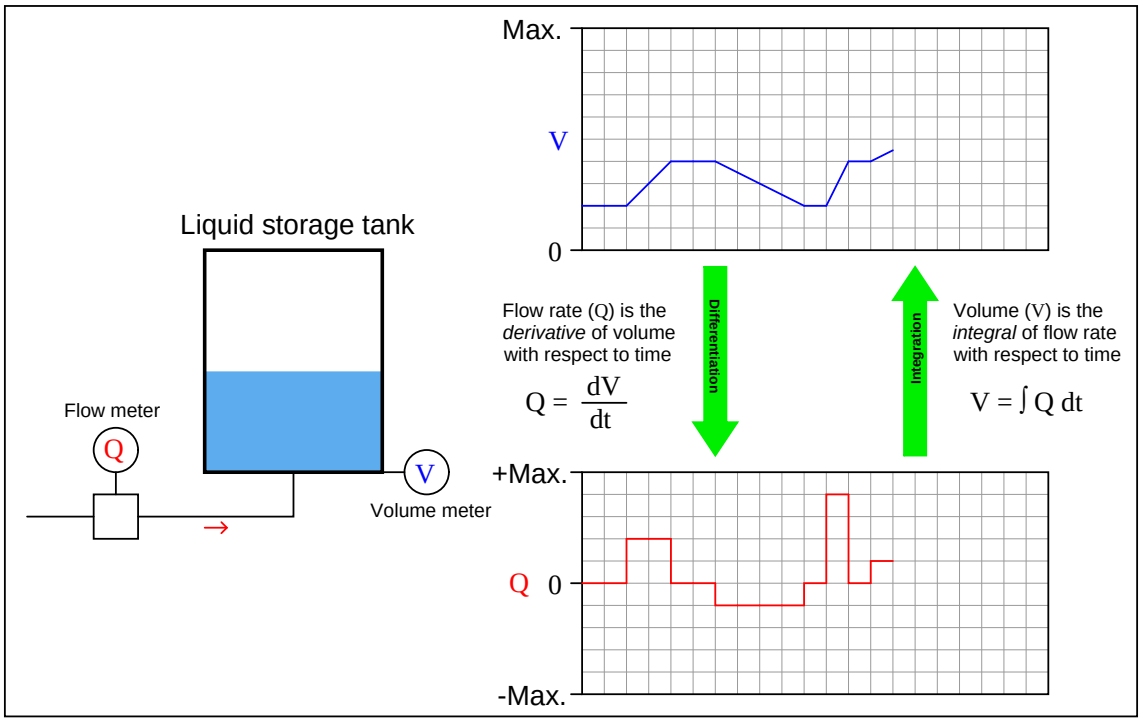
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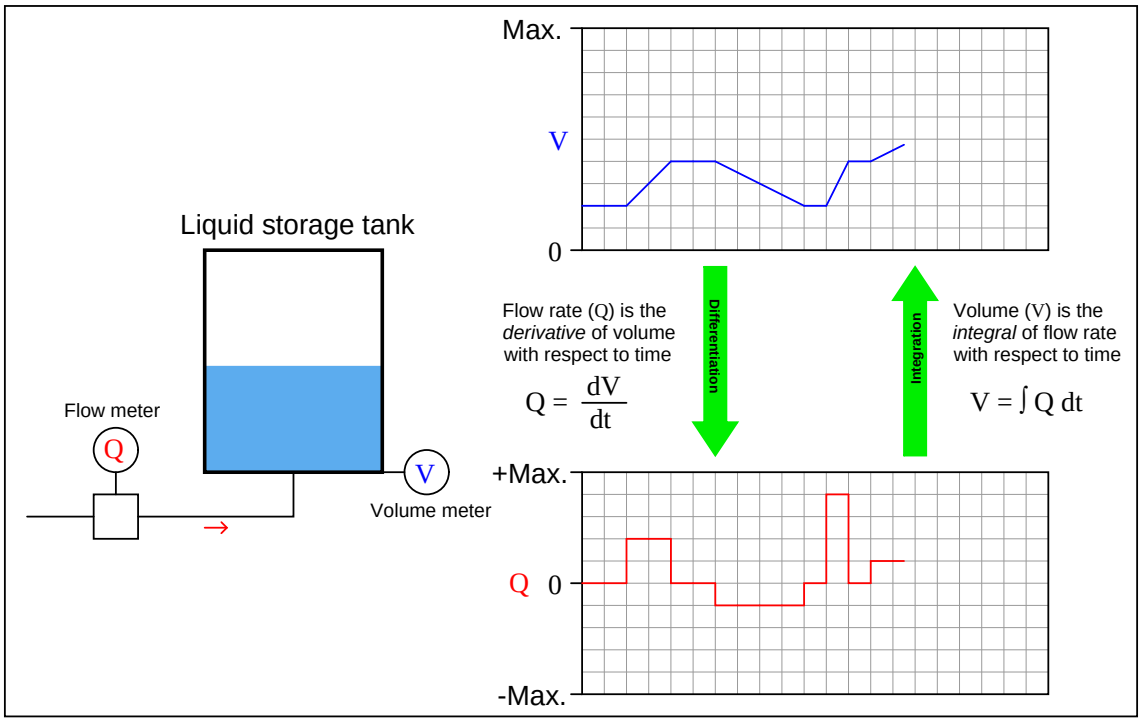
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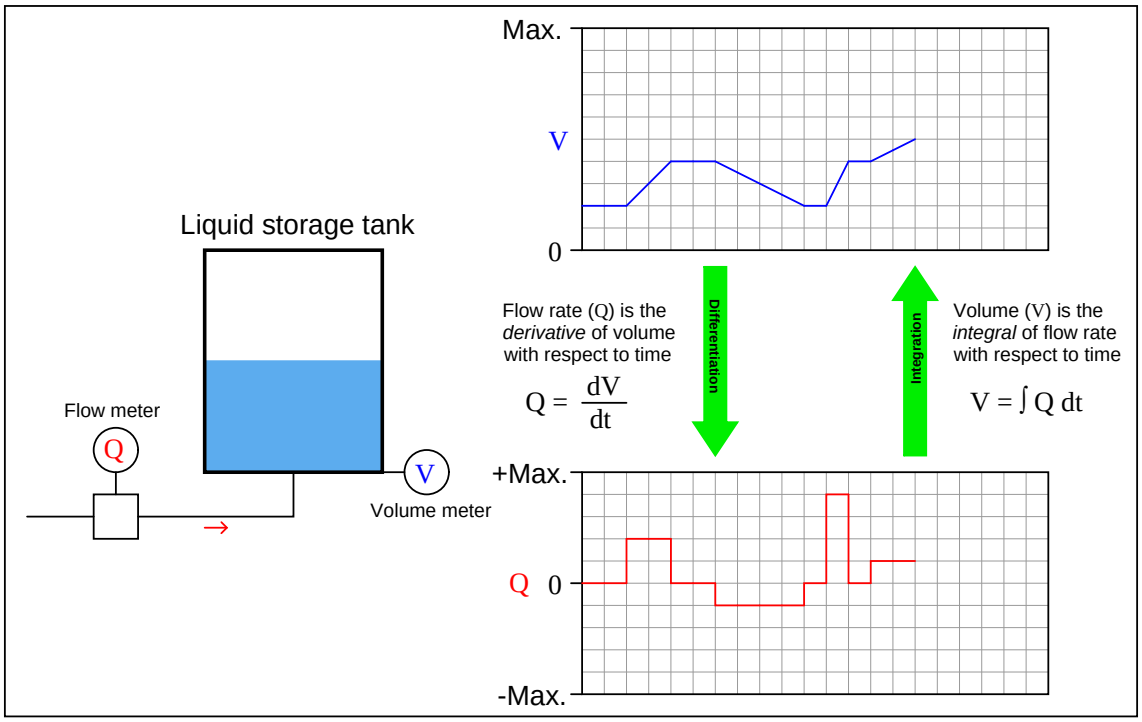
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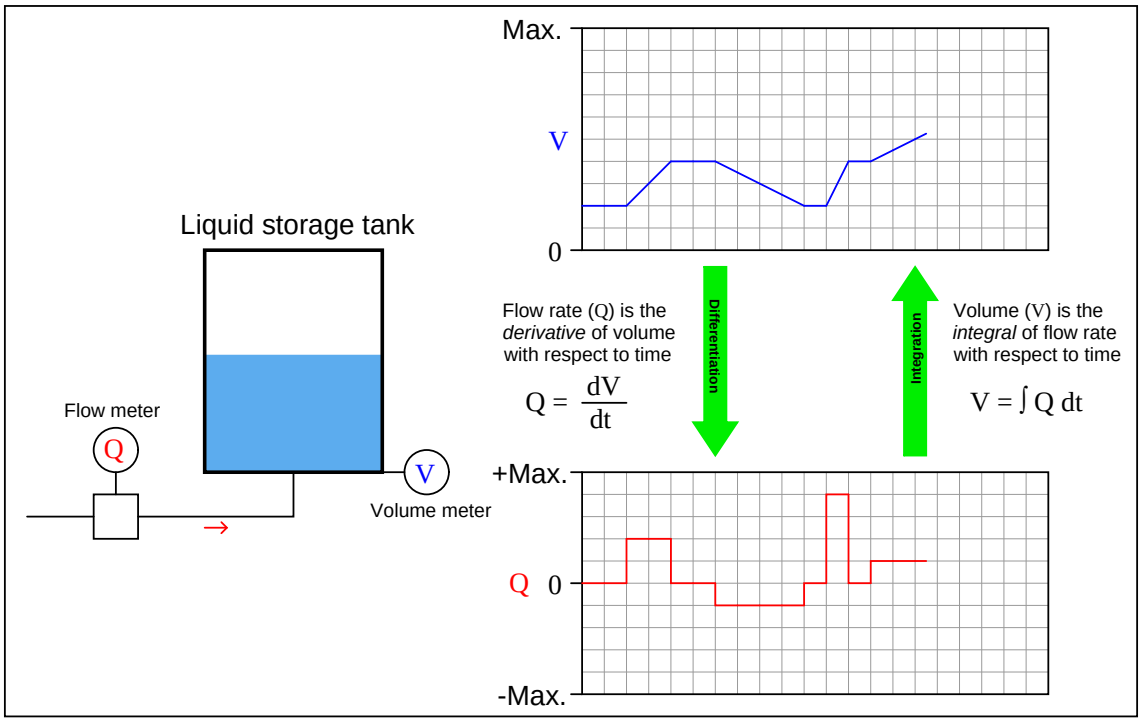
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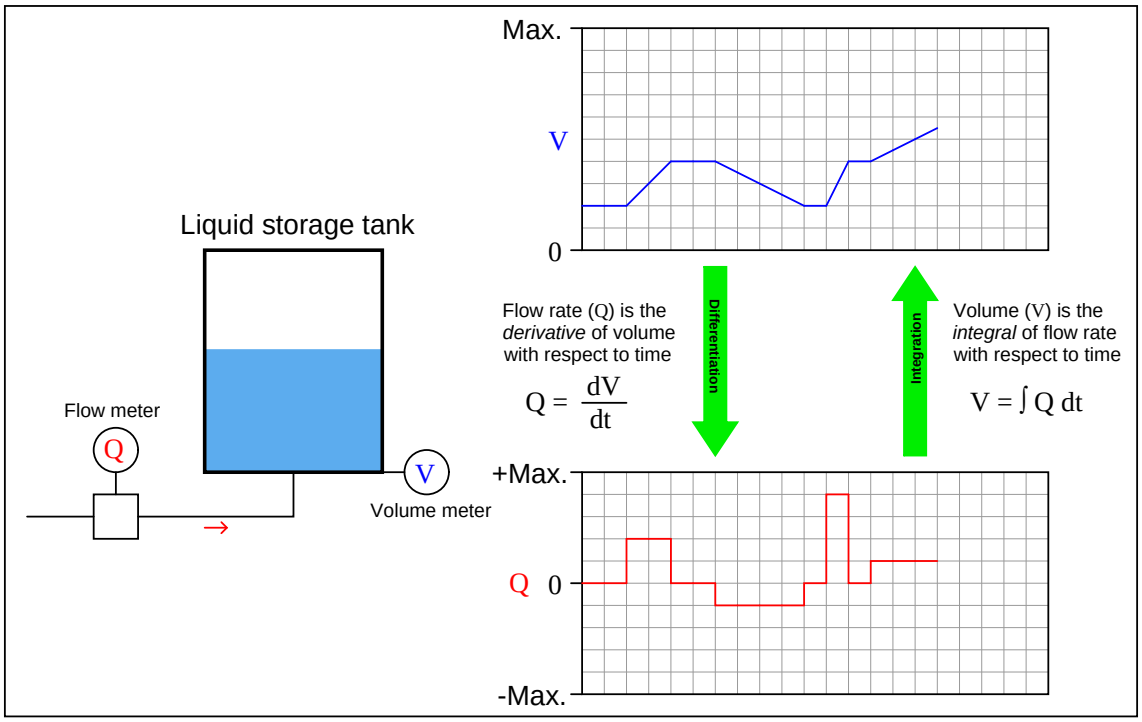
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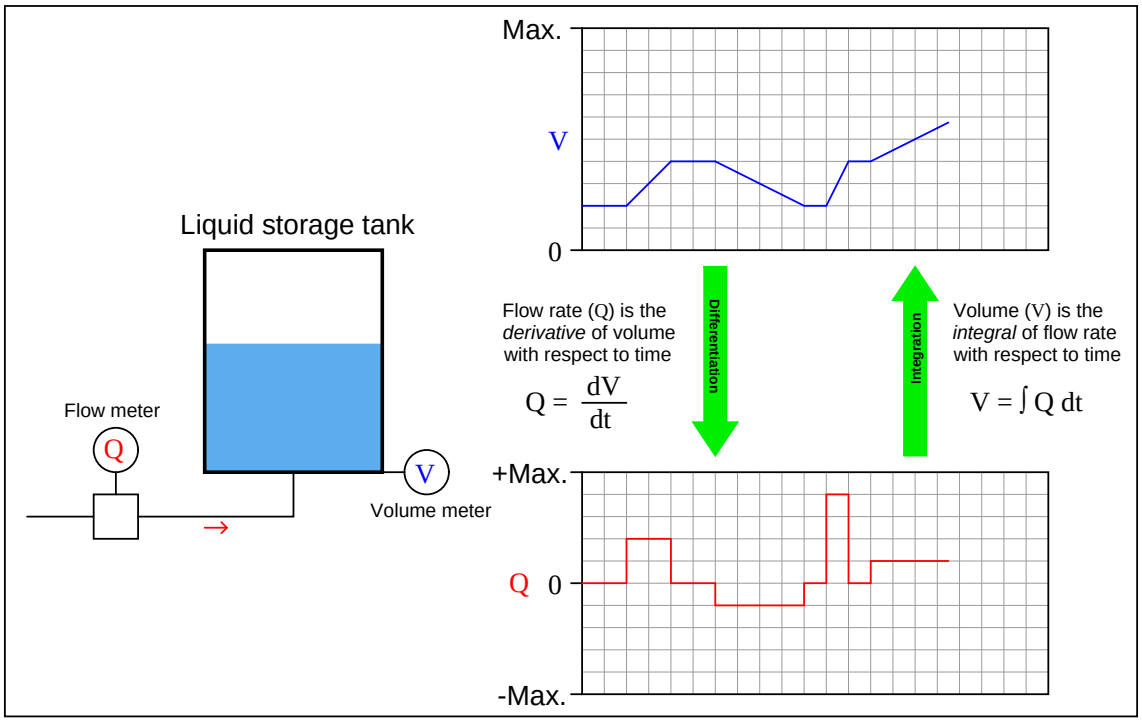
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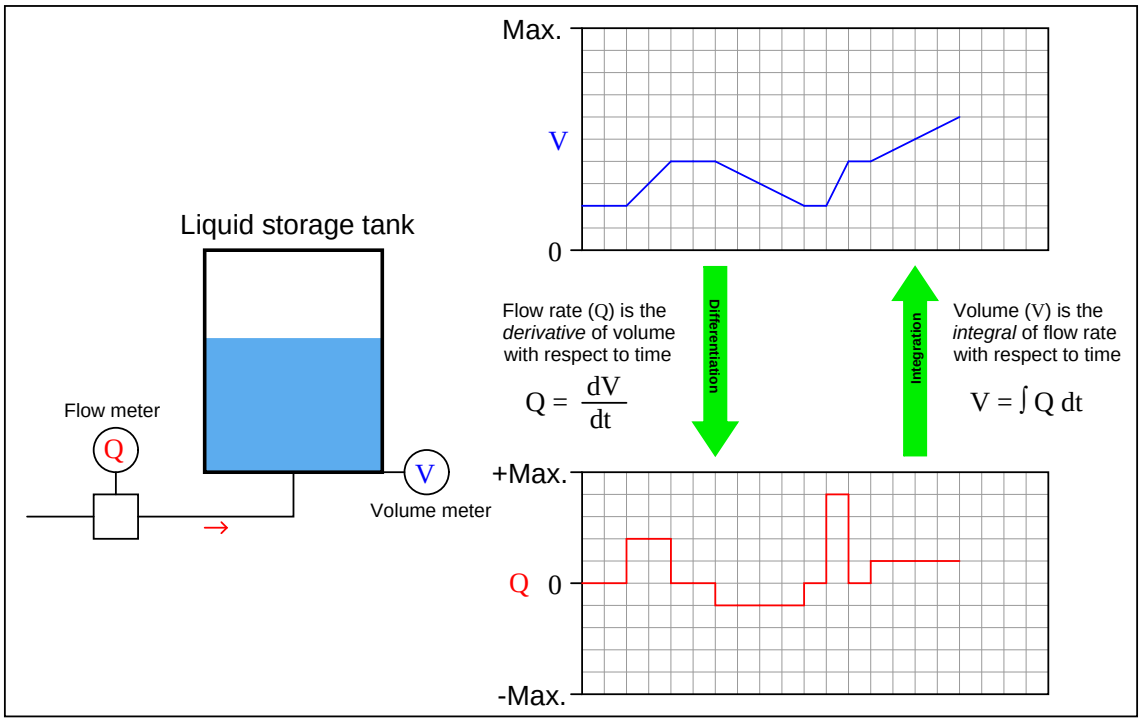
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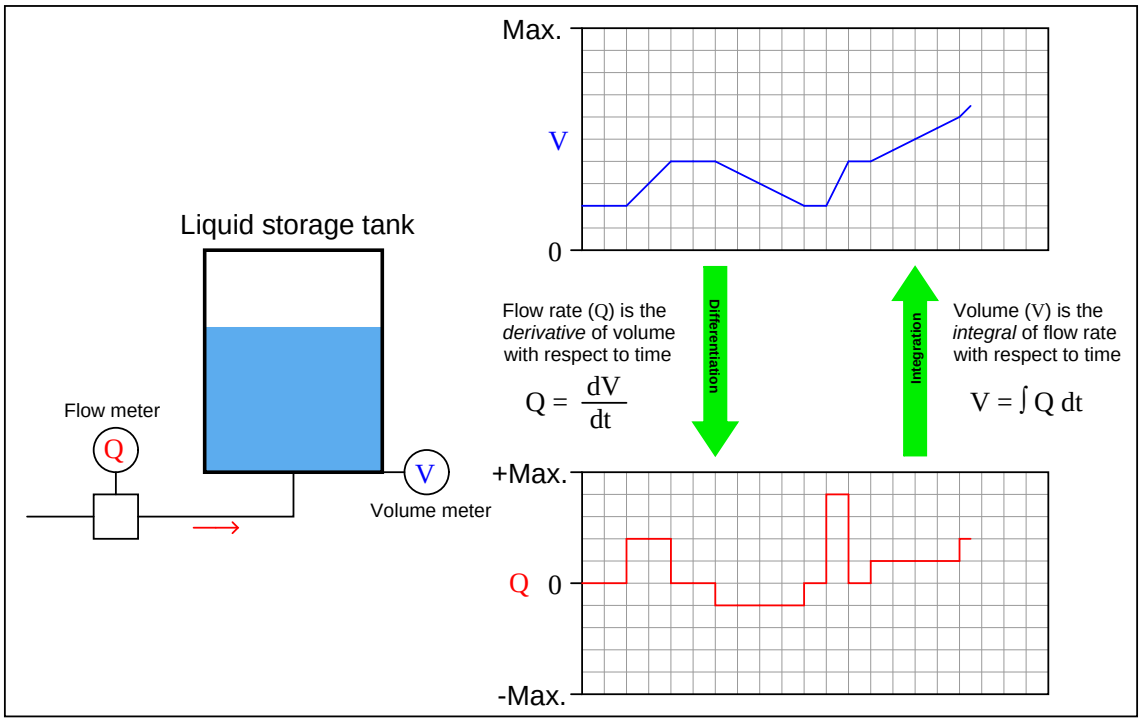
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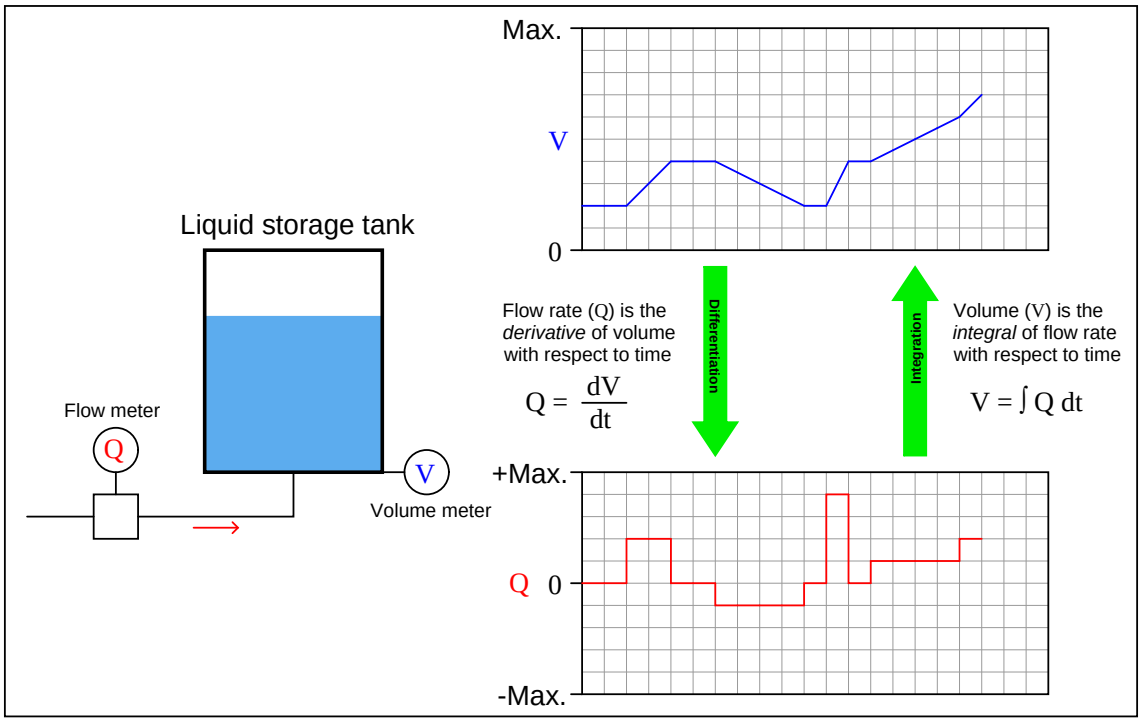
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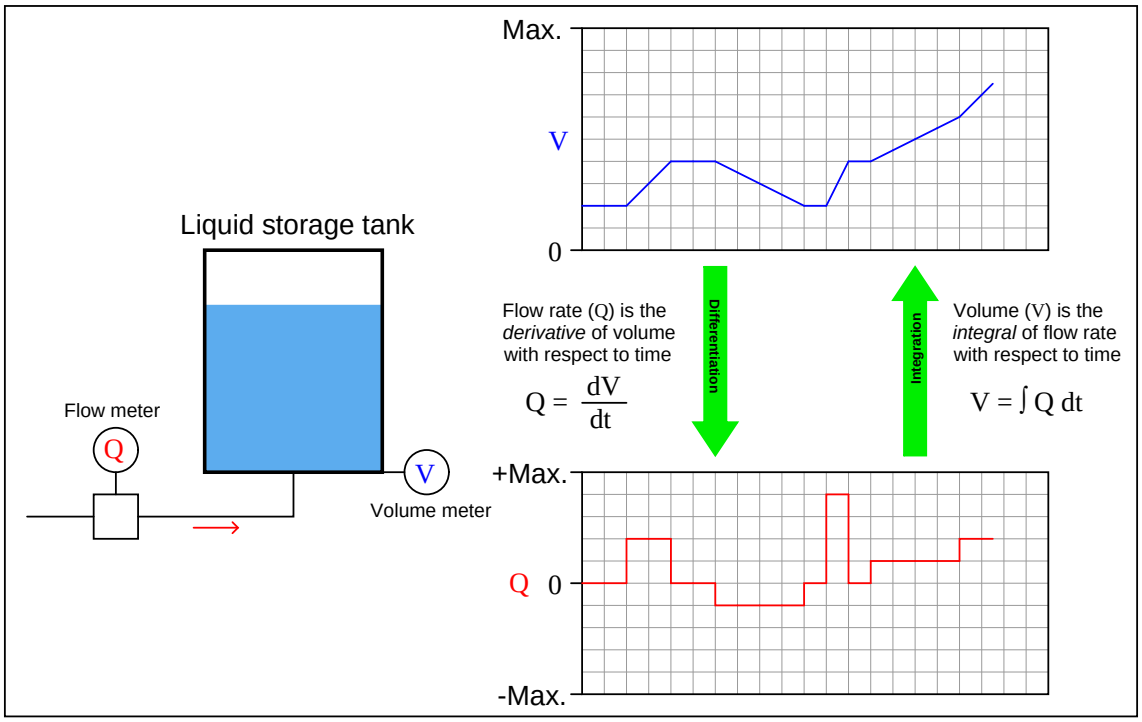
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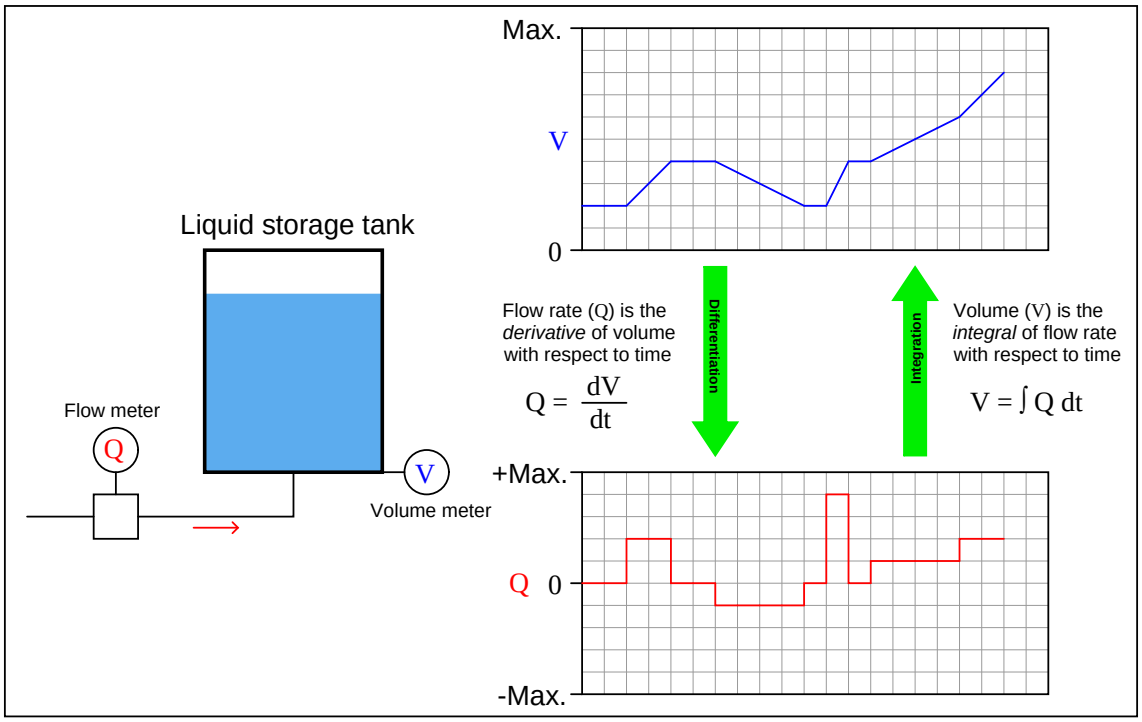
Flow rate (Q) is the *derivative* of volume with respect to time
Q = dV/dt
Differentiation
Integration
Volume (V) is the *integral* of flow rate with respect to time
V = ∫ Q dt
+Max.
Q
0
-Max.


Max.
V
0
Liquid storage tank
Flow meter
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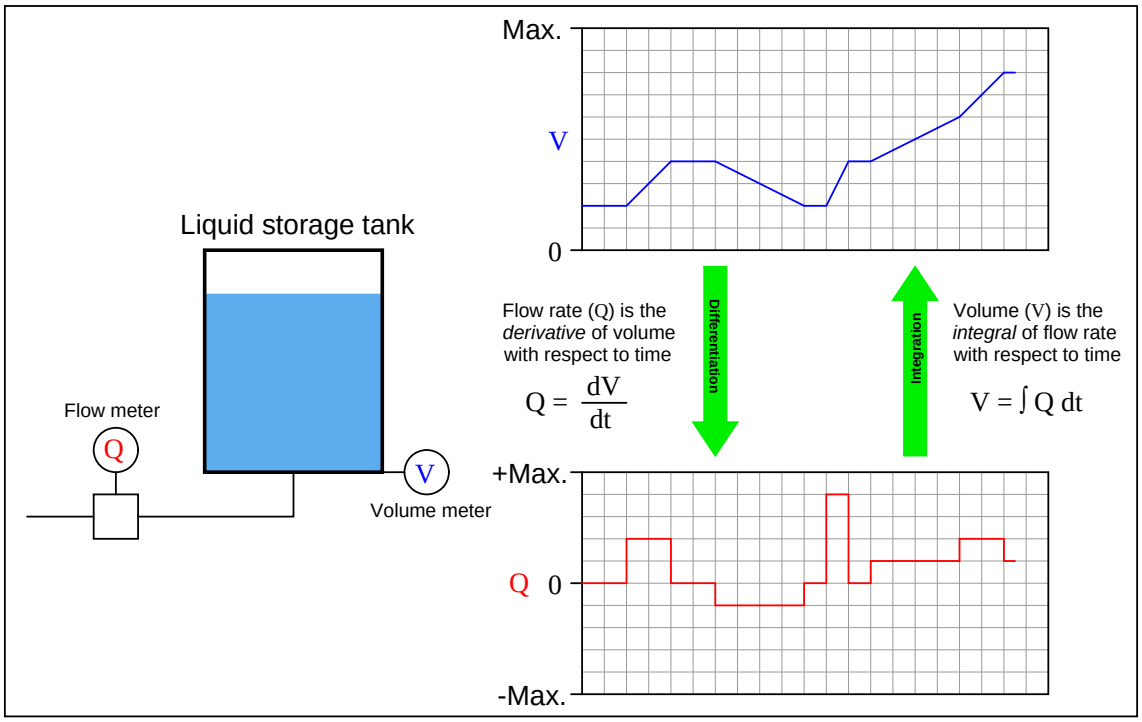
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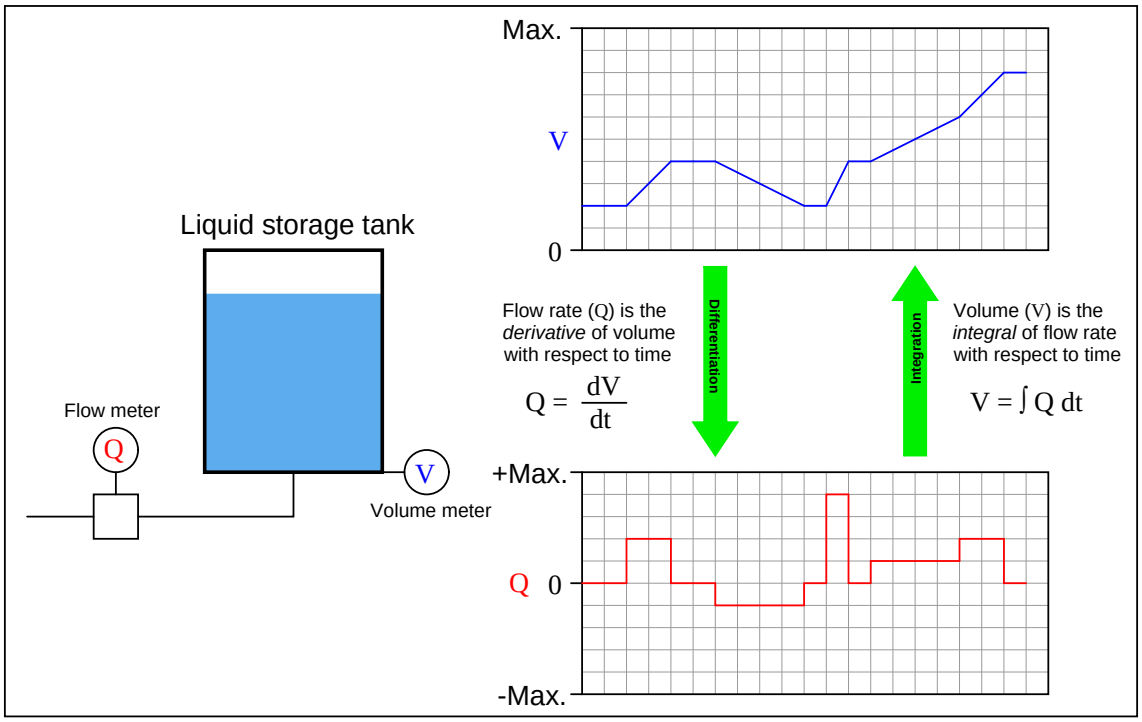
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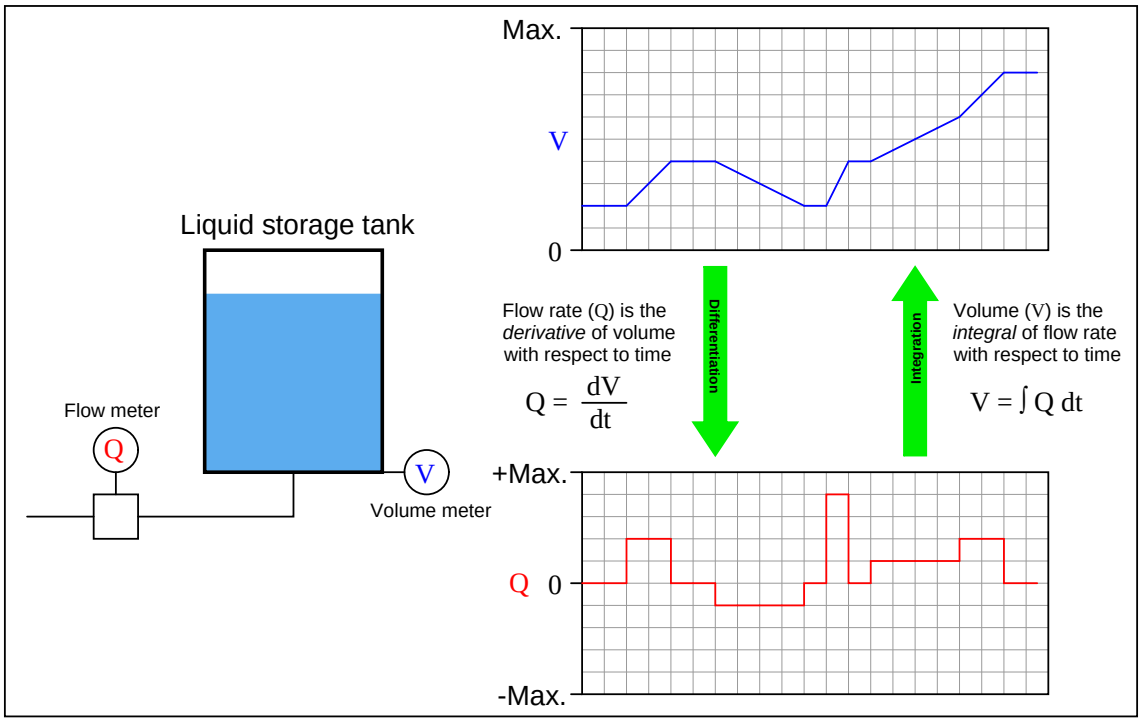
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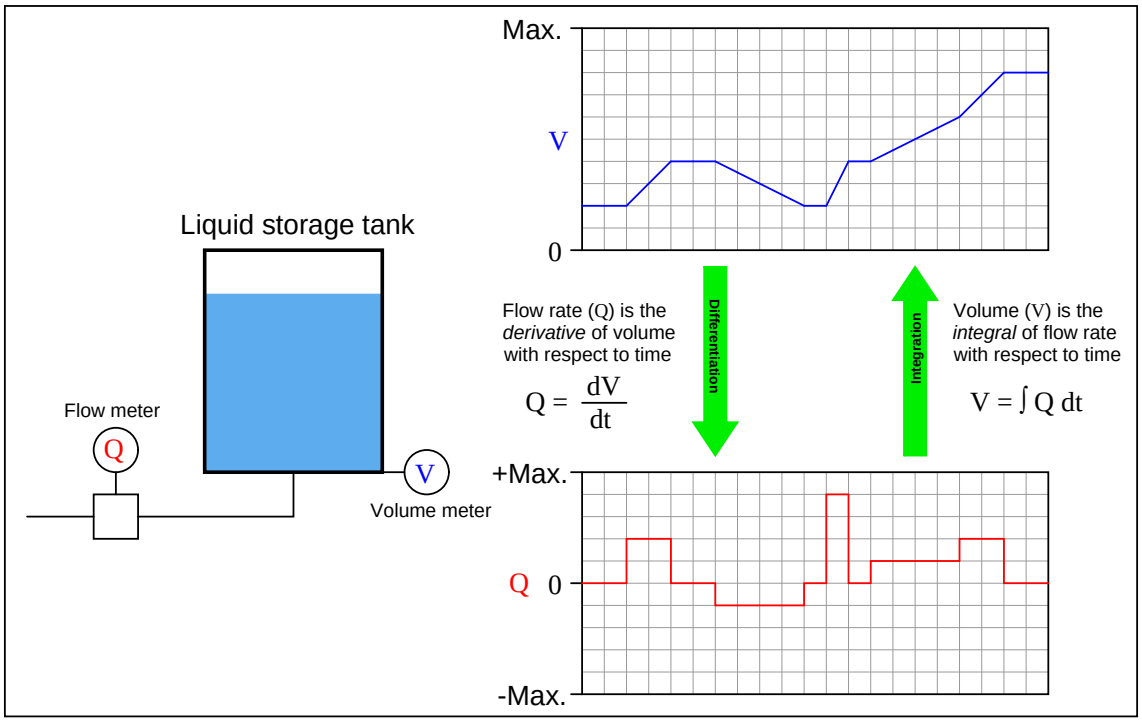
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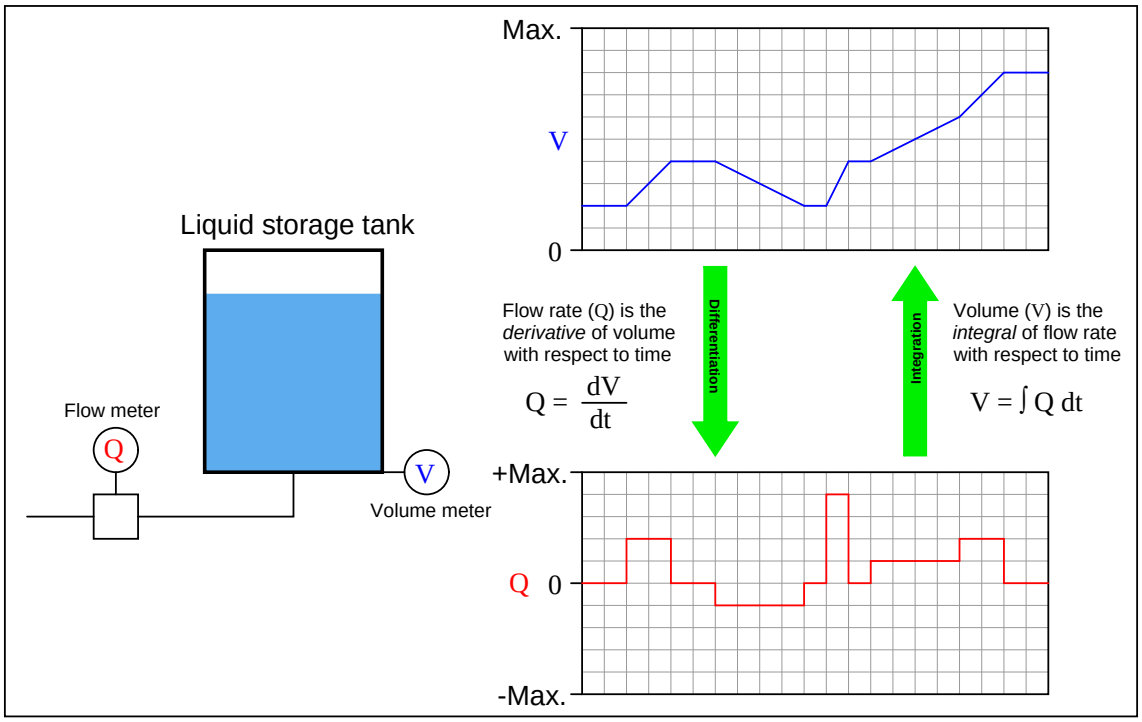
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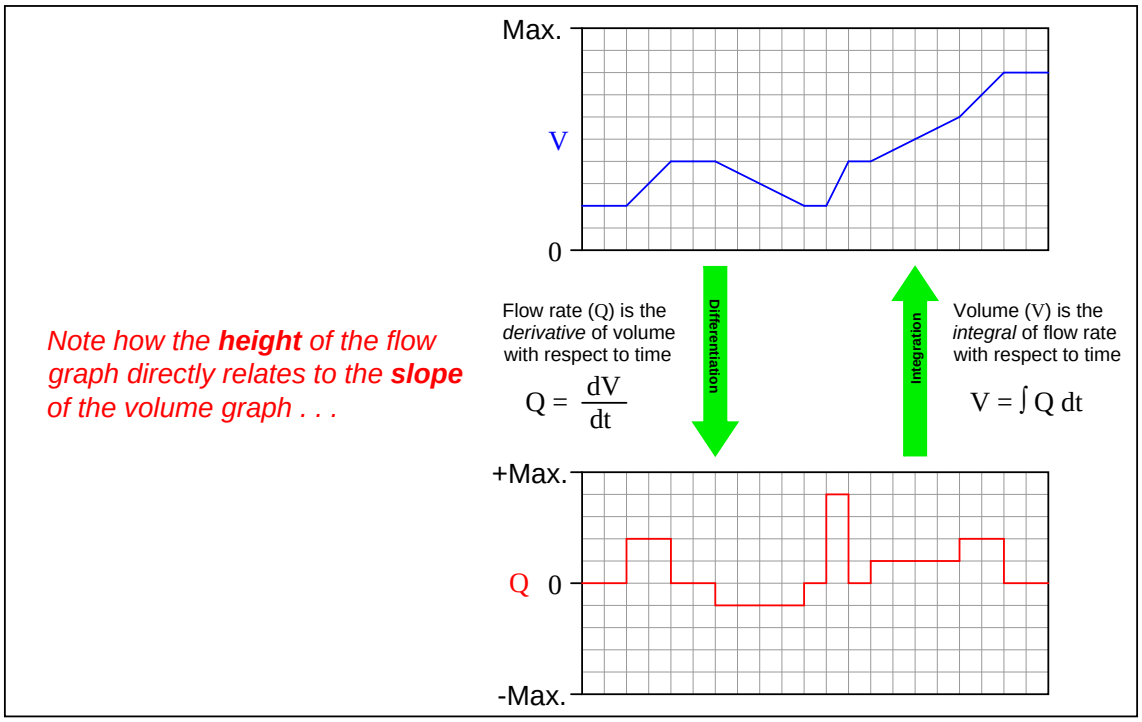
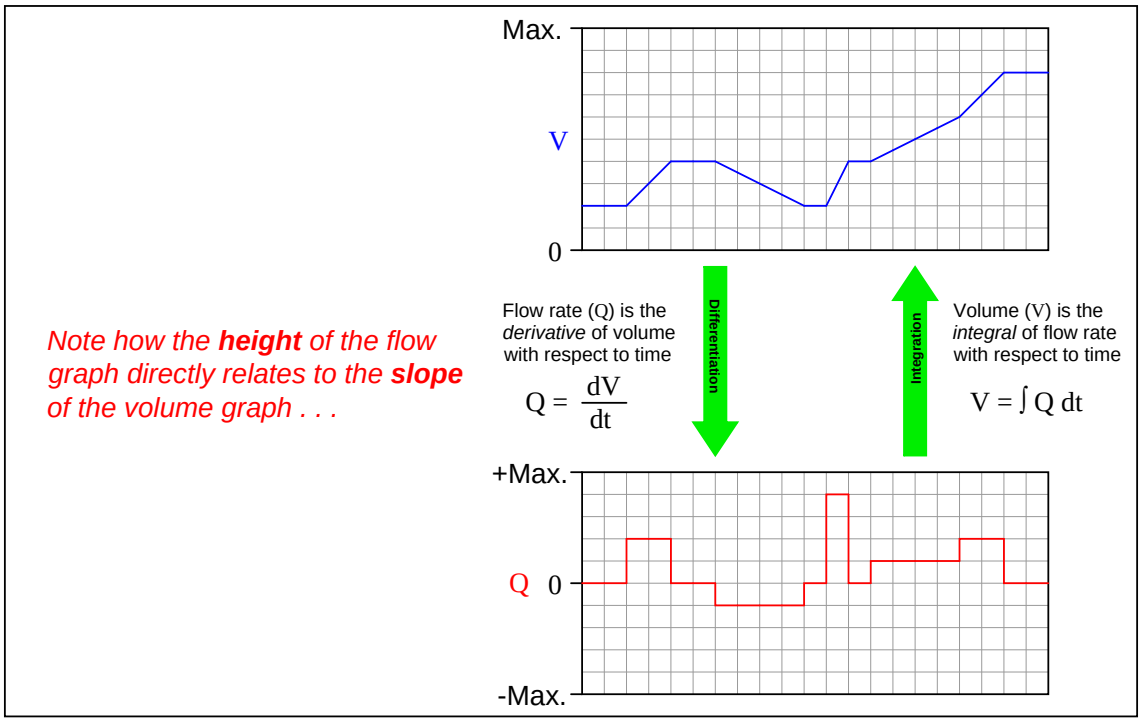


*Note how the **height** of the flow graph directly relates to the **slope** of the volume graph . . .*



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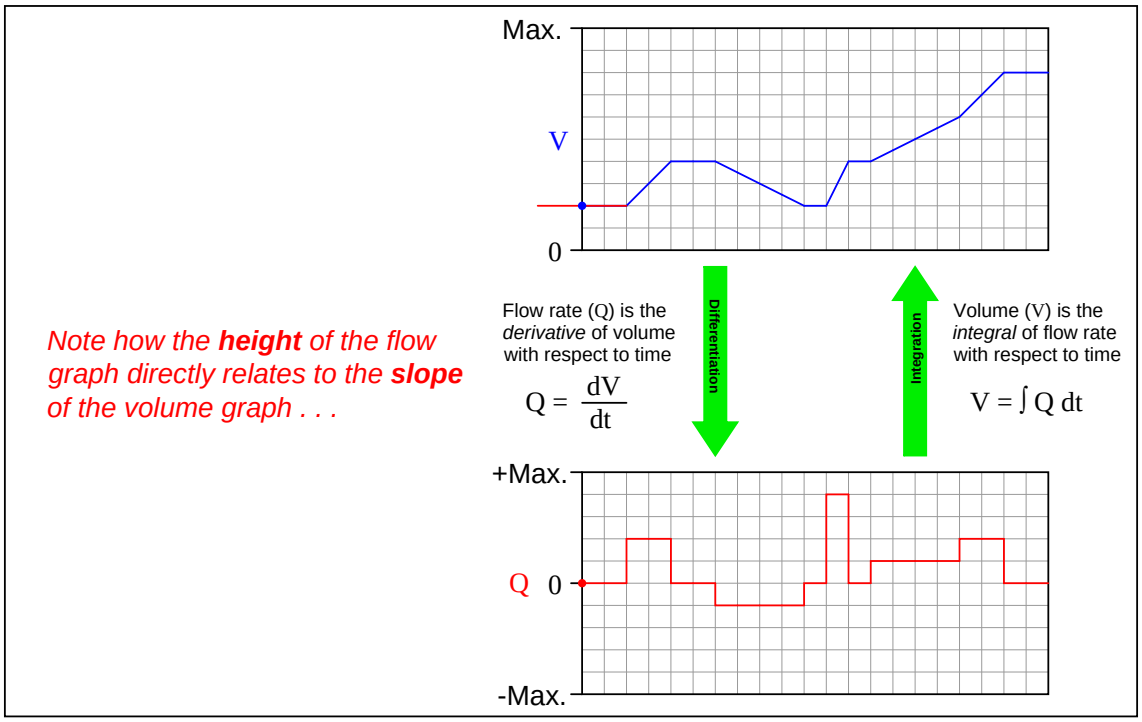

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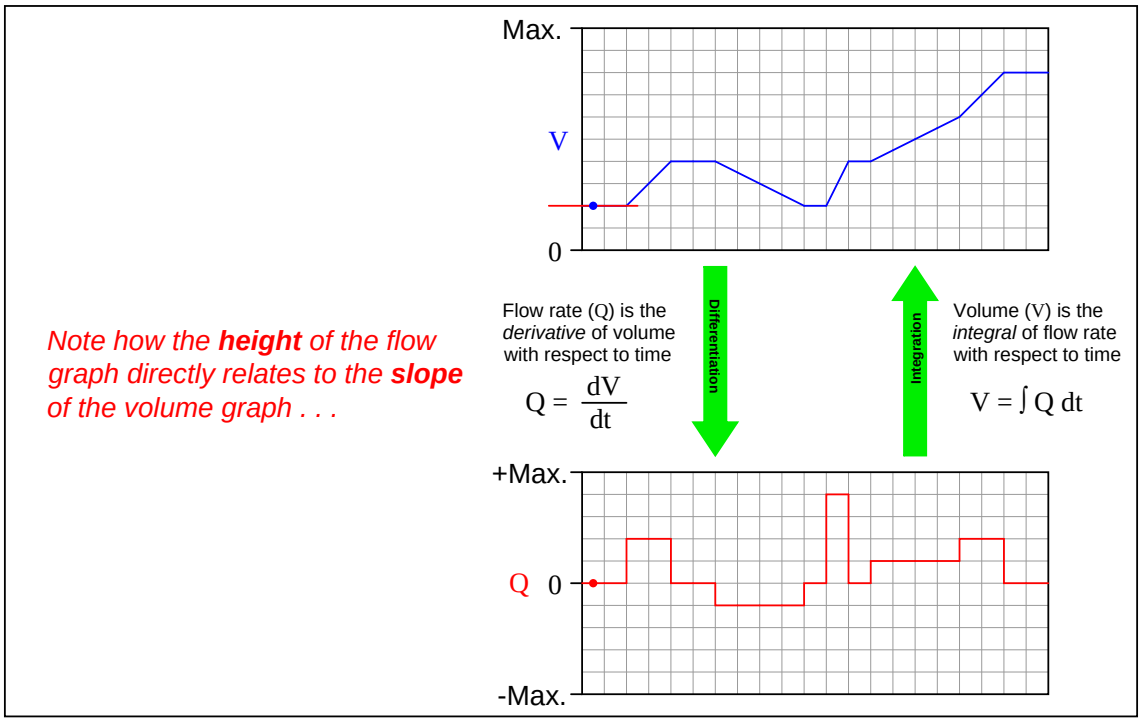

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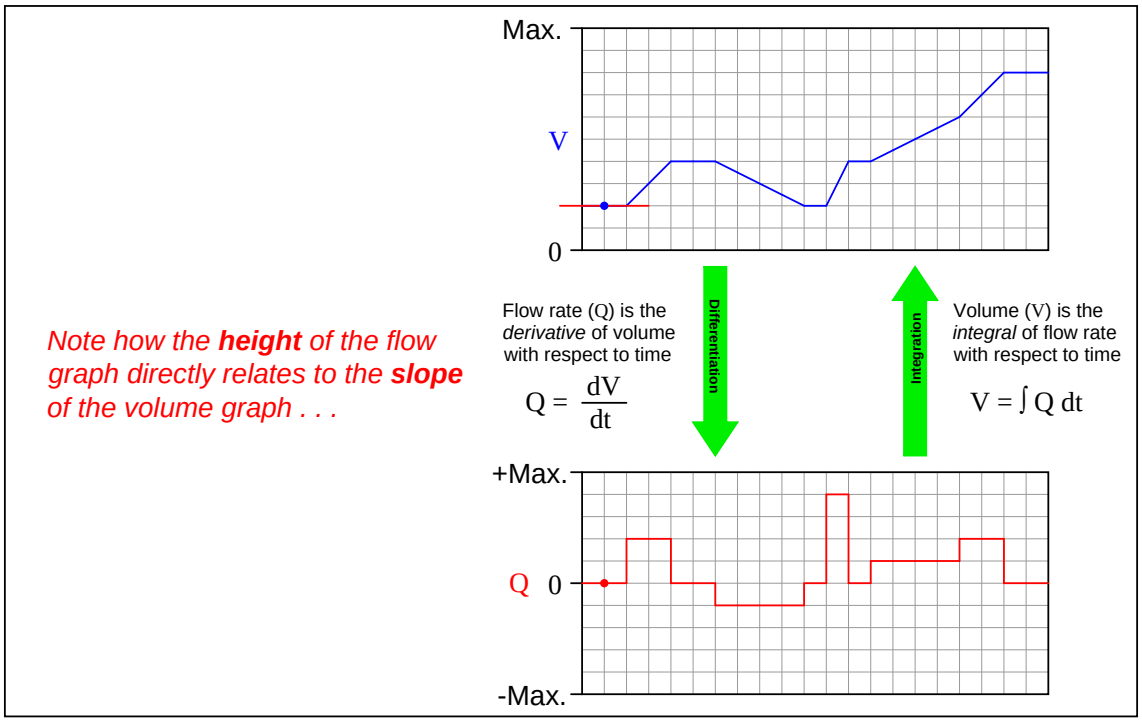

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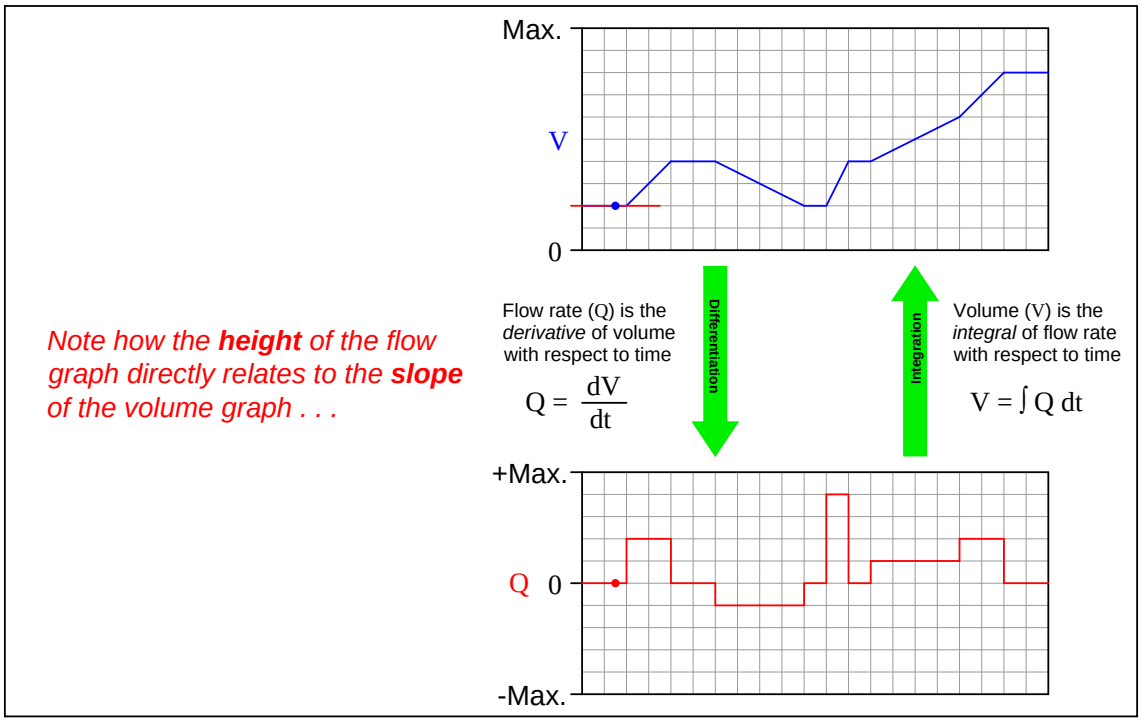

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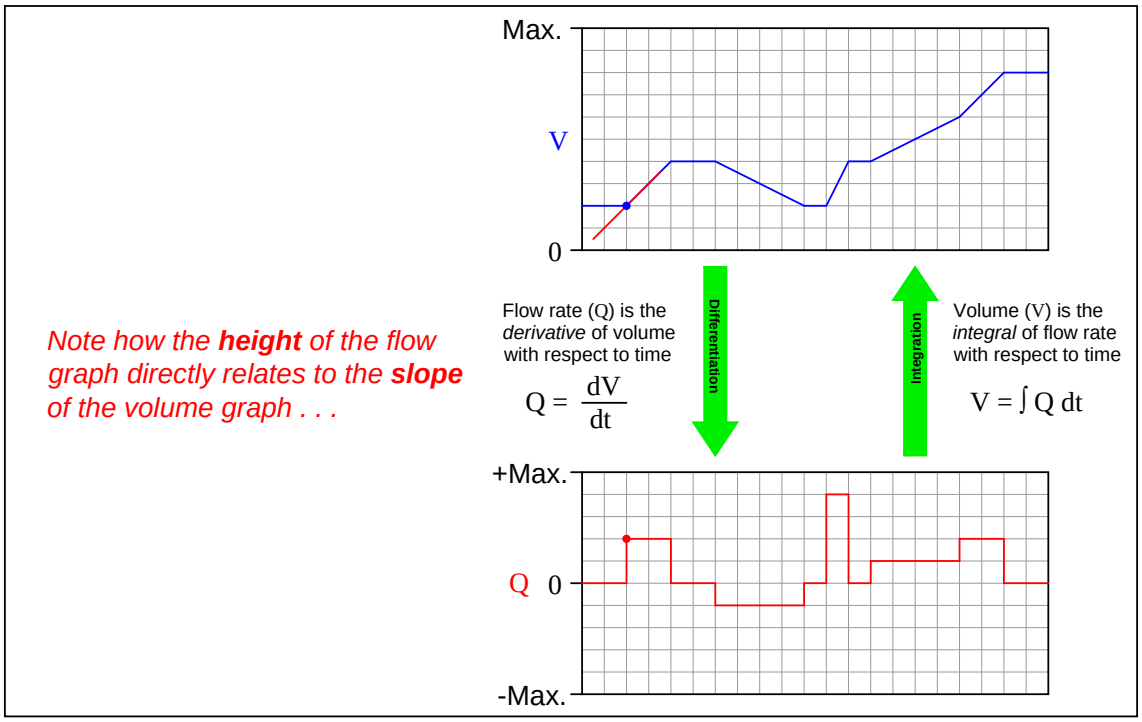

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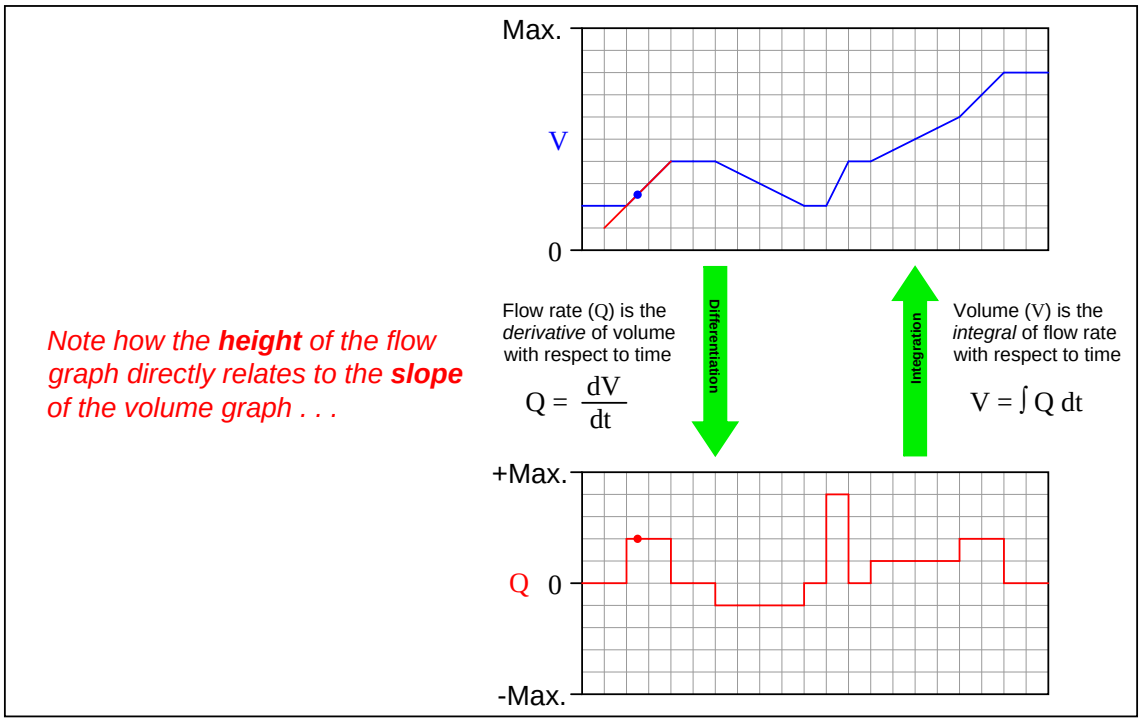

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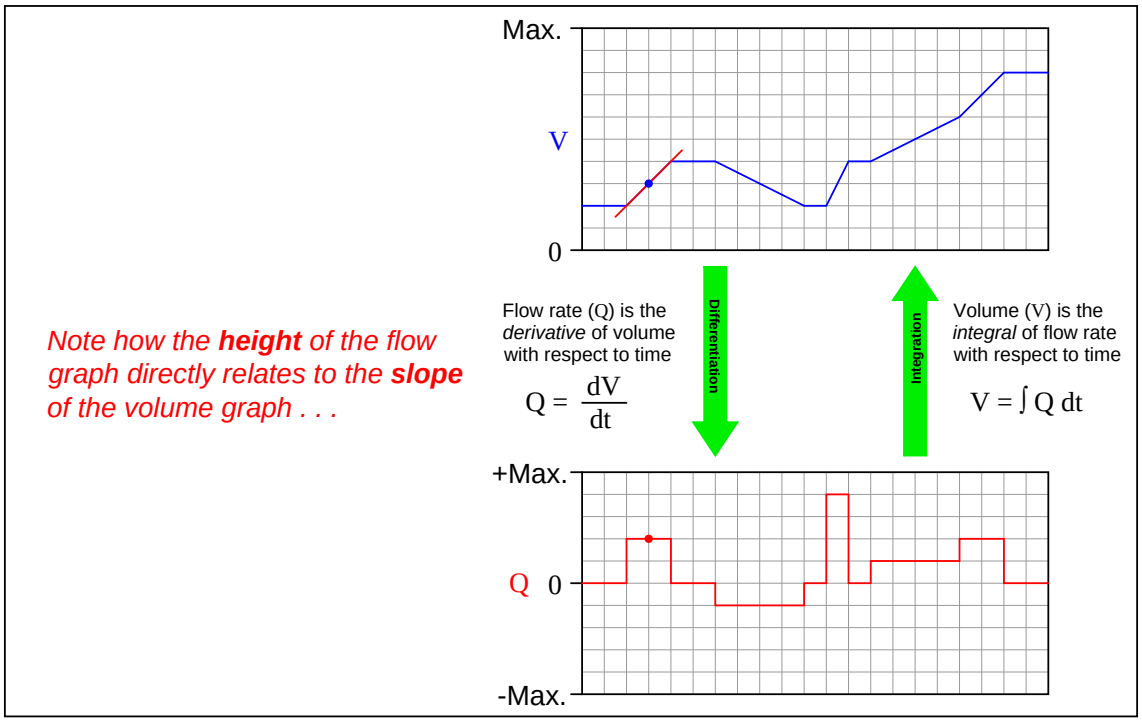

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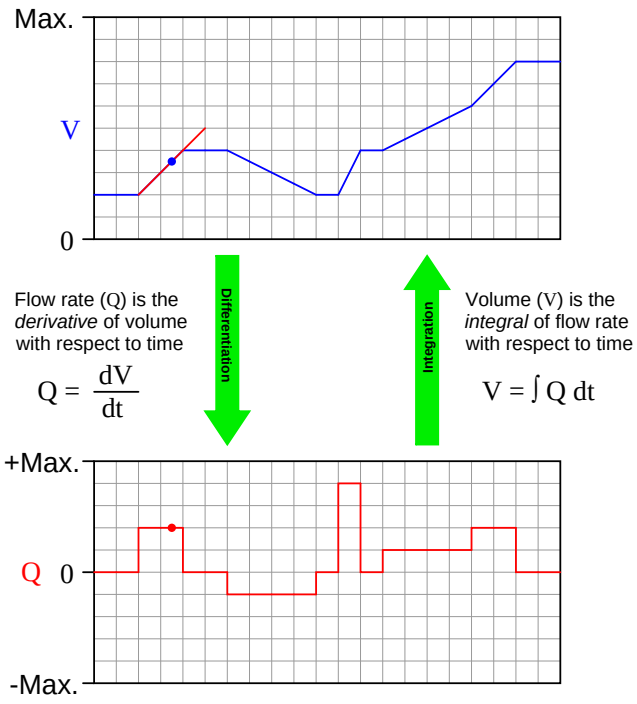

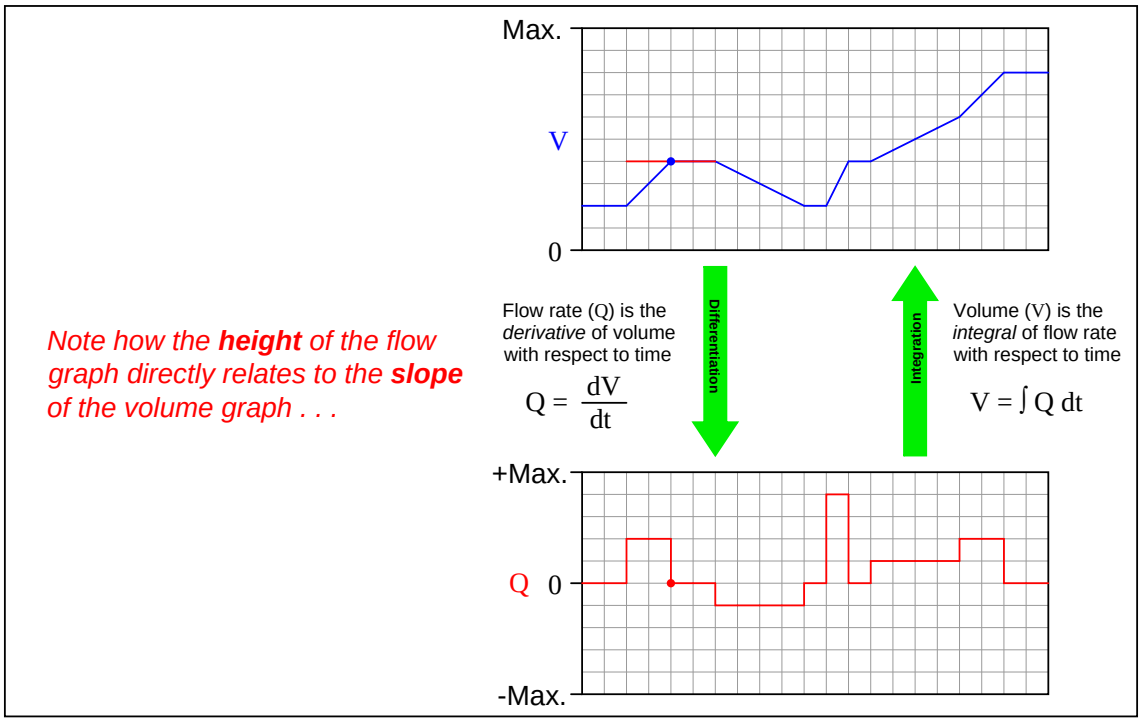

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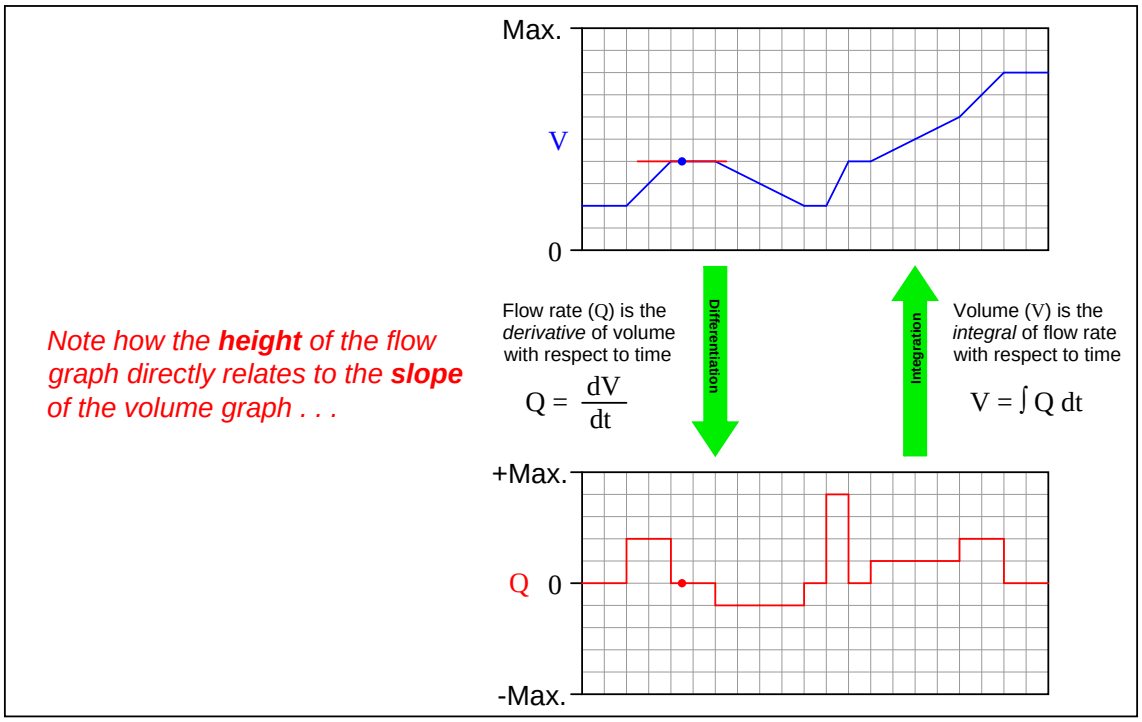

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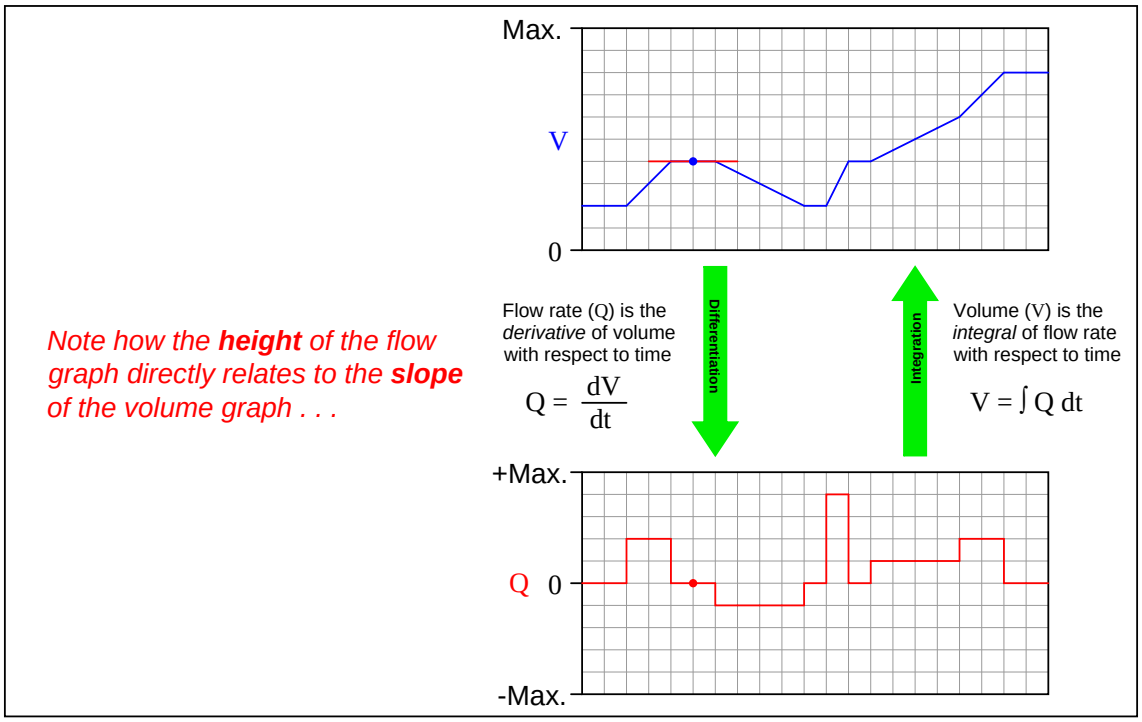

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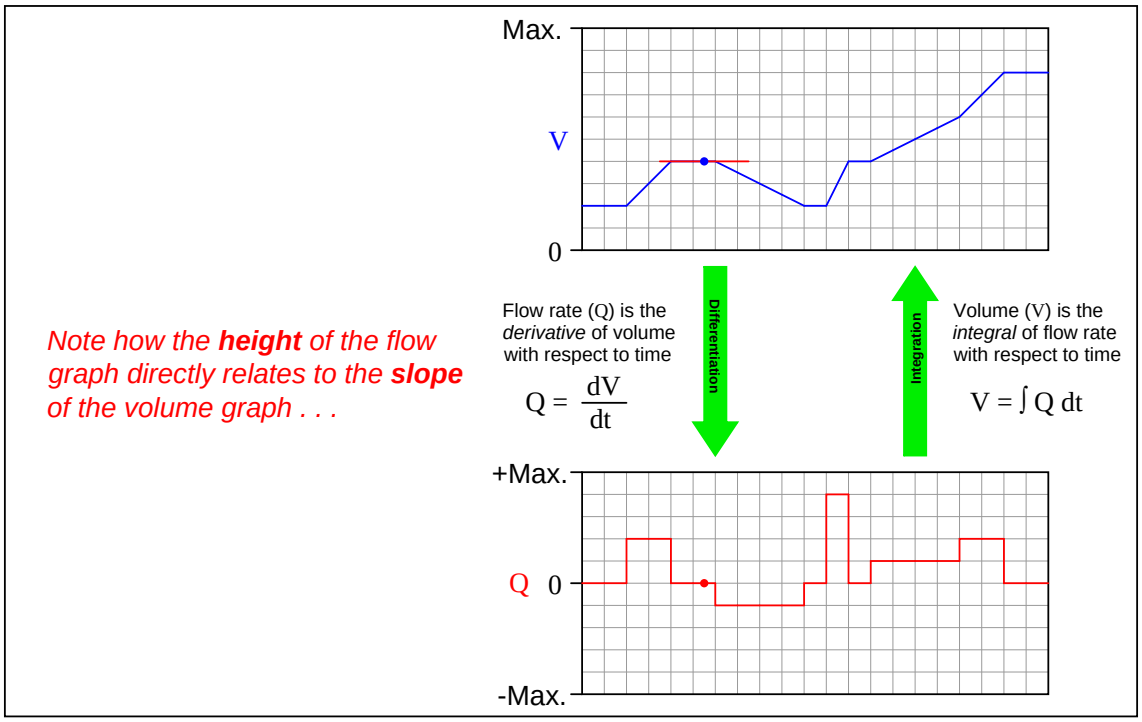

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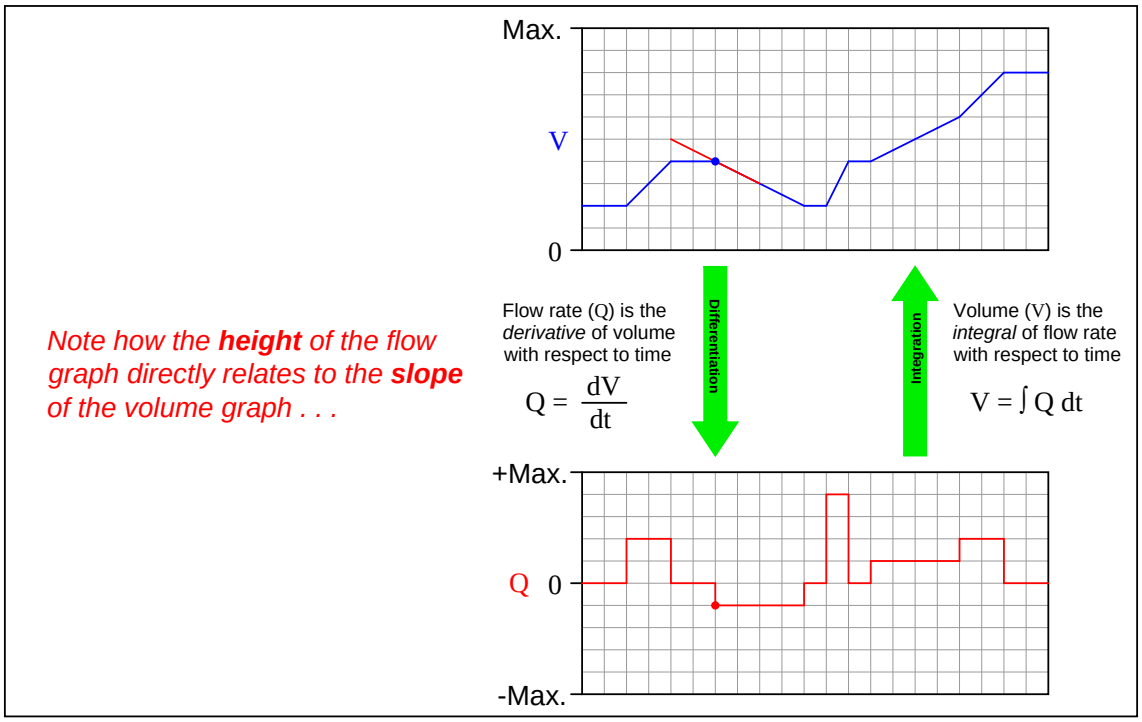

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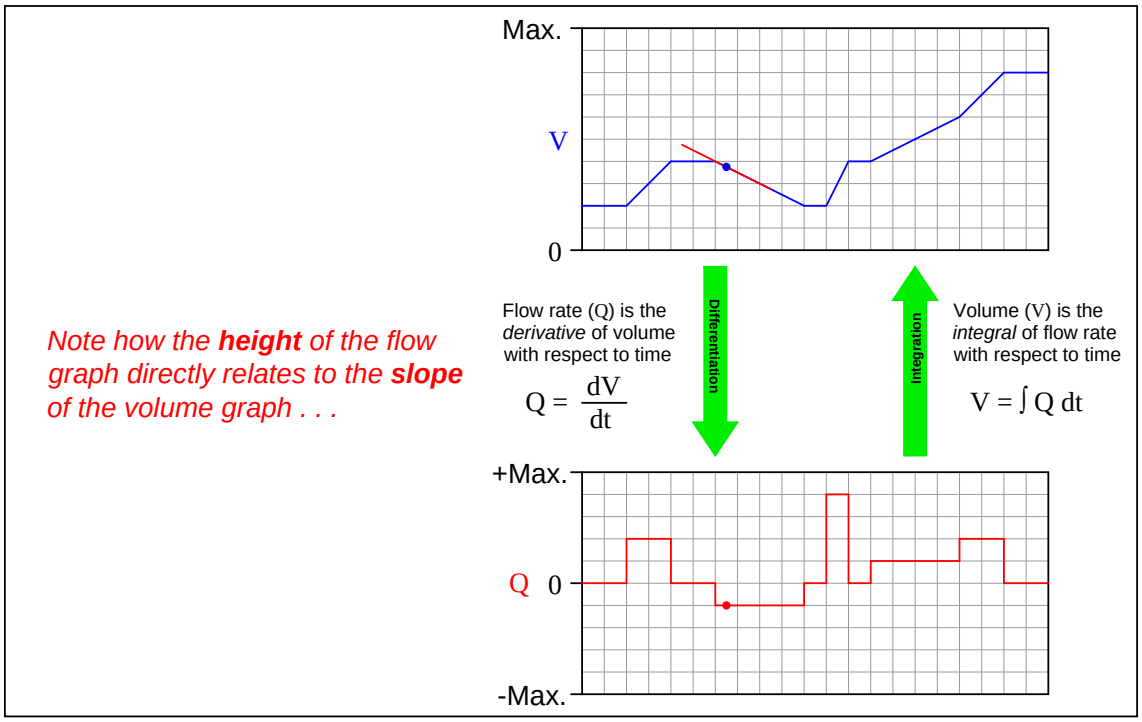

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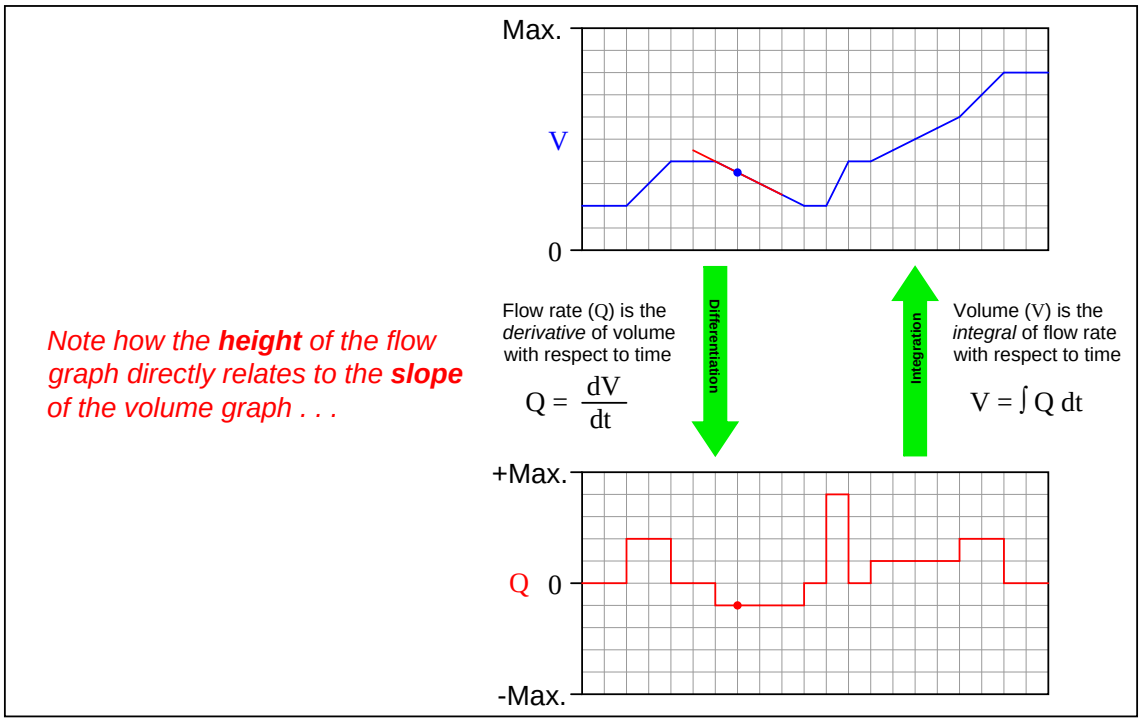

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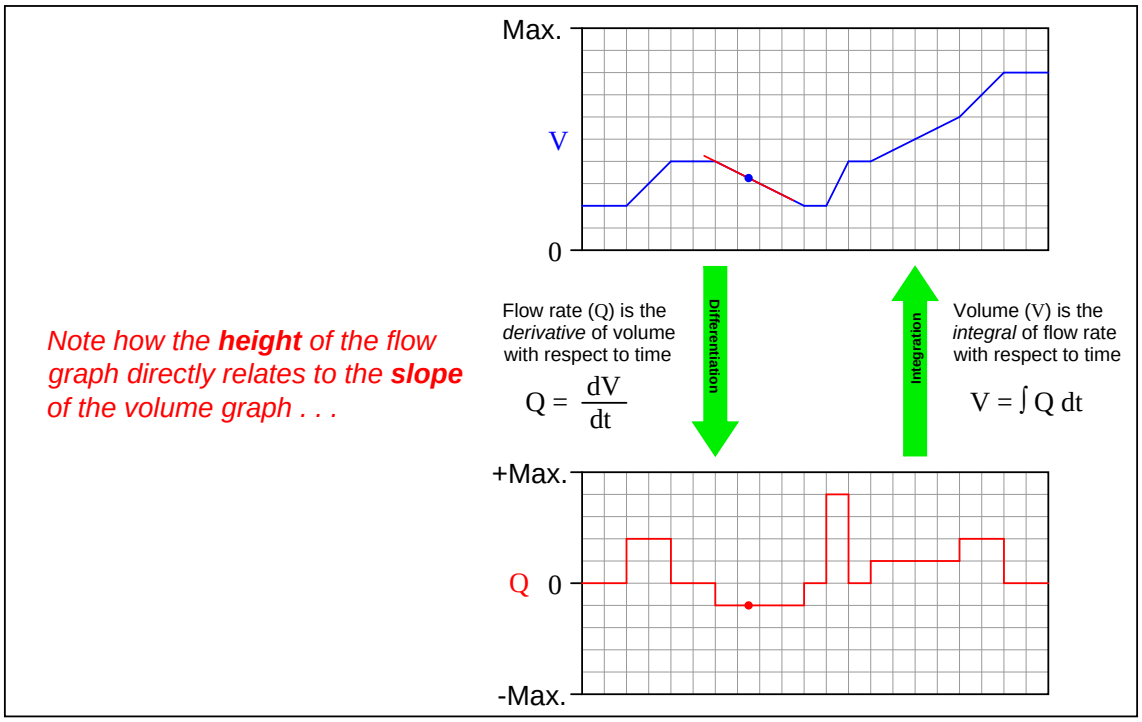

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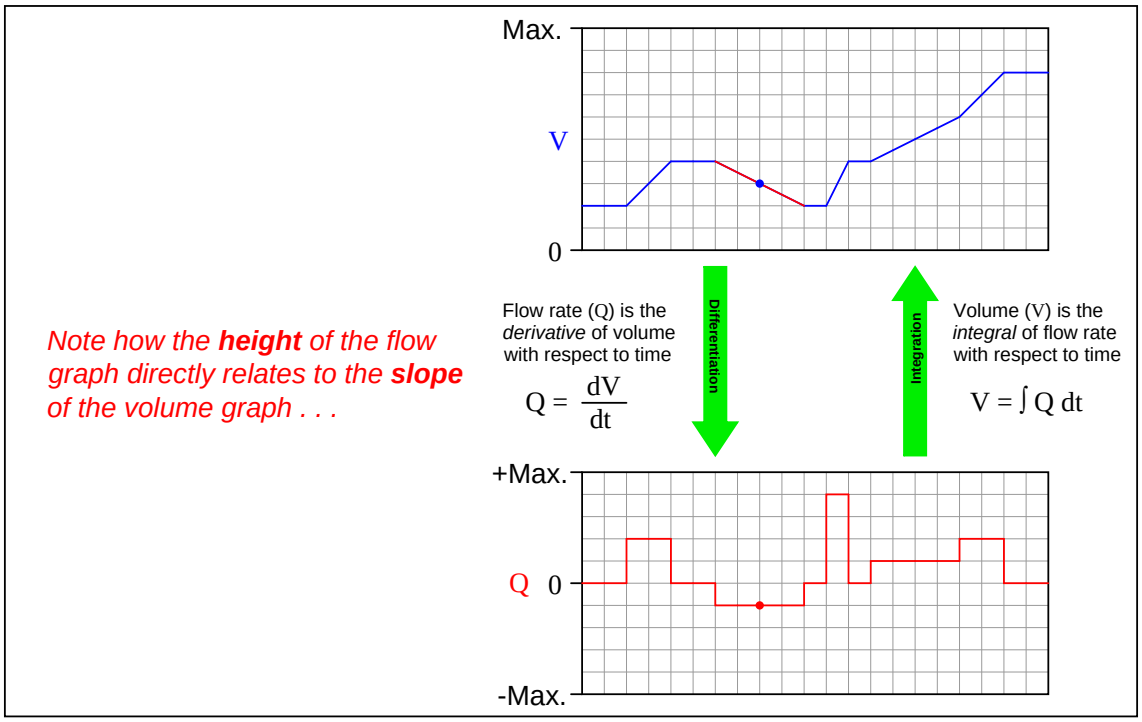

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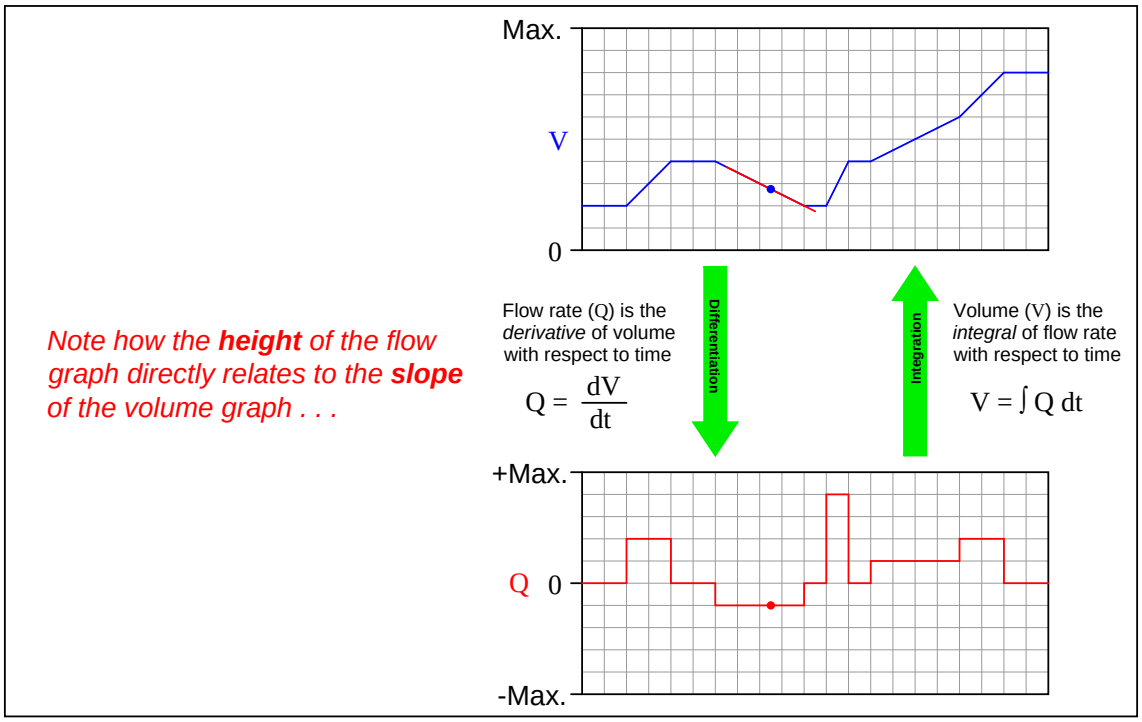

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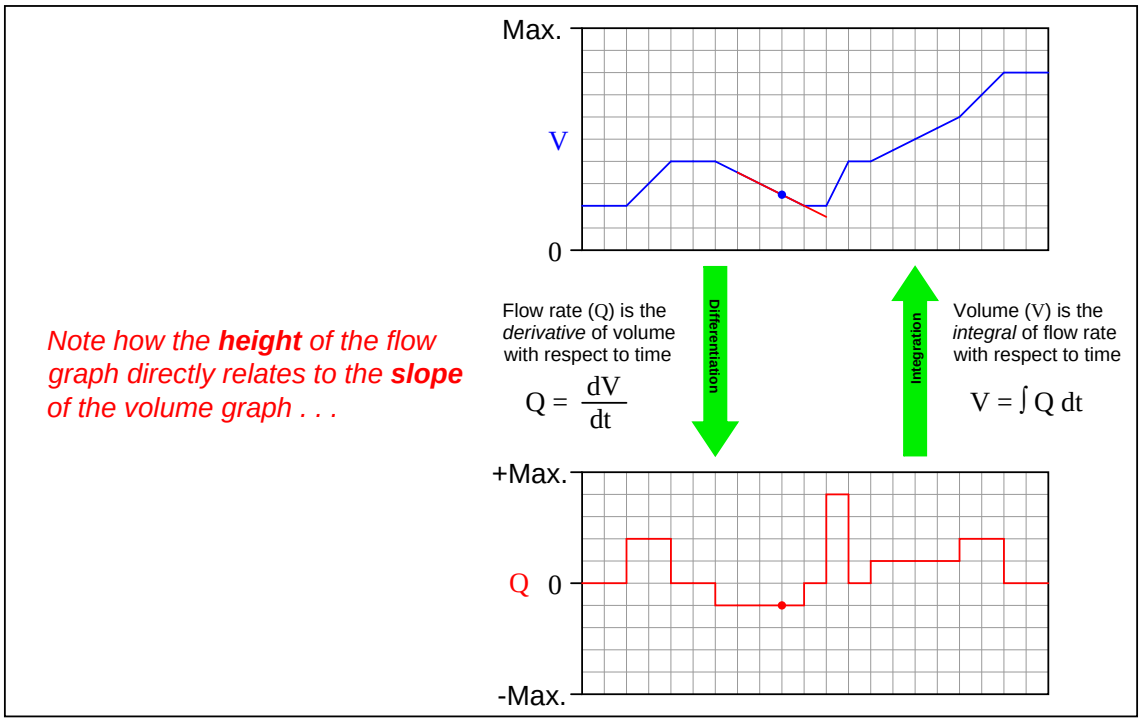

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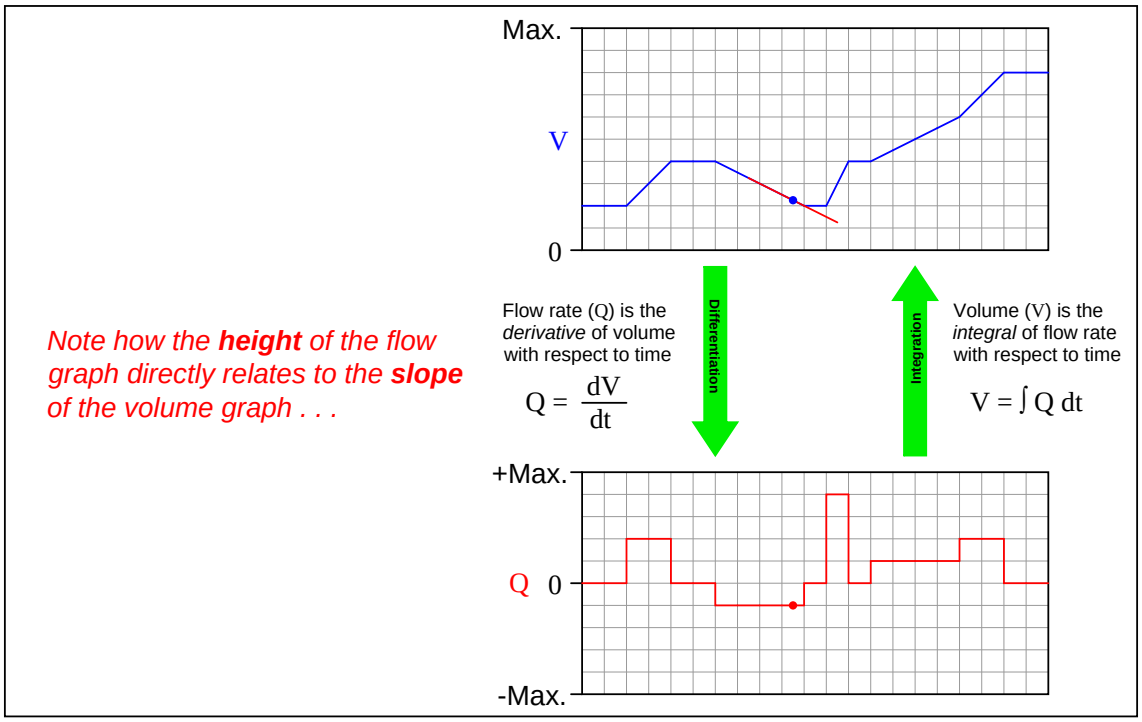

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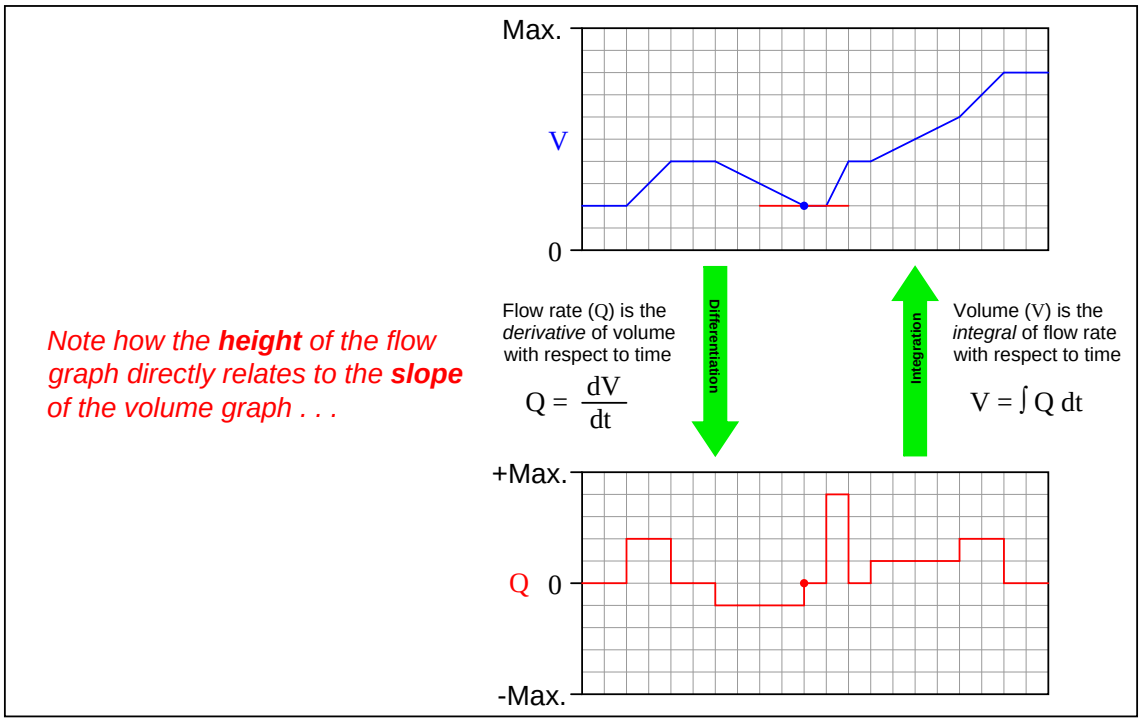

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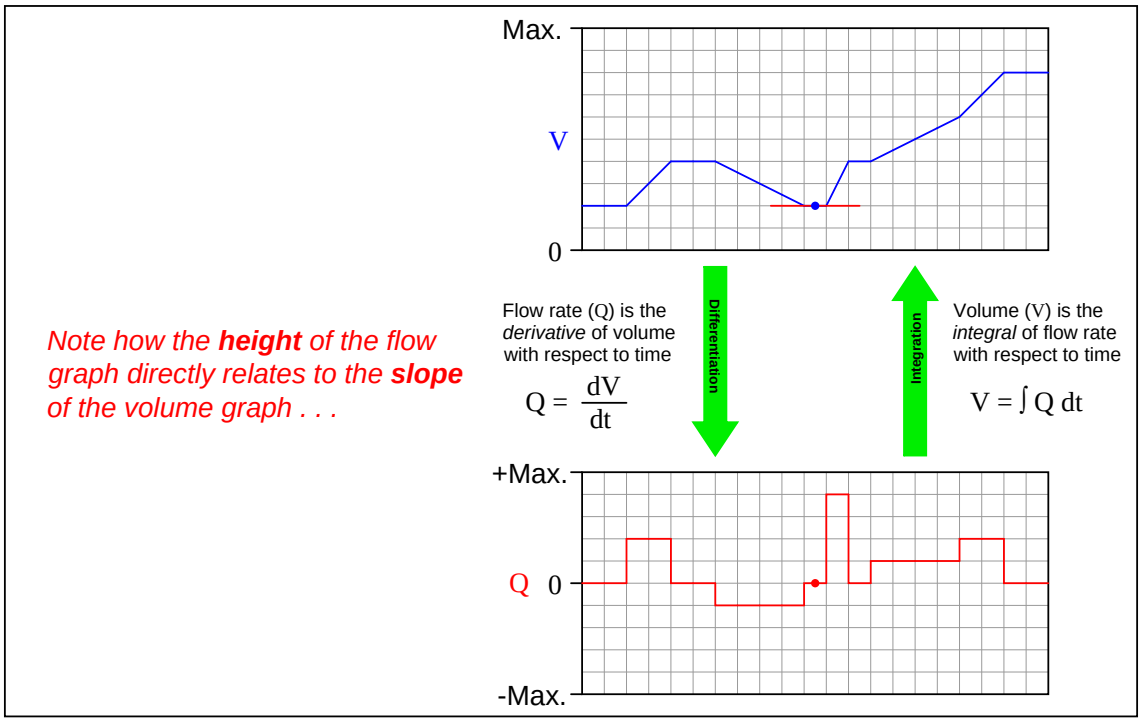

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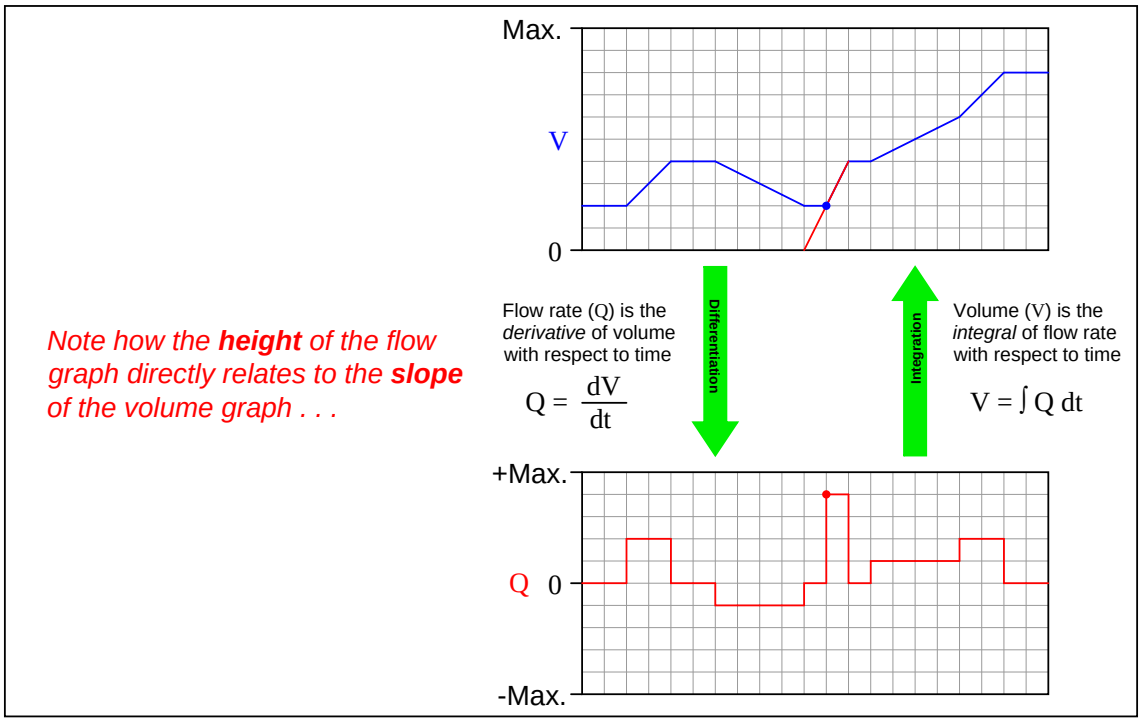

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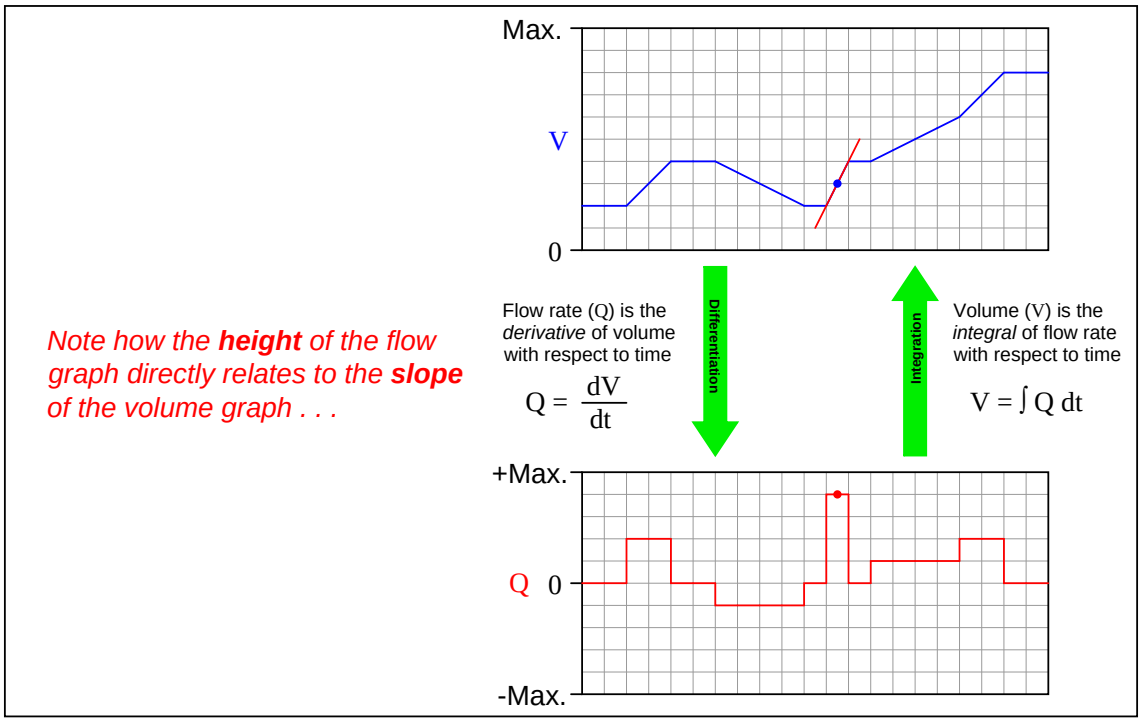

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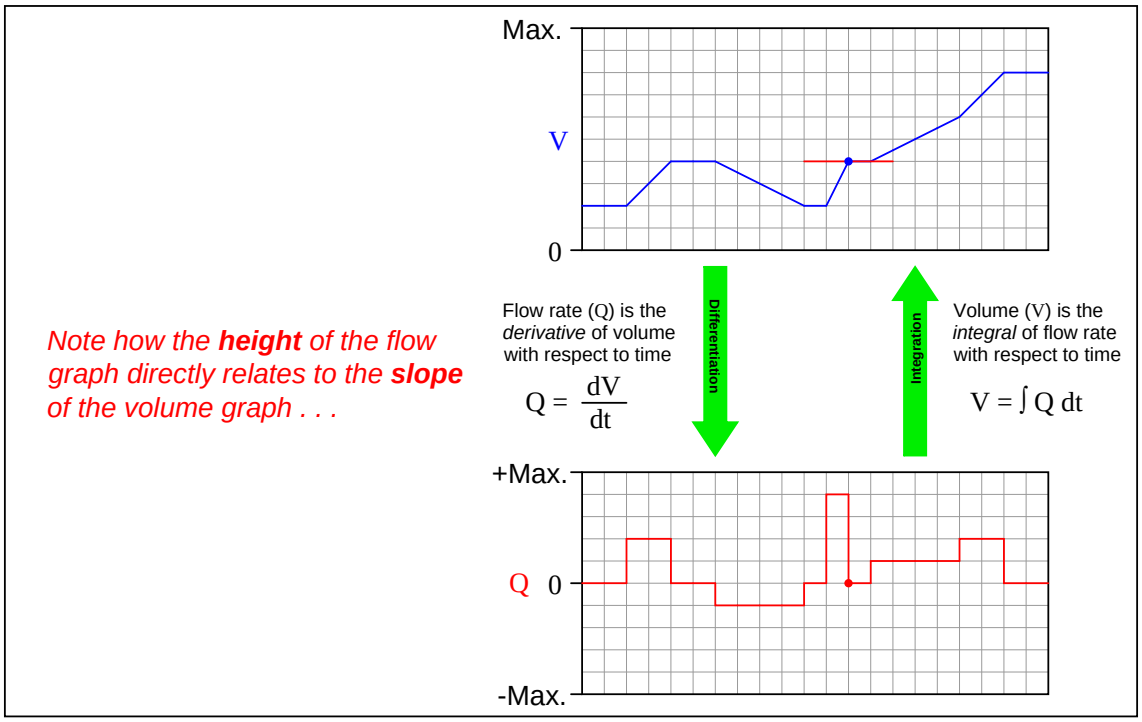

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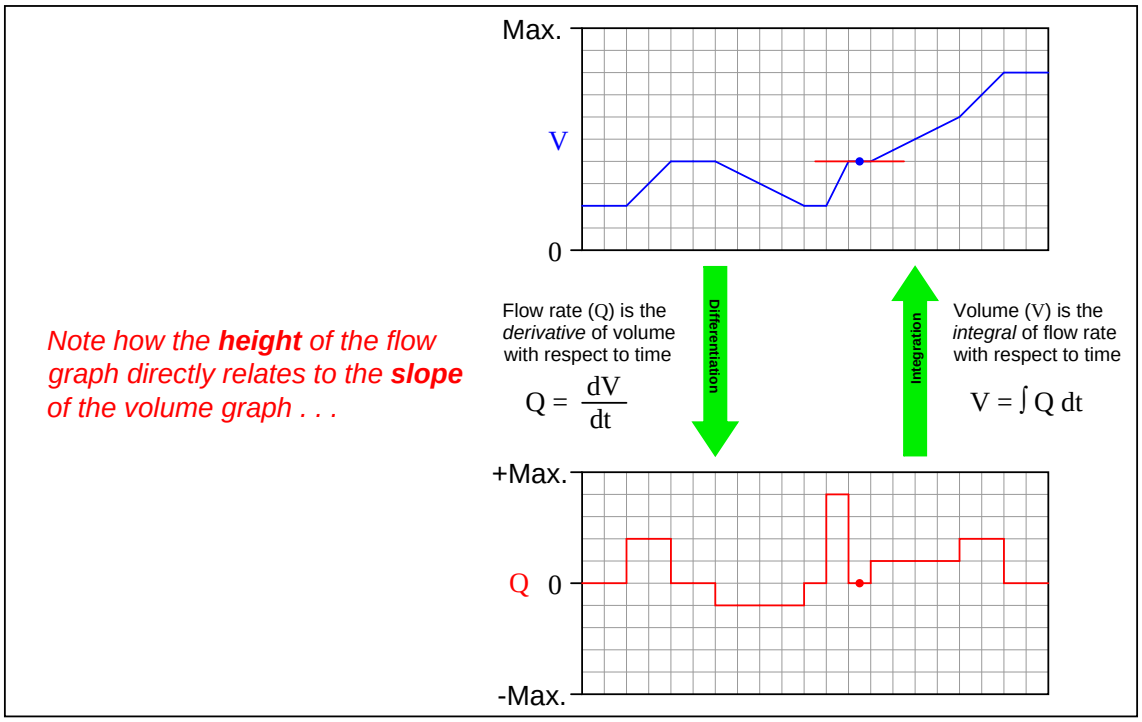

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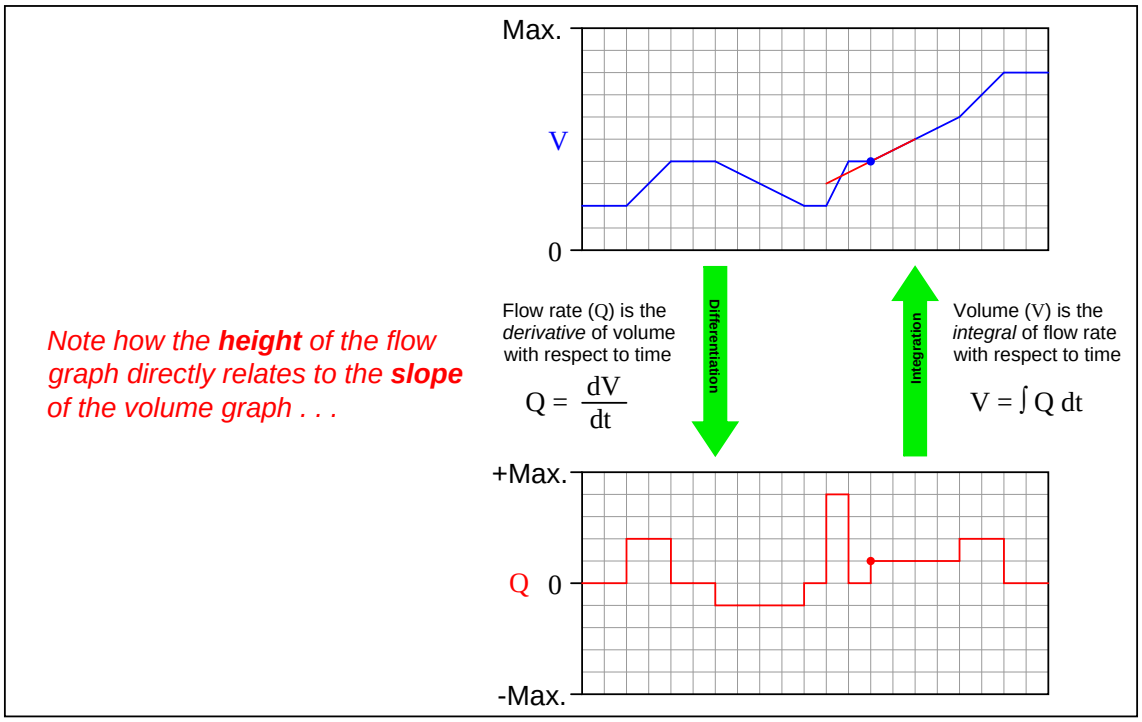

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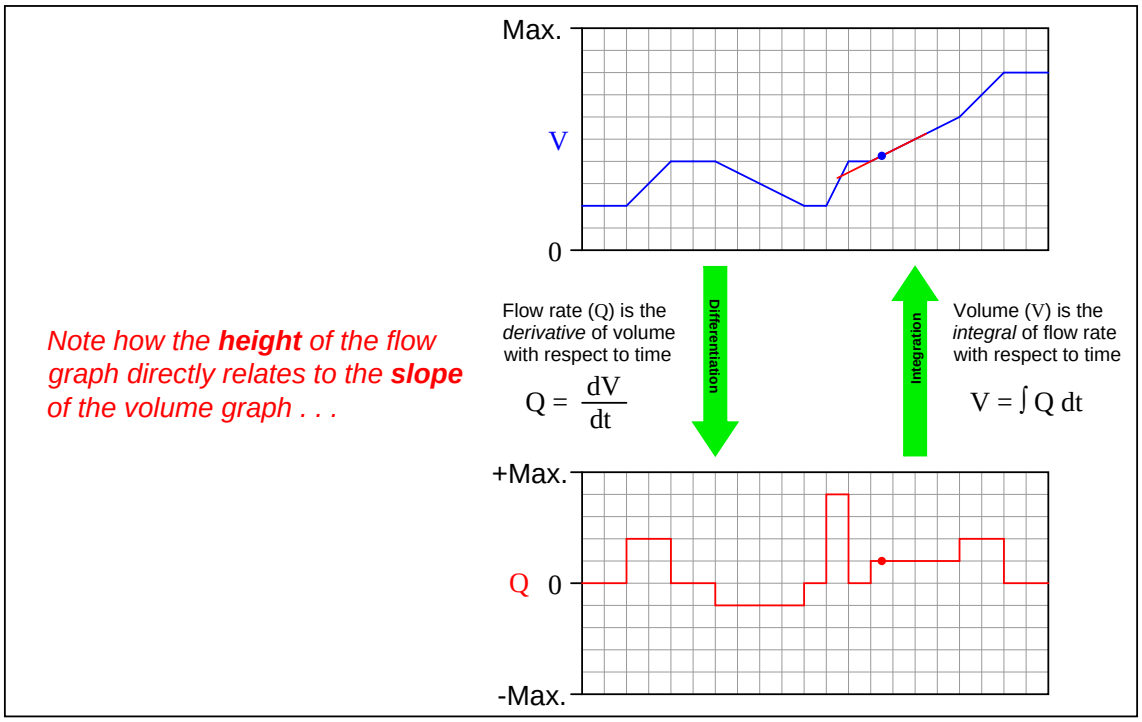

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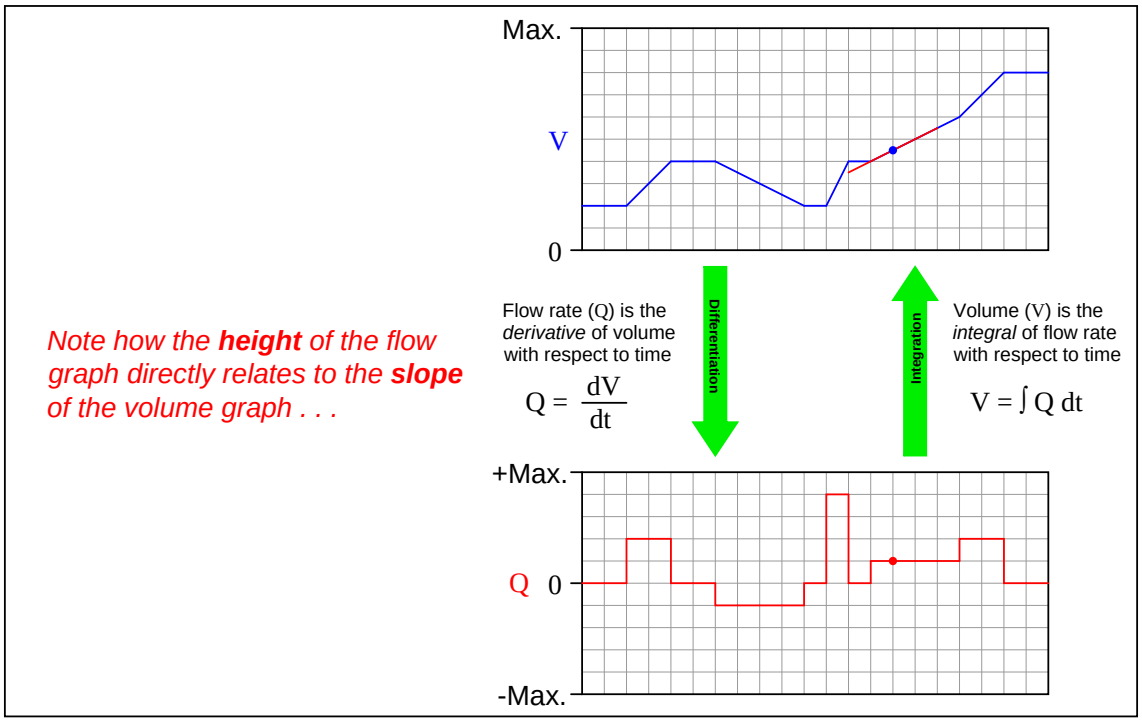

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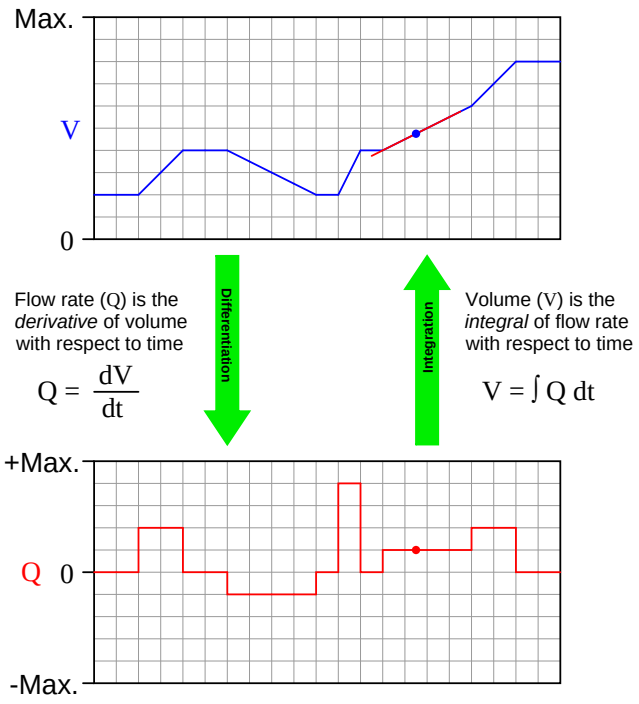

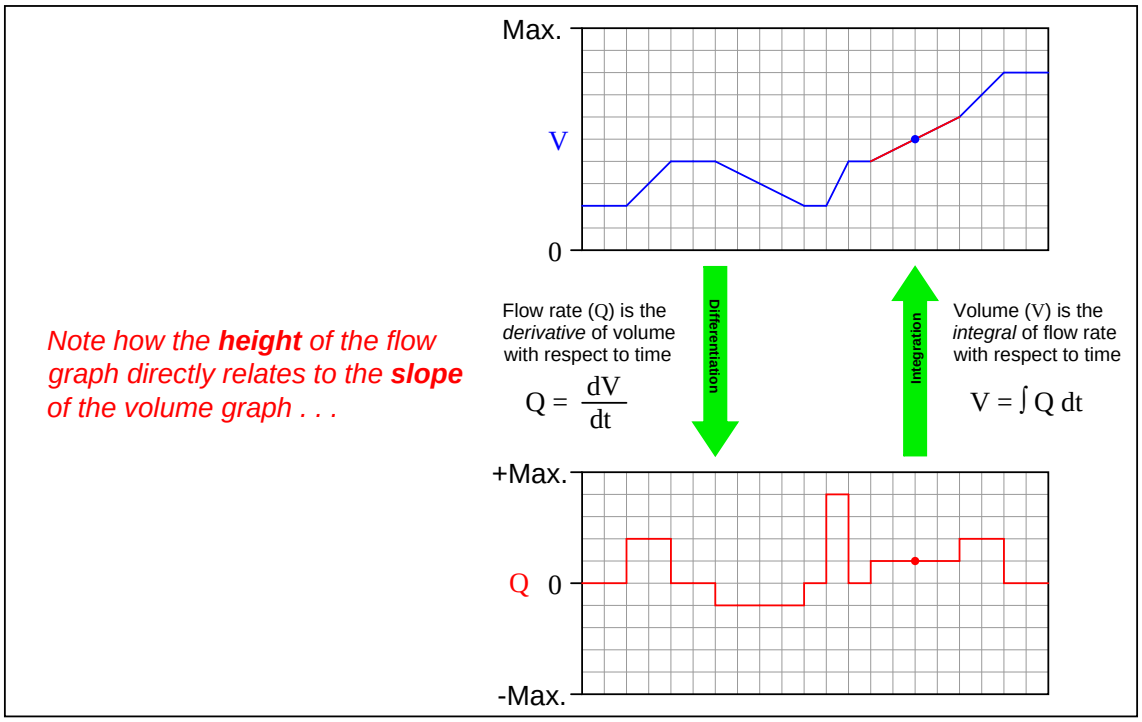

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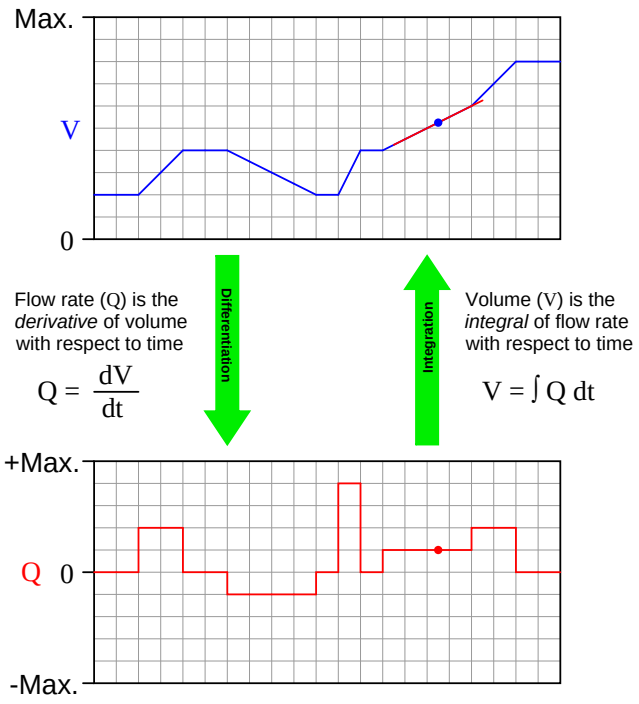

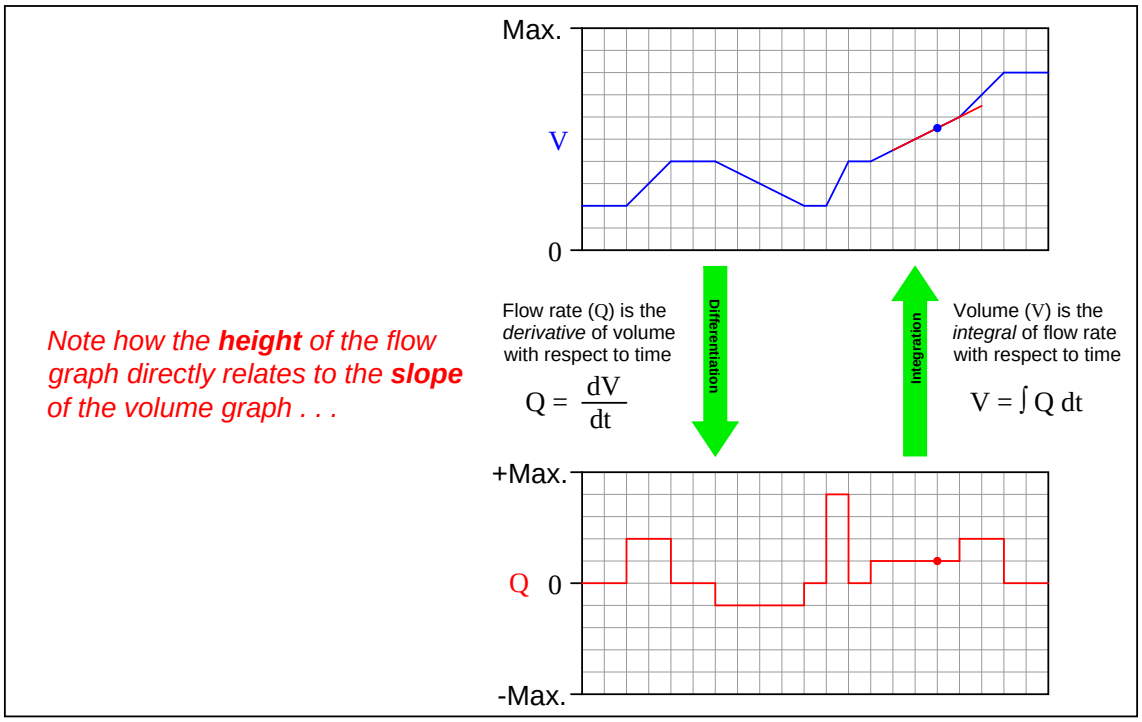

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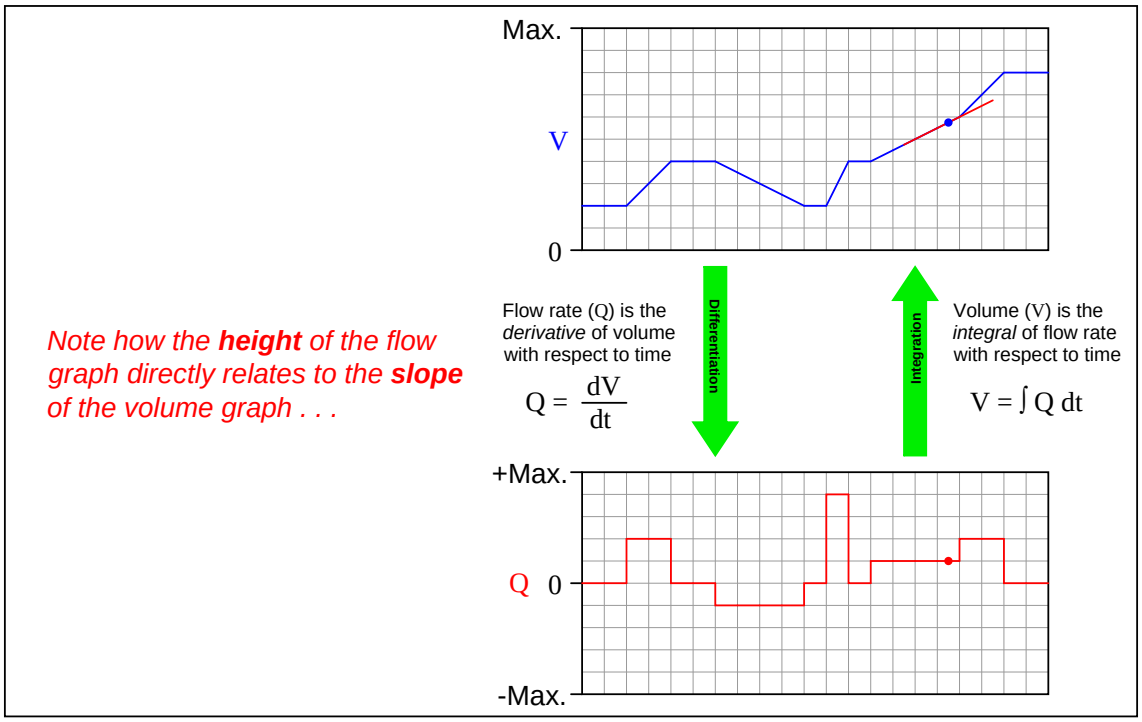

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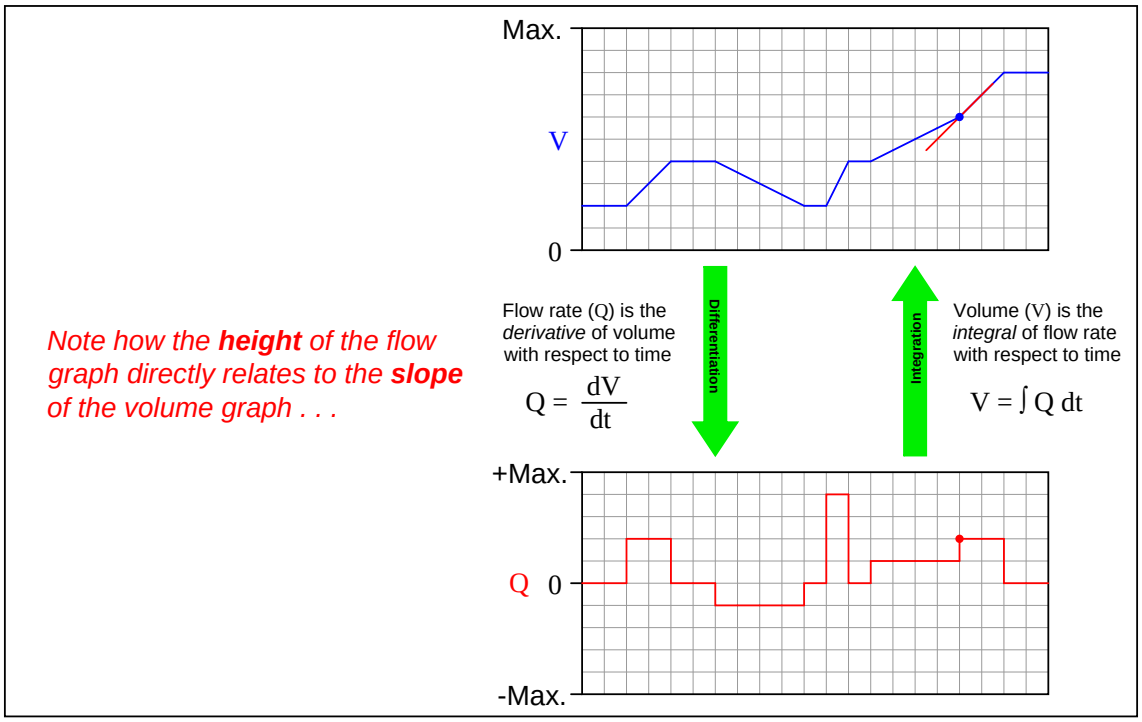

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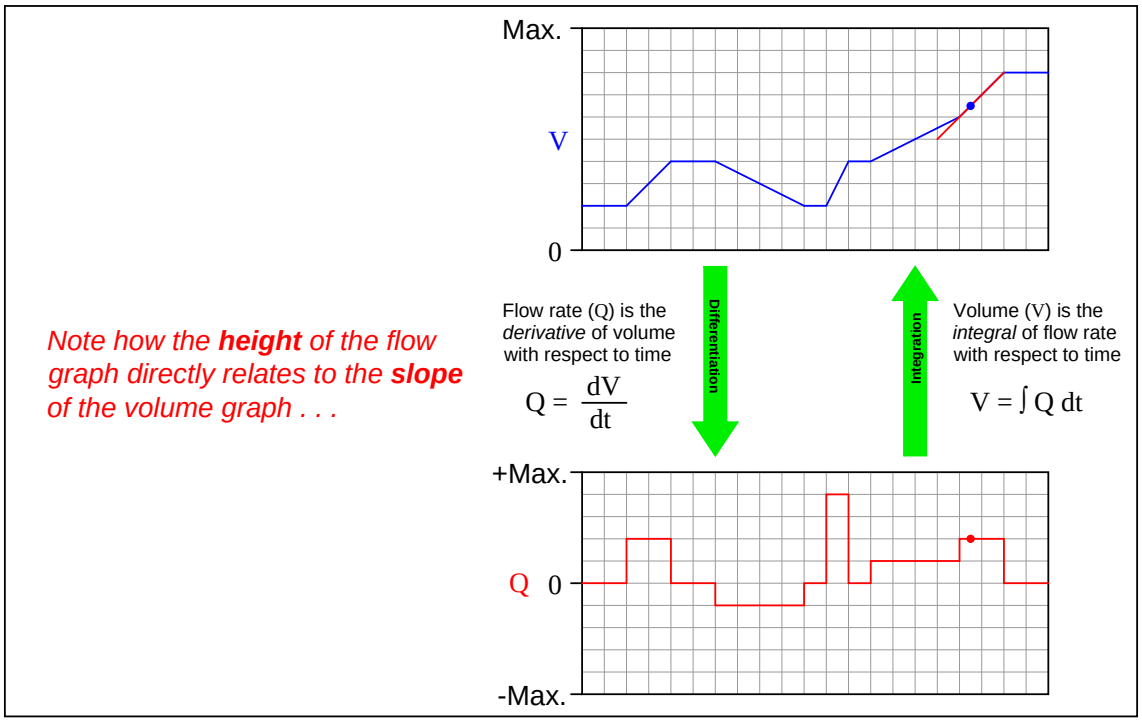

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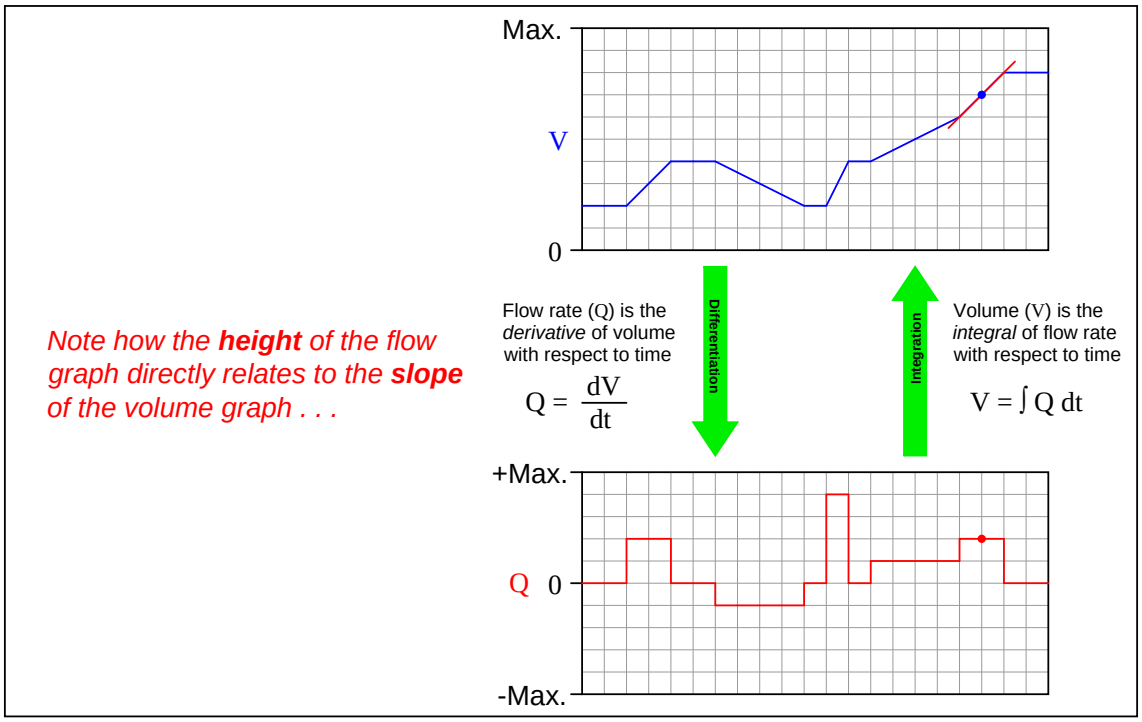

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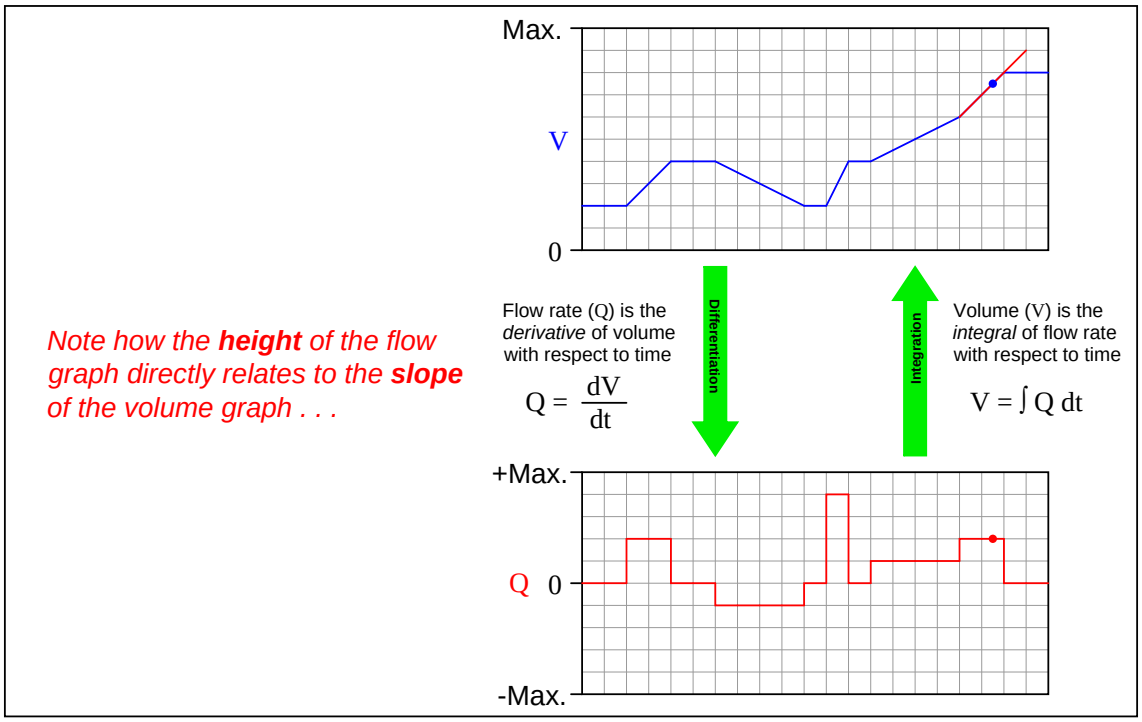

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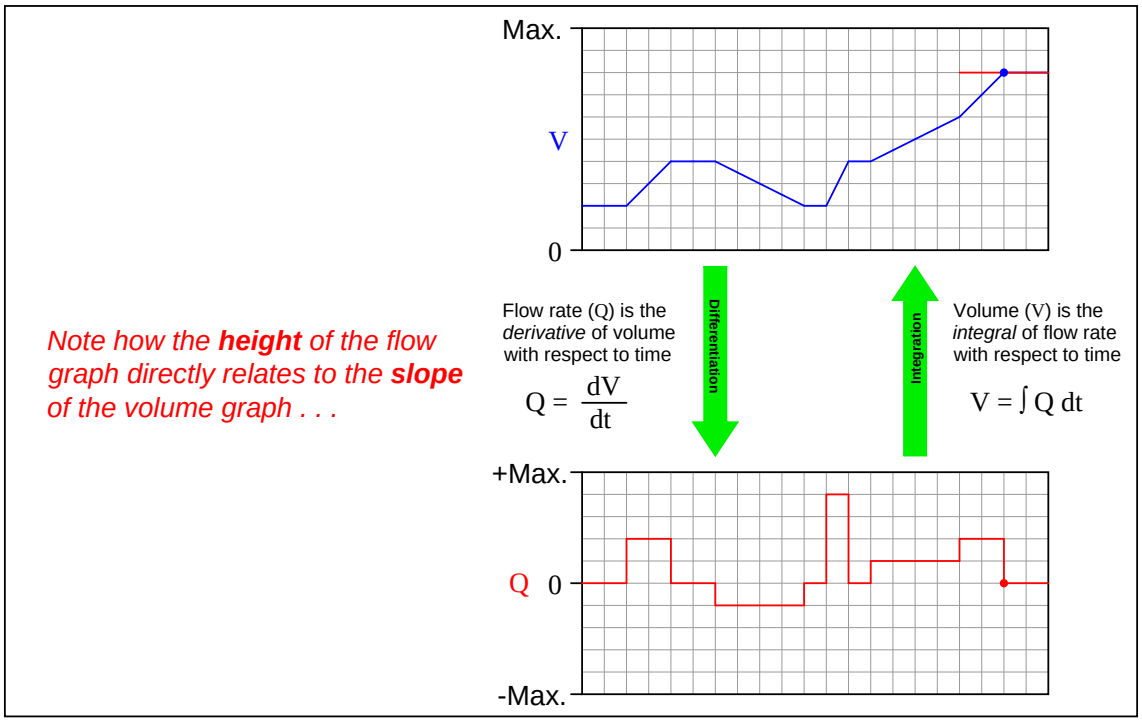

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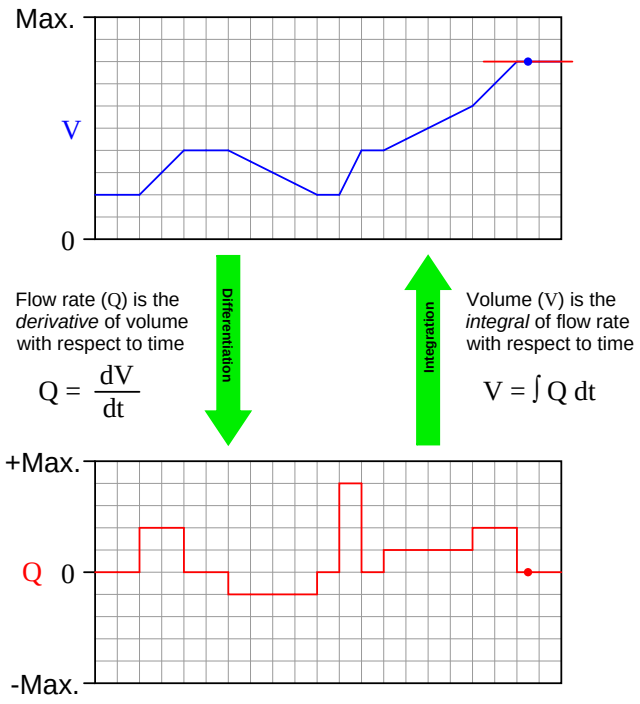

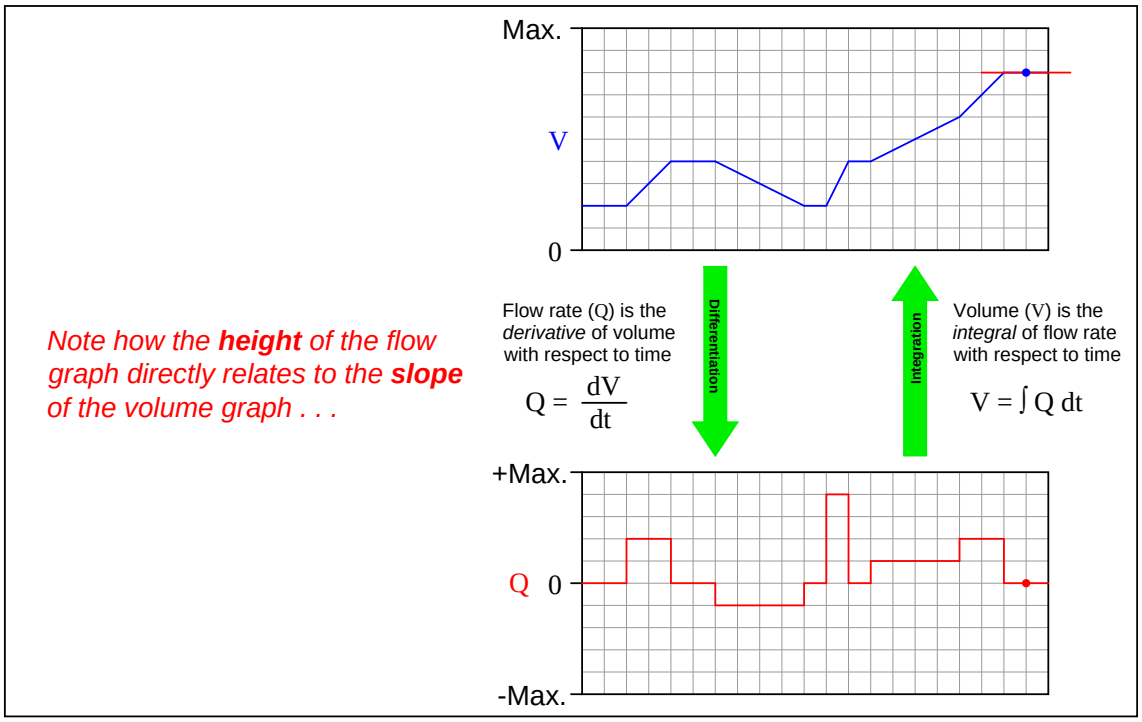

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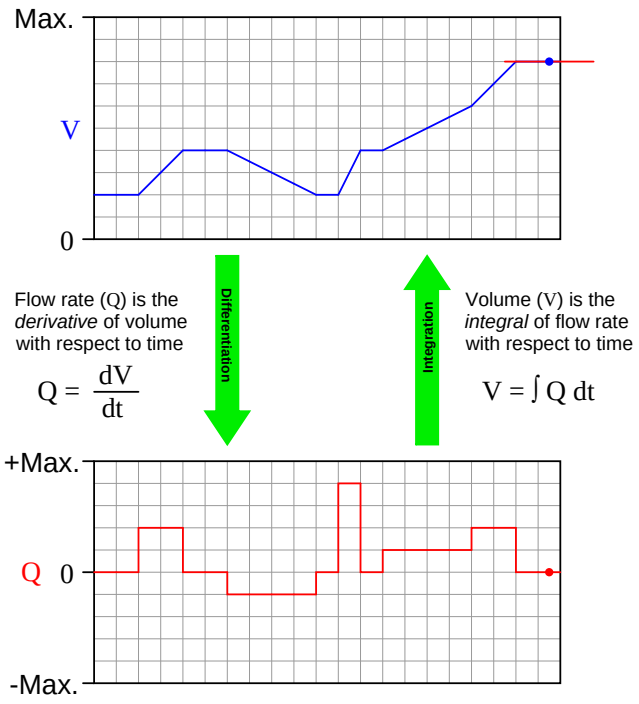

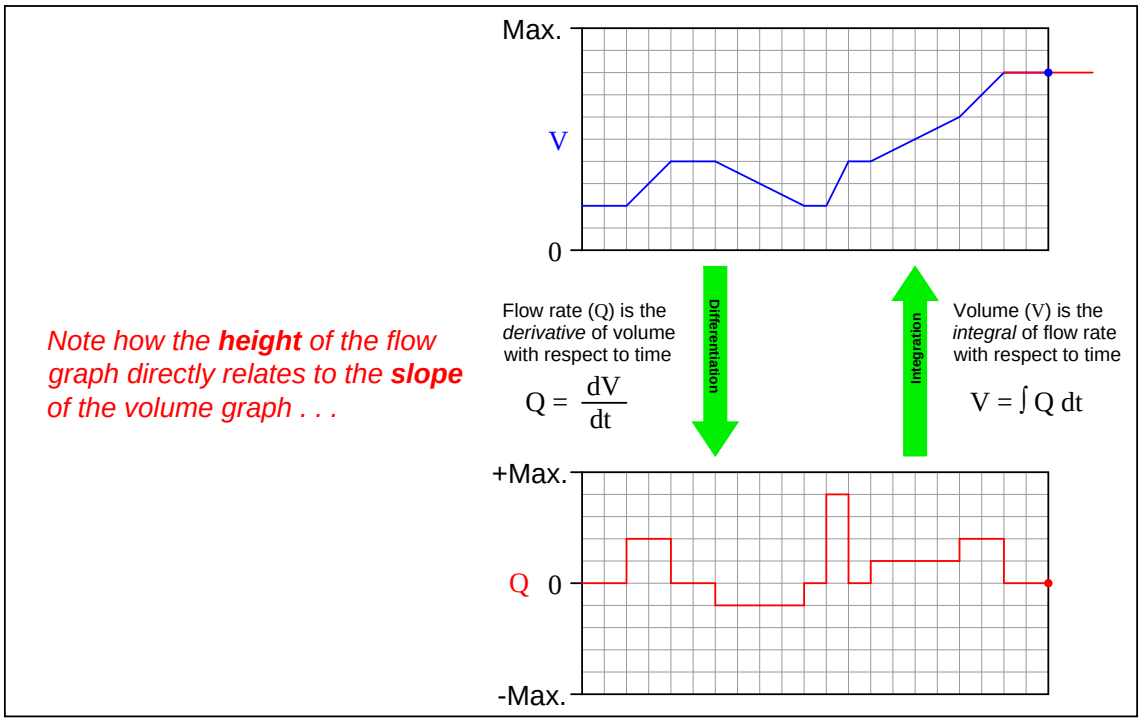

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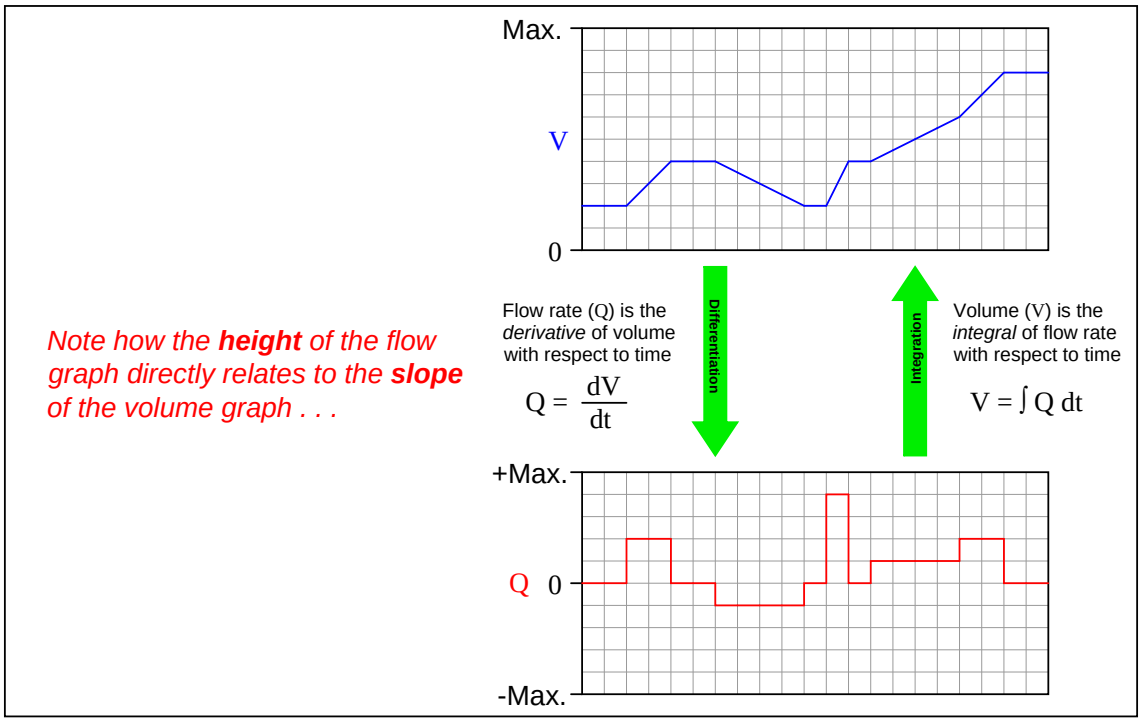

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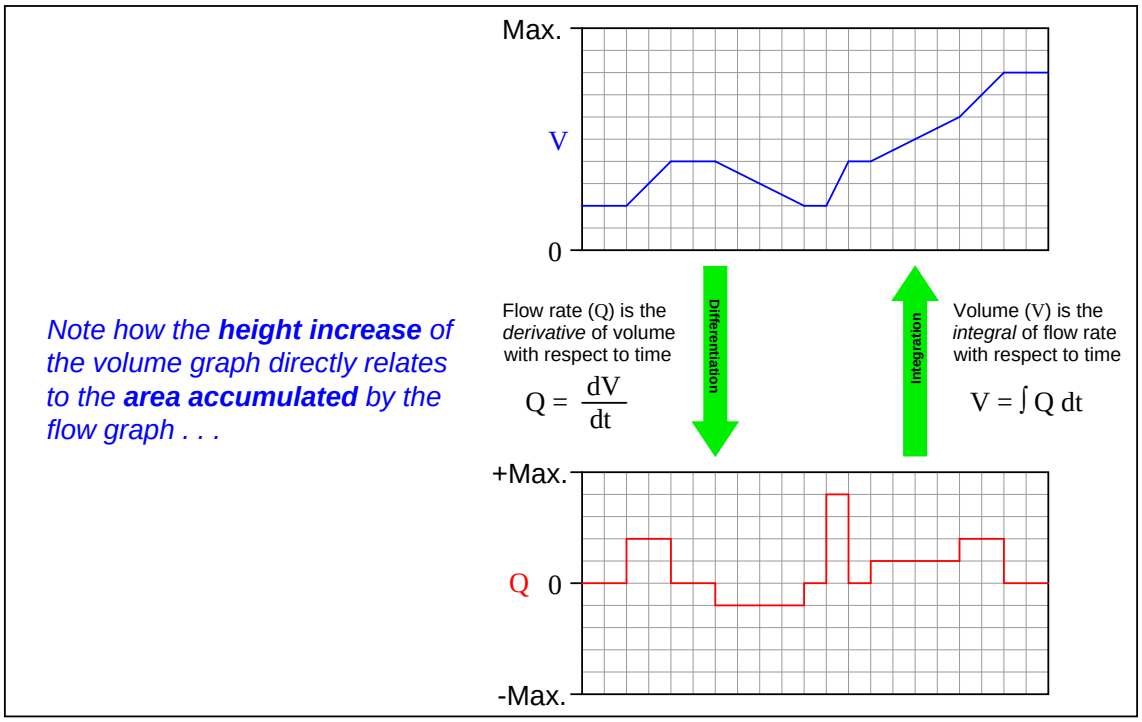

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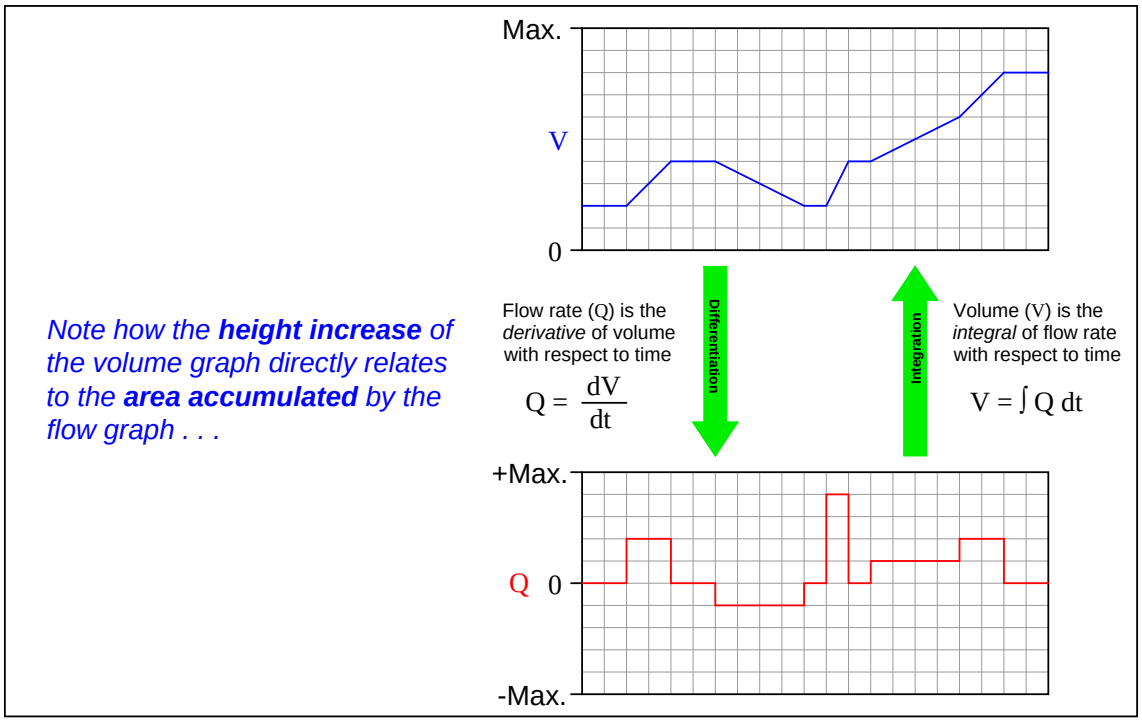

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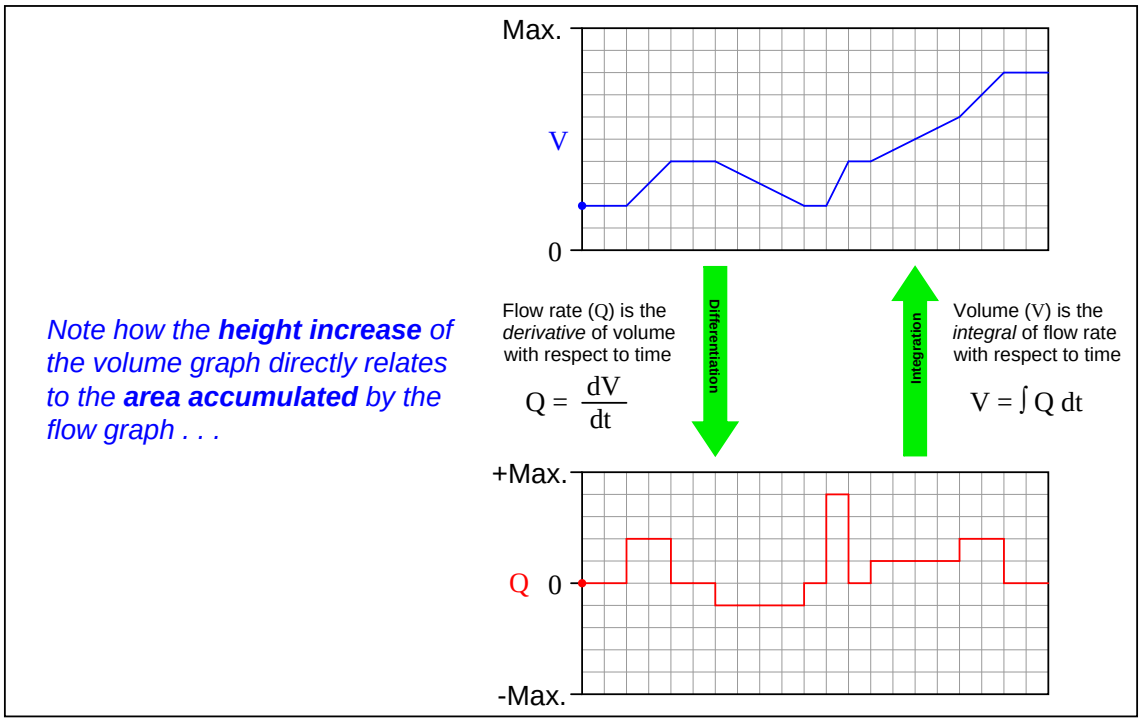

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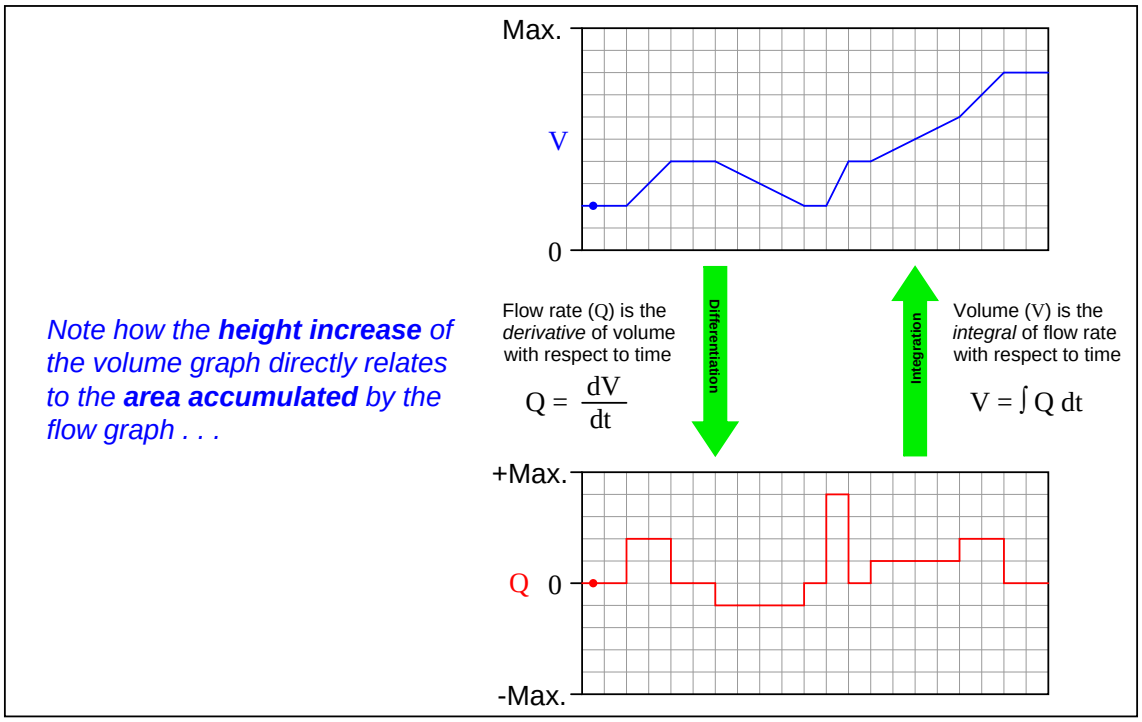

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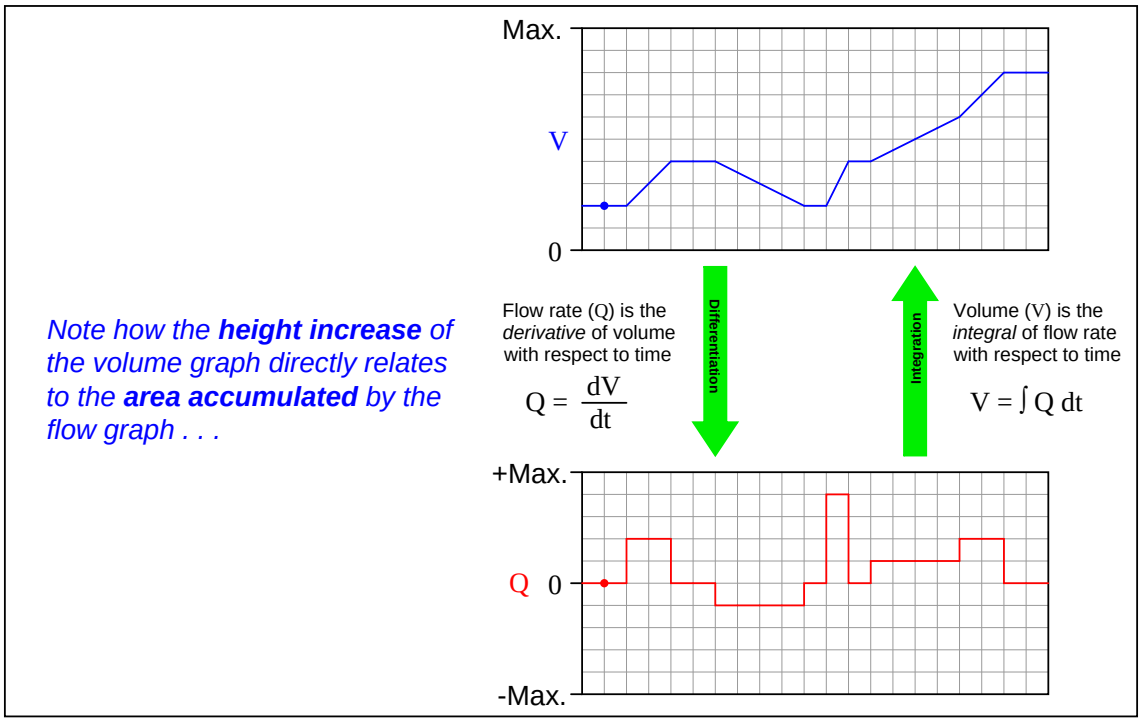

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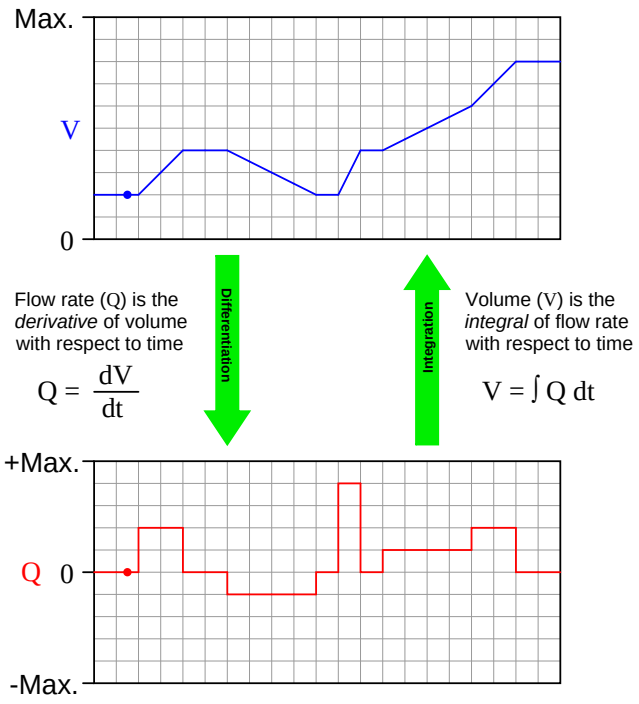

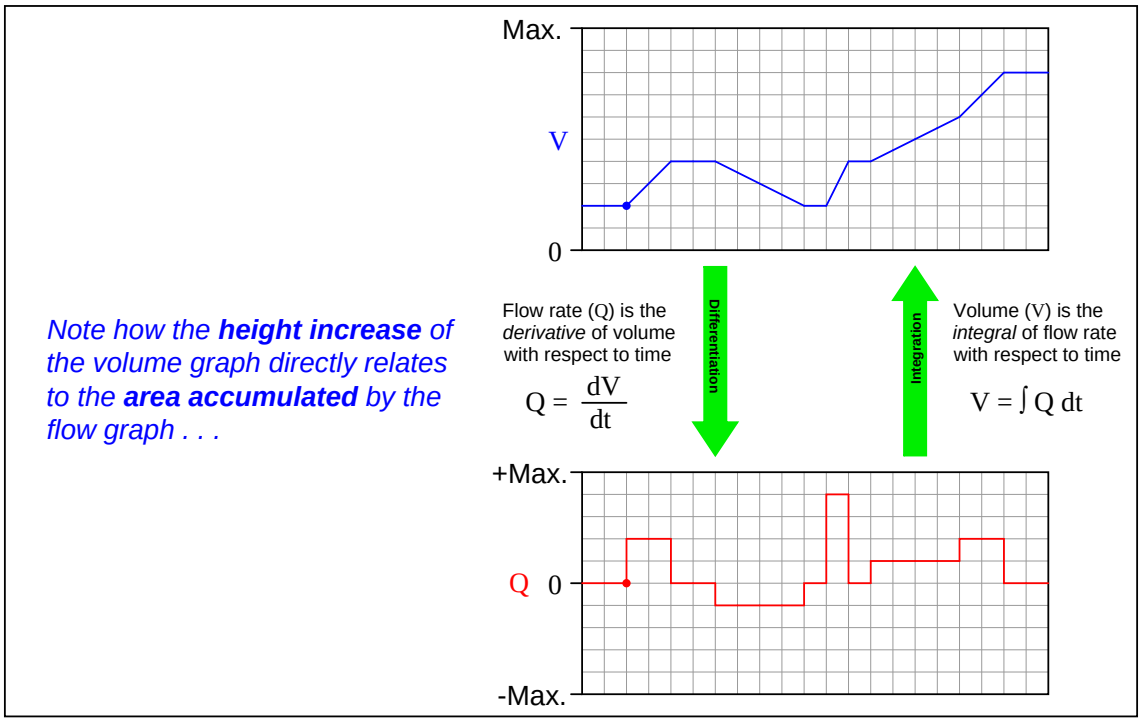

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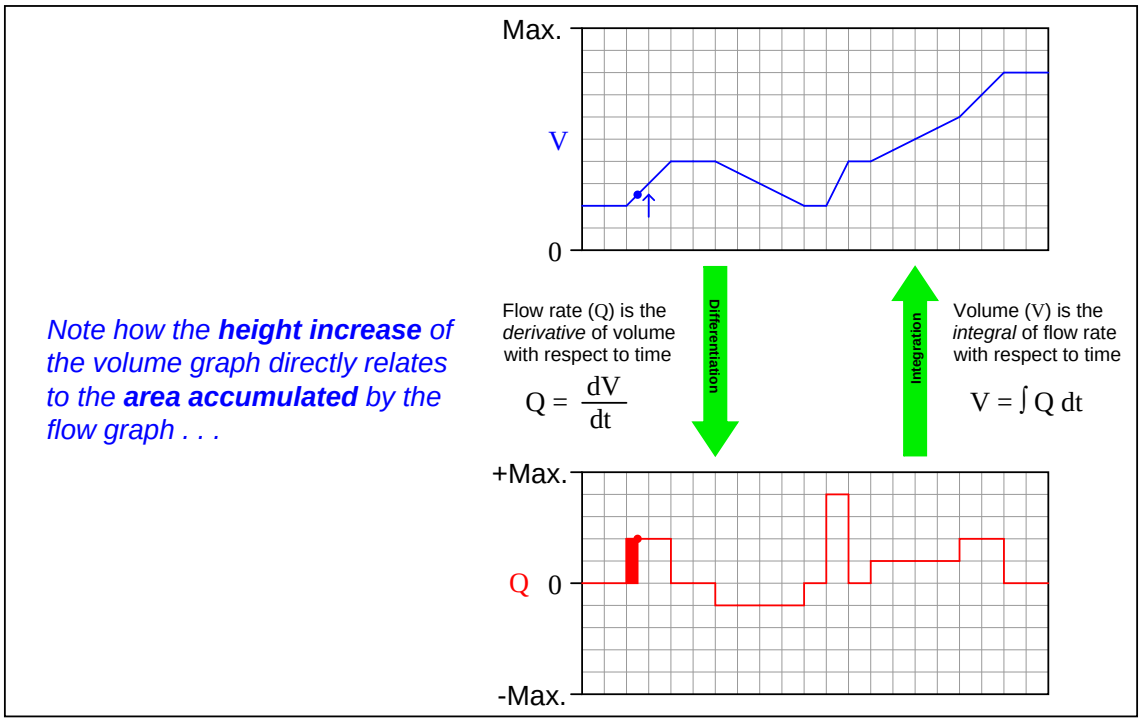

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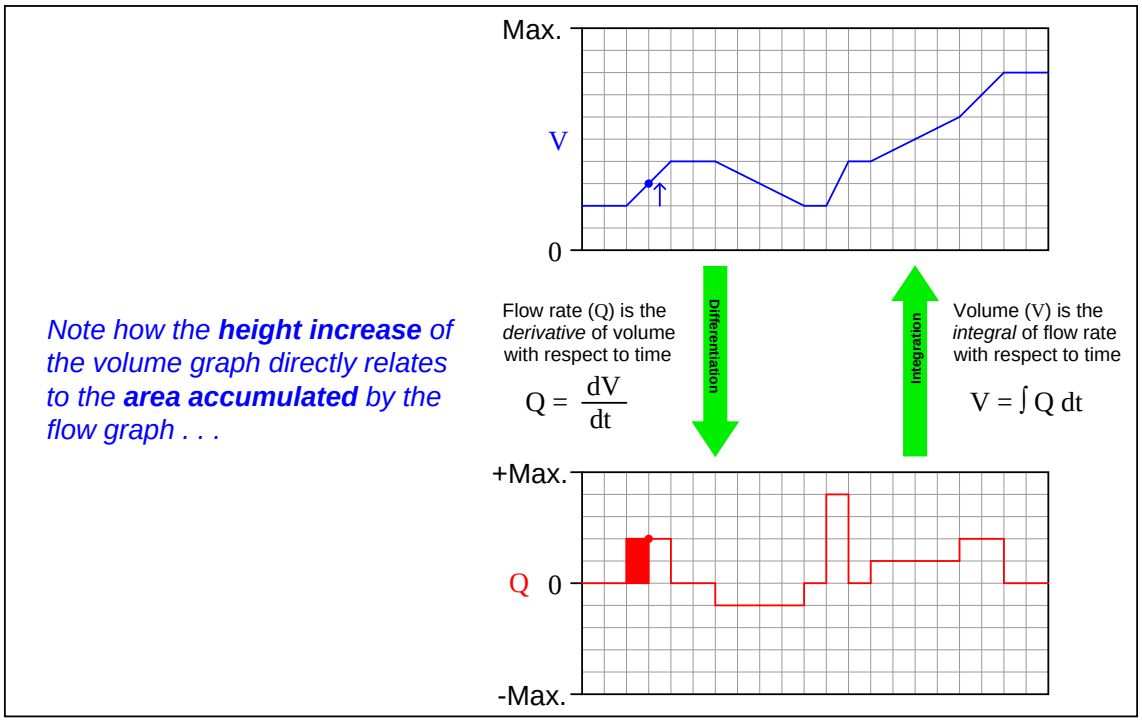

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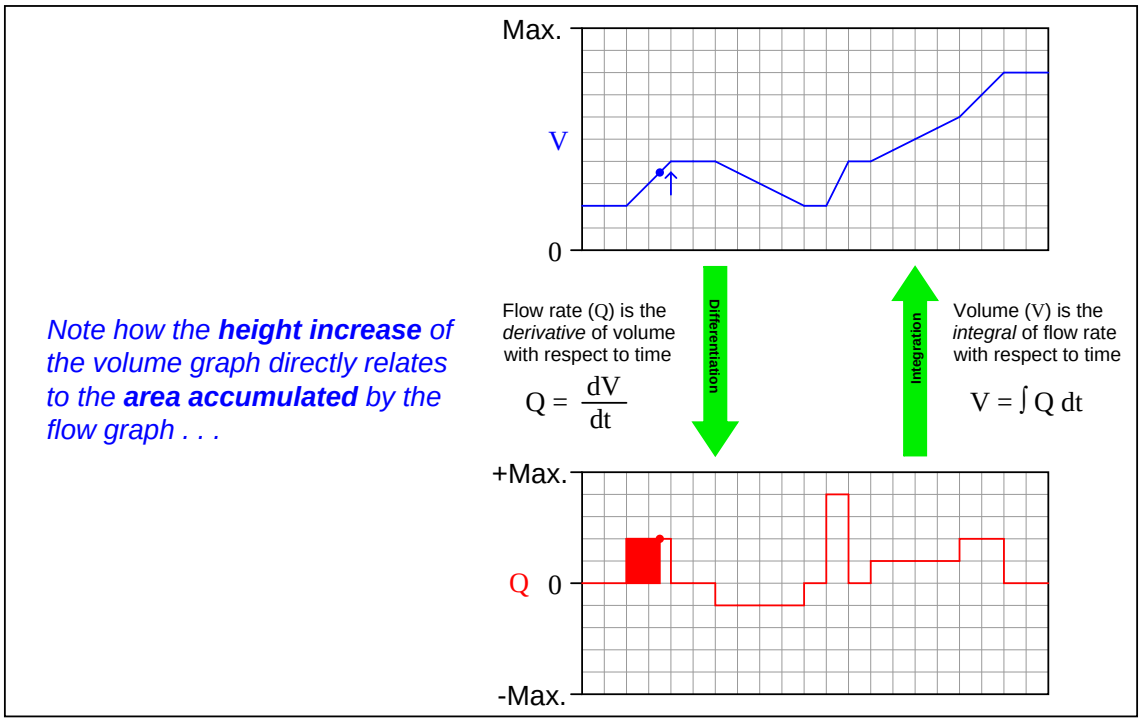

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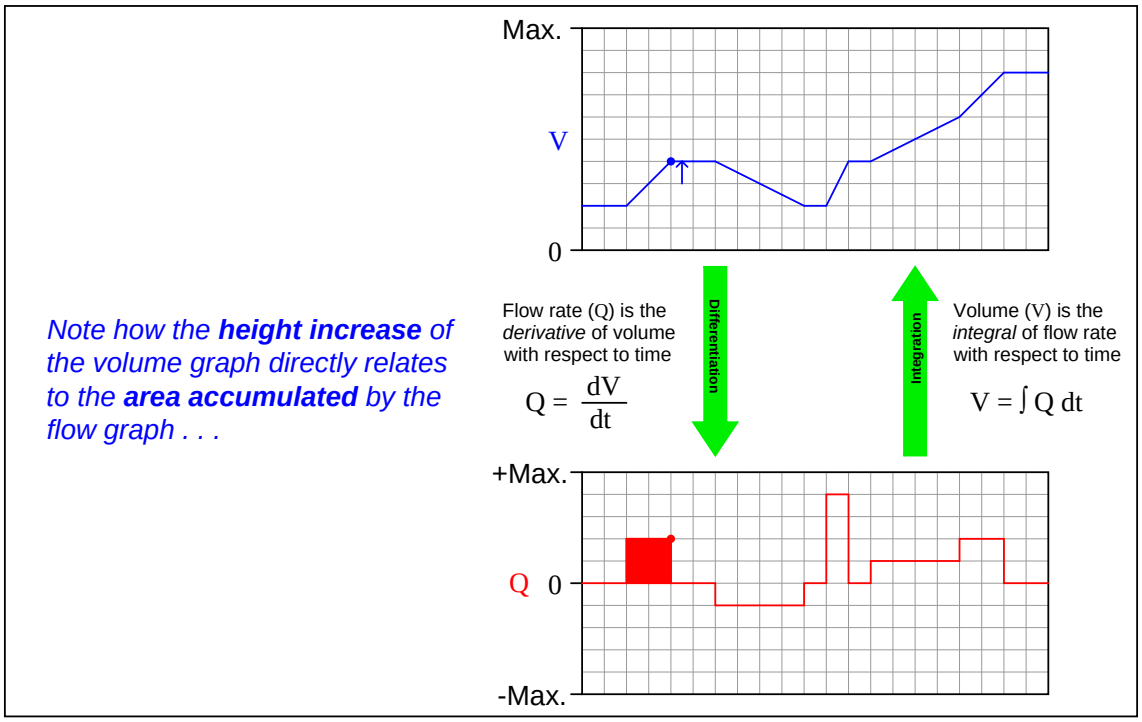

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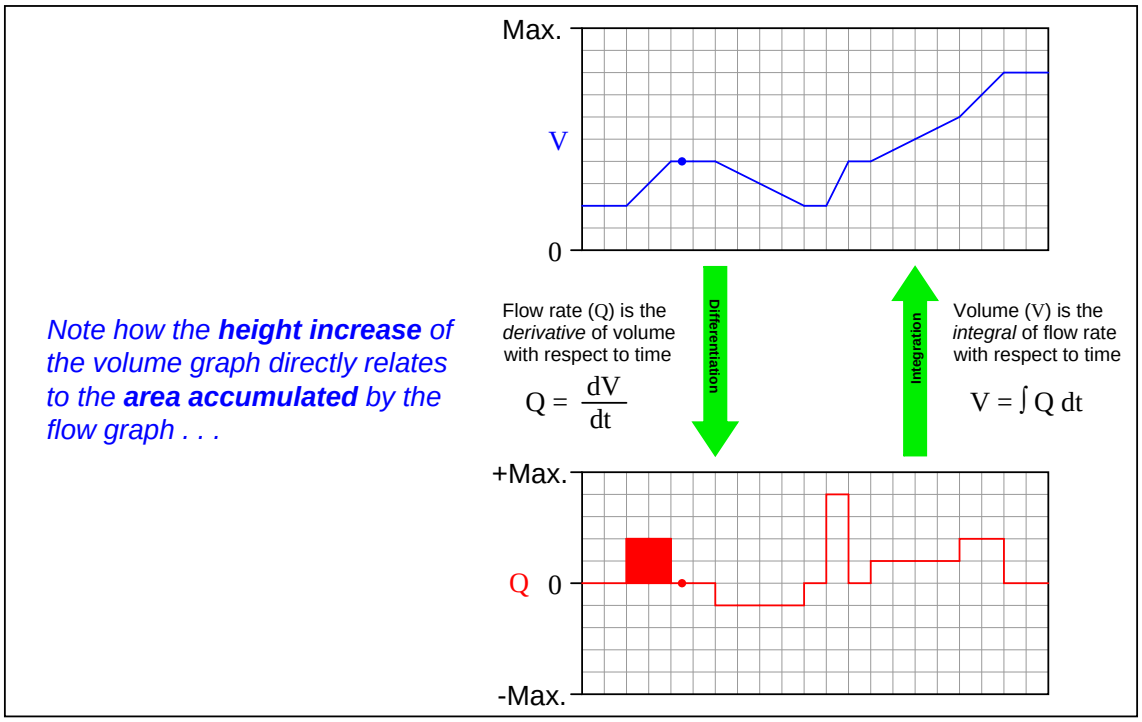

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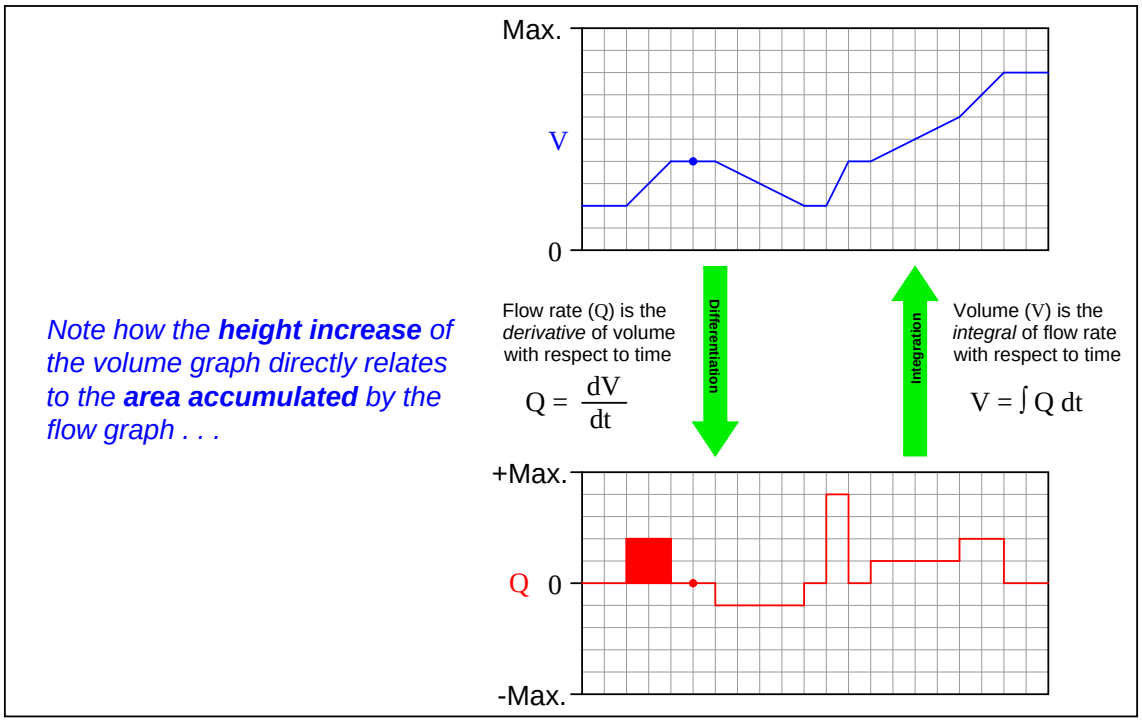

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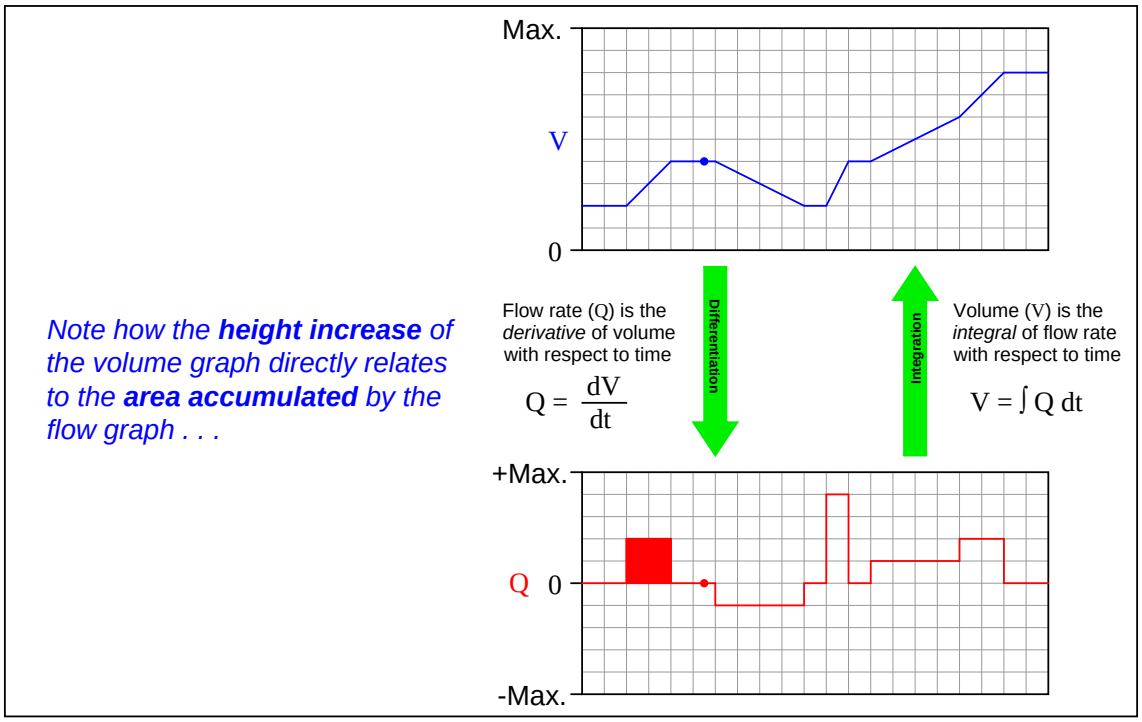

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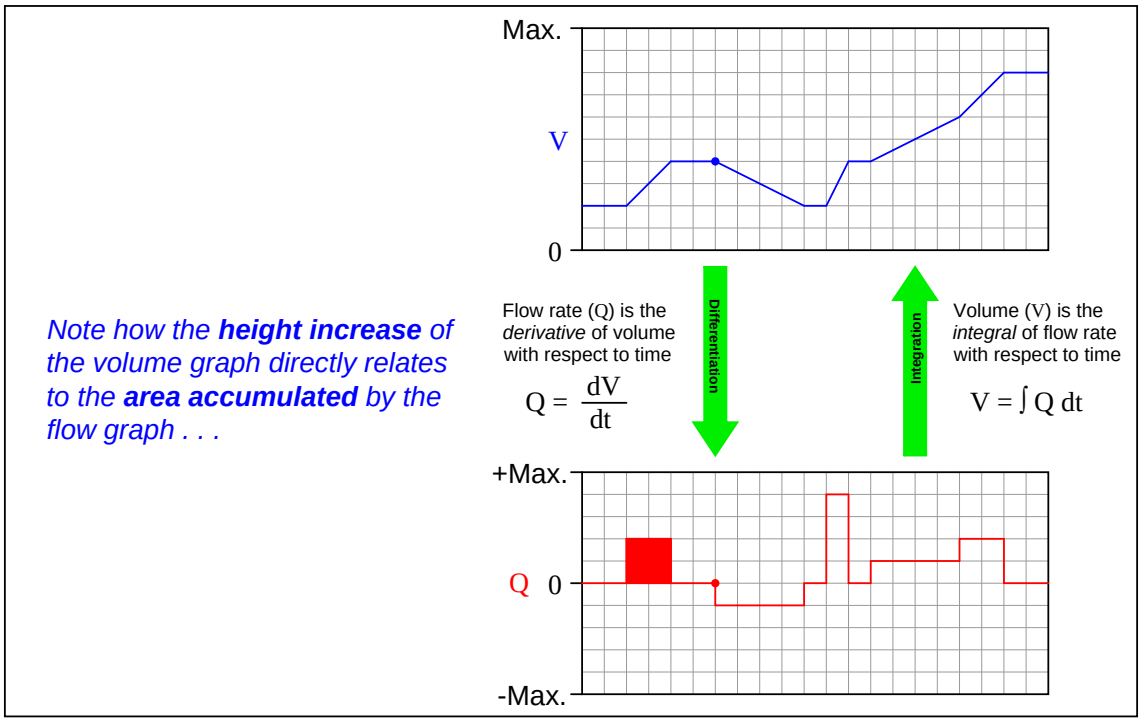

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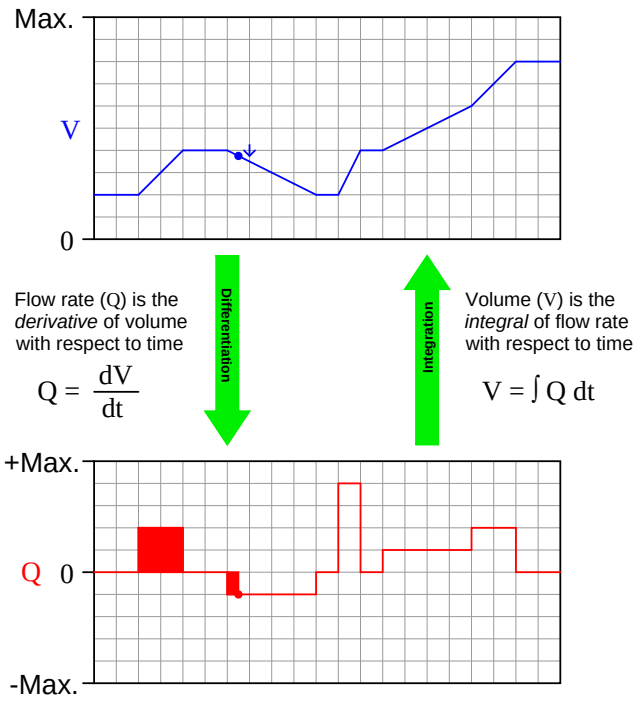

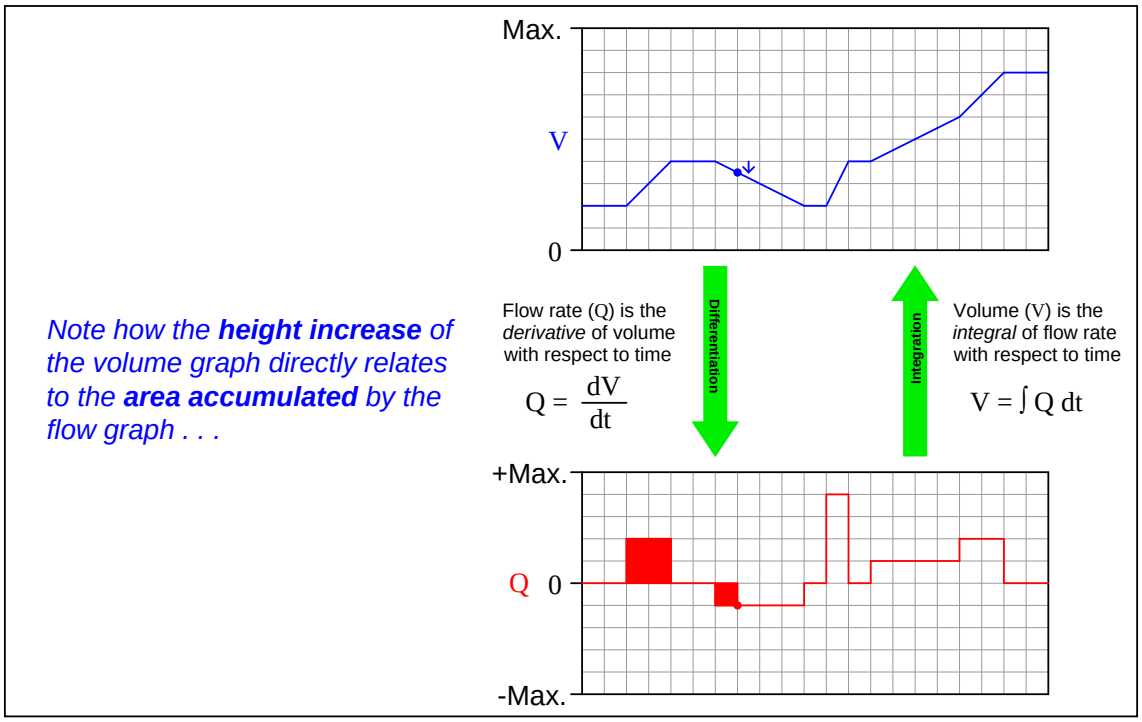

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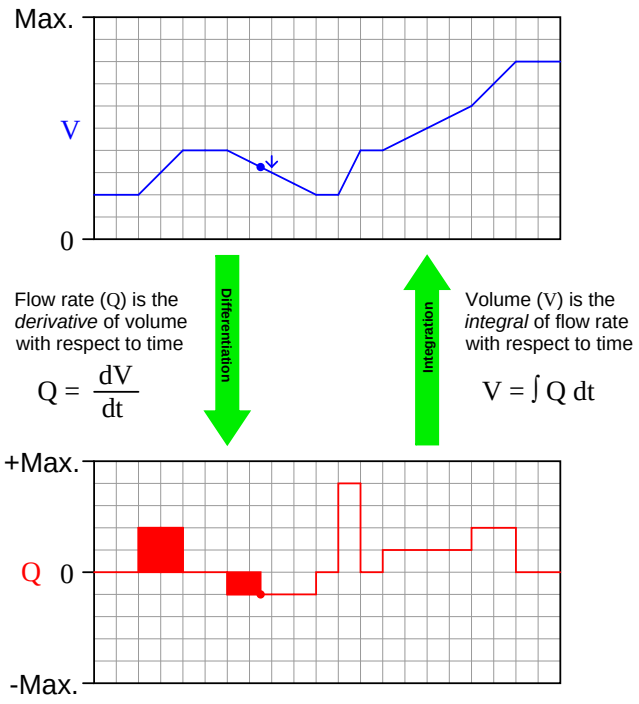

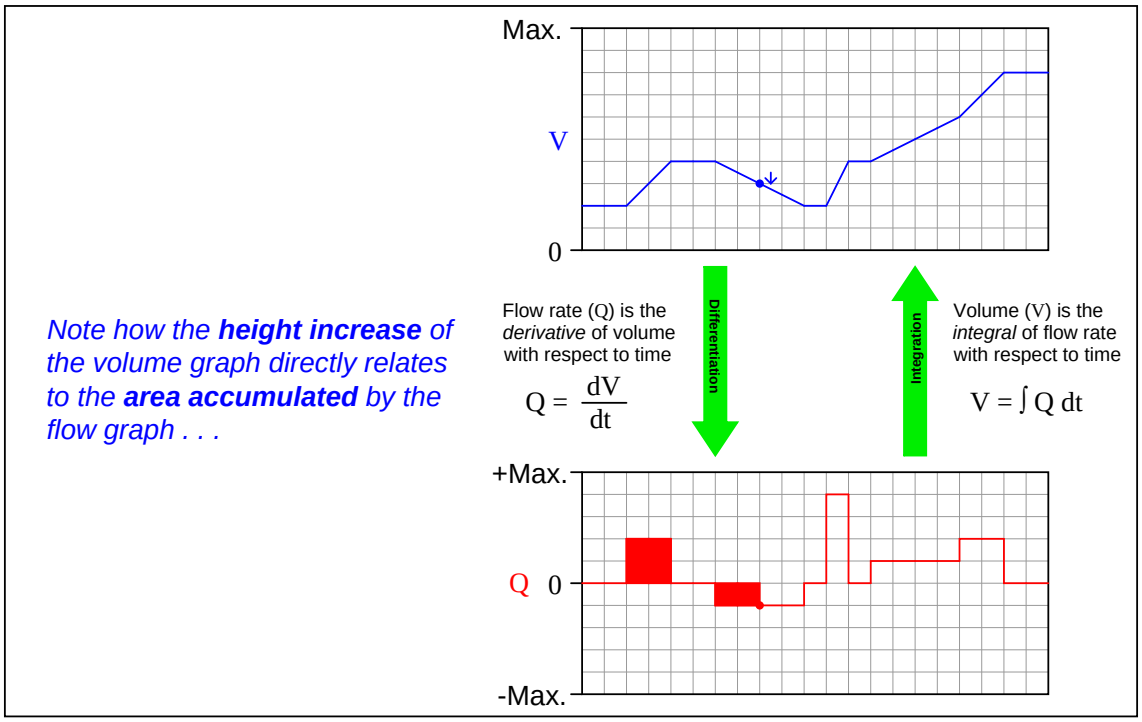

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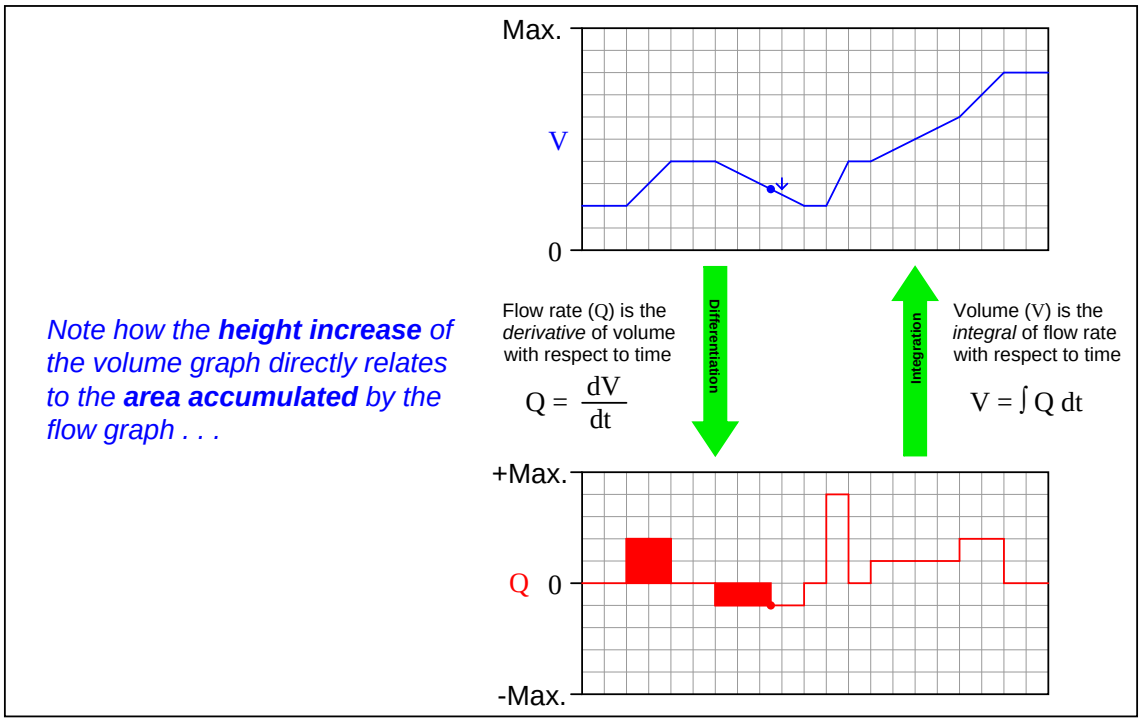

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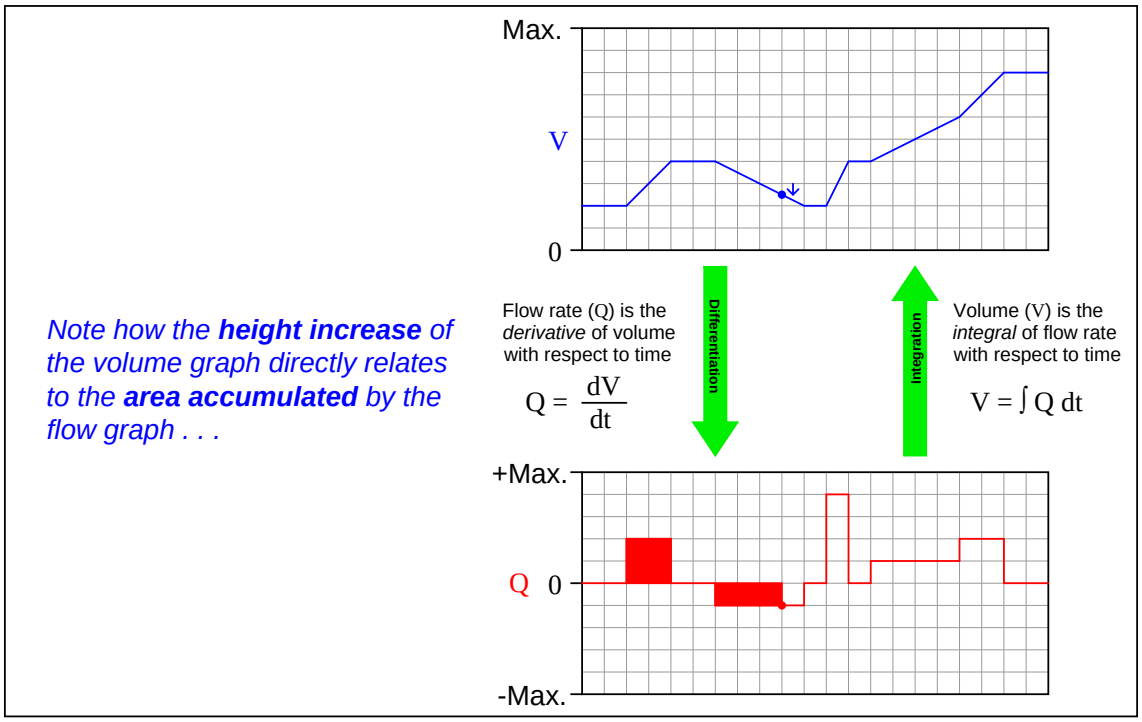

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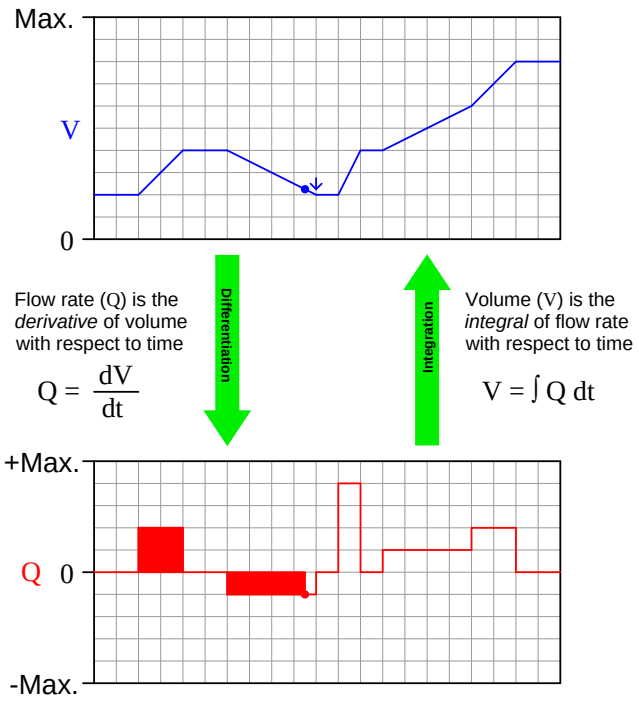

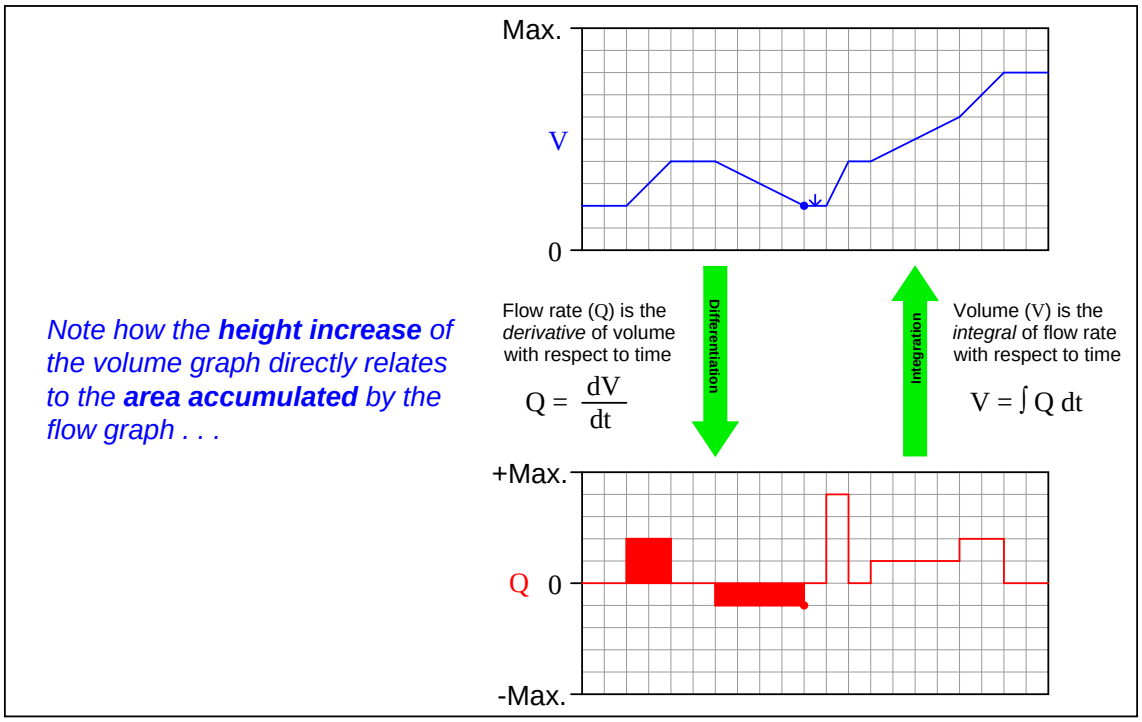

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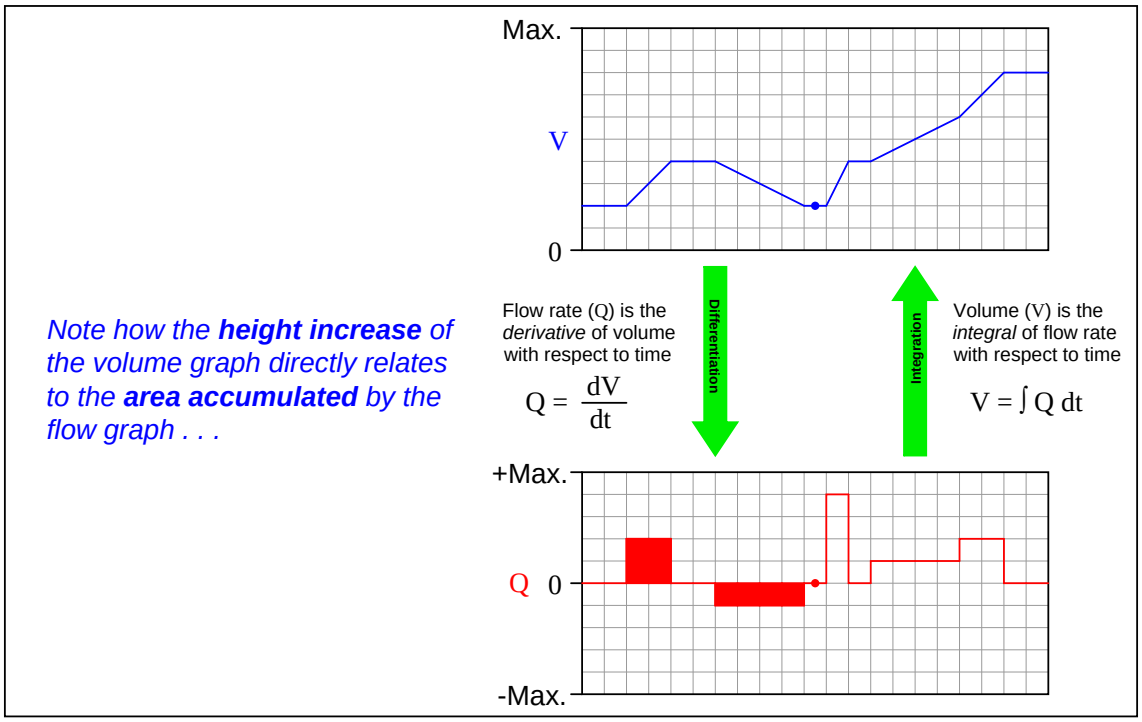

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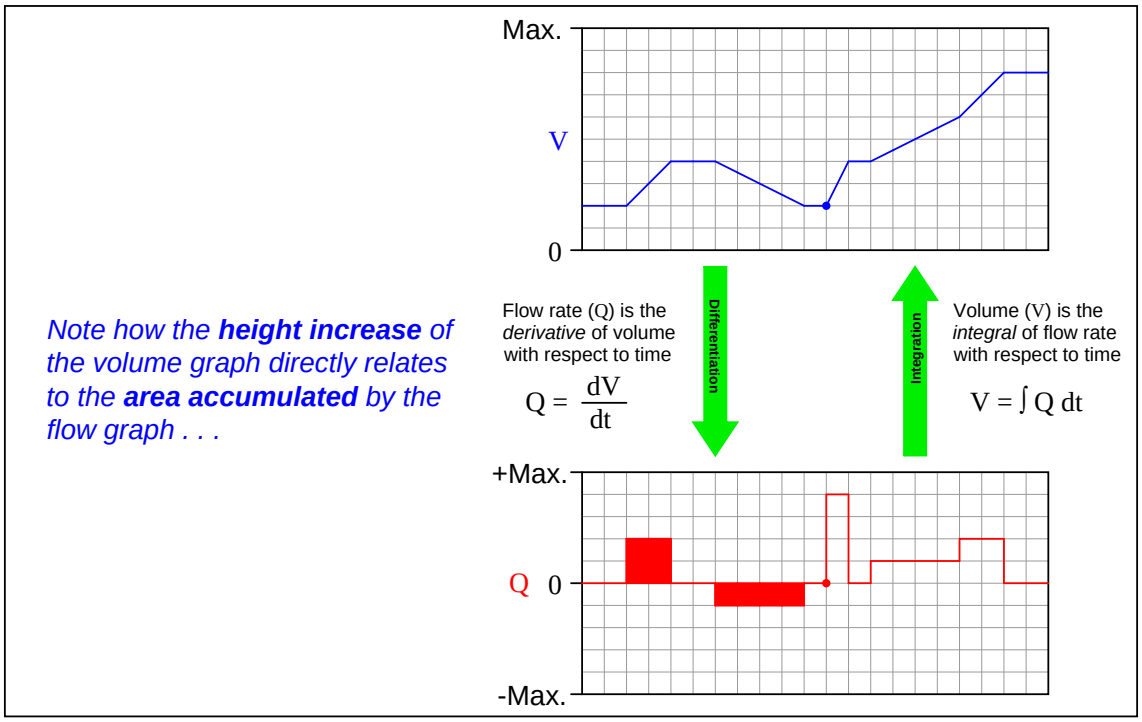

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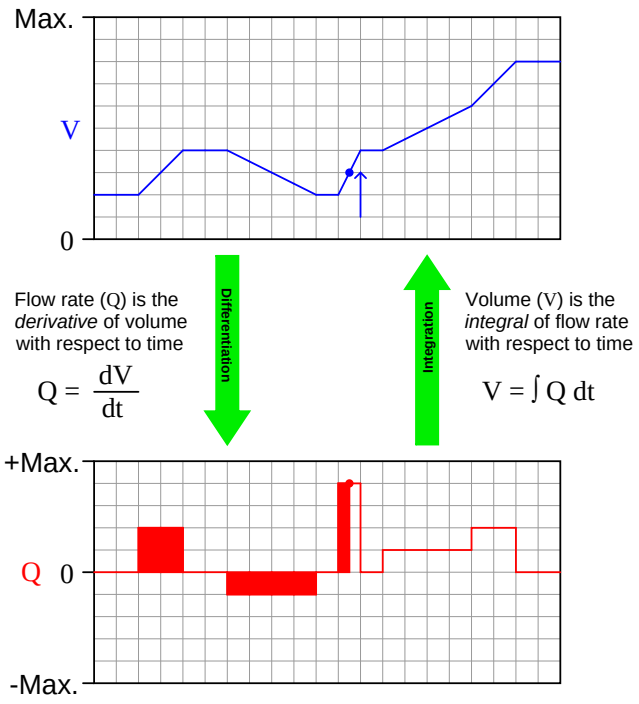

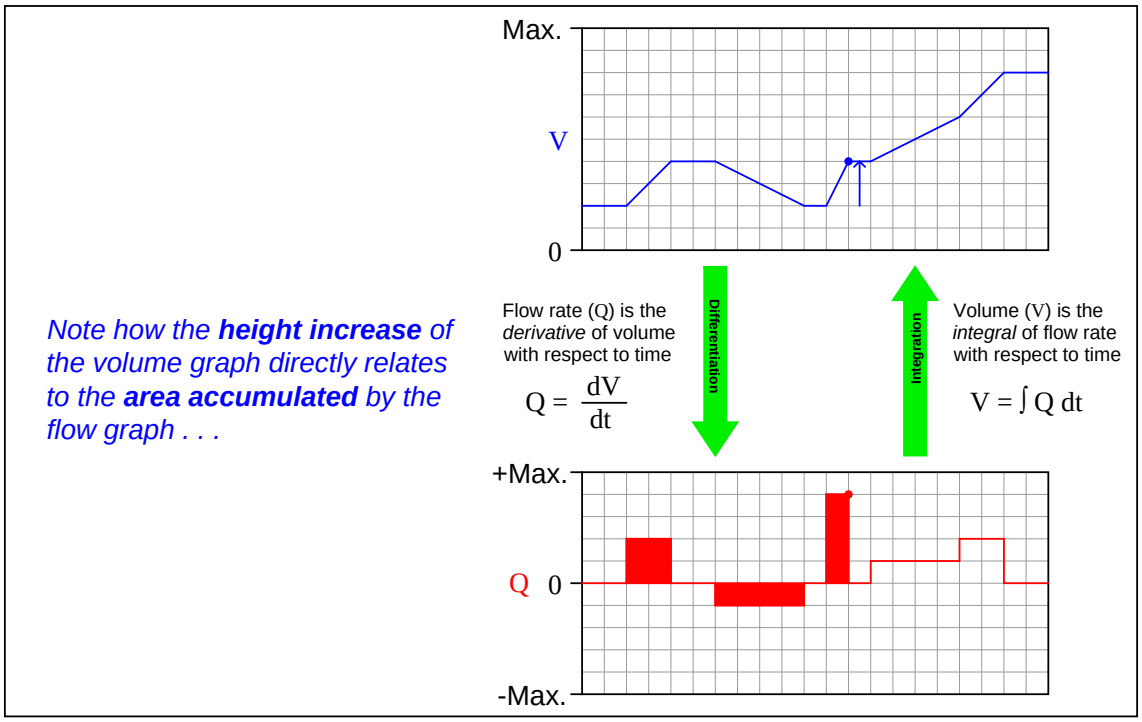

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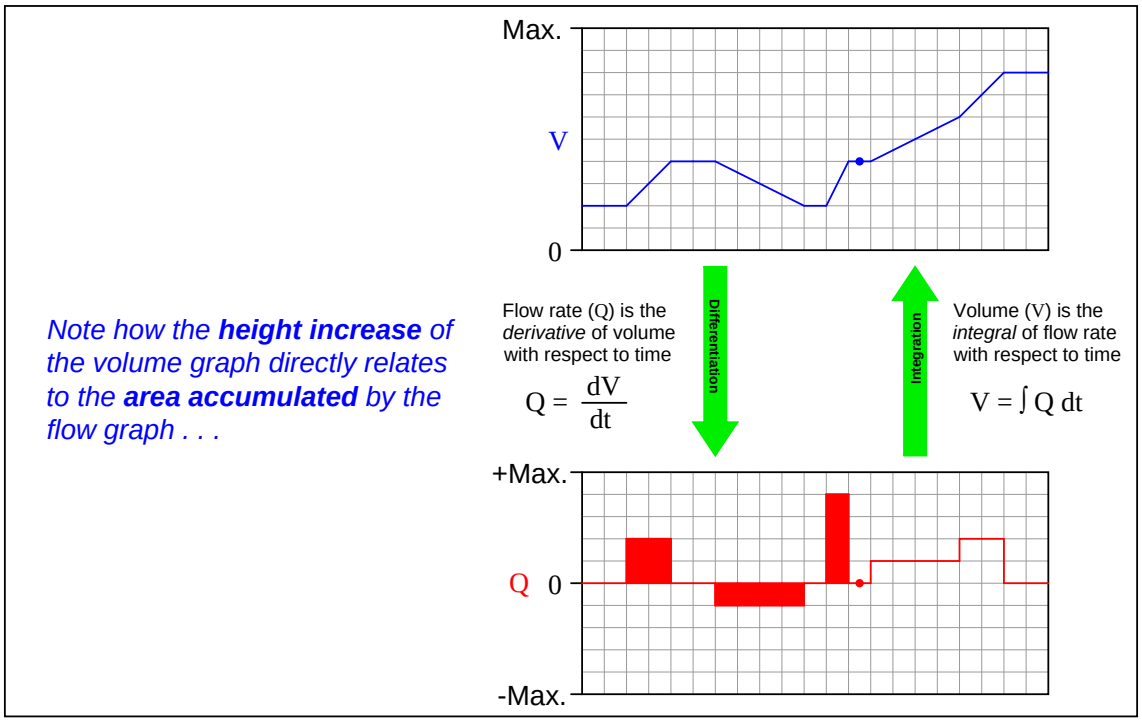

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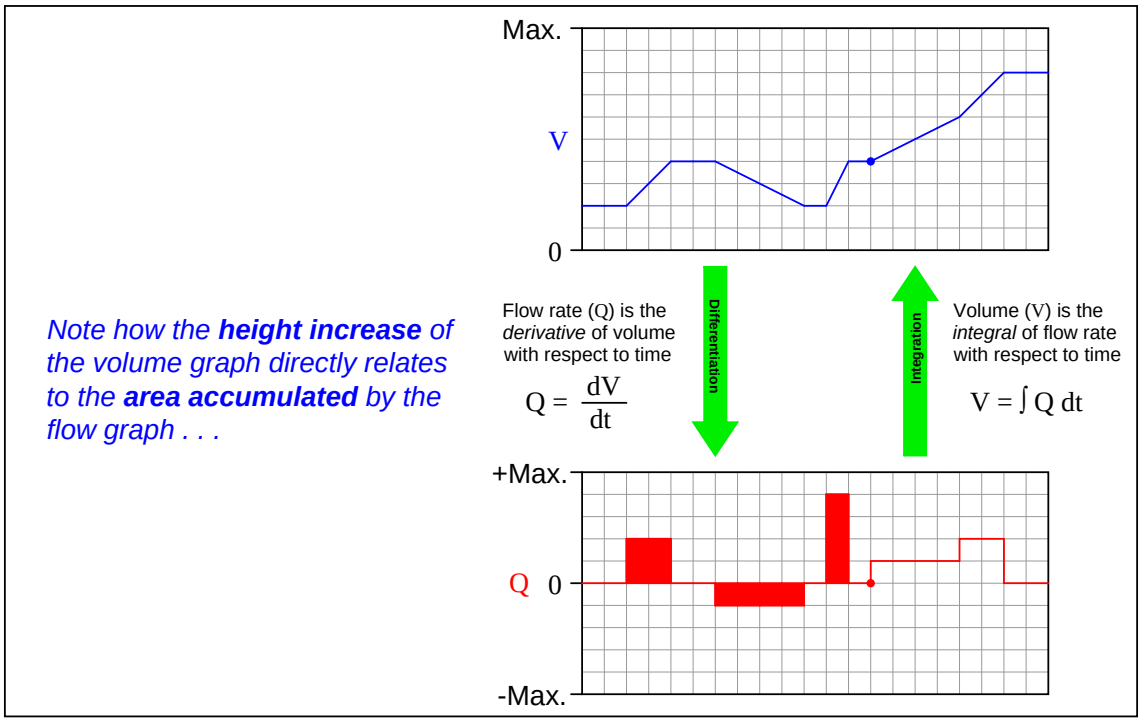

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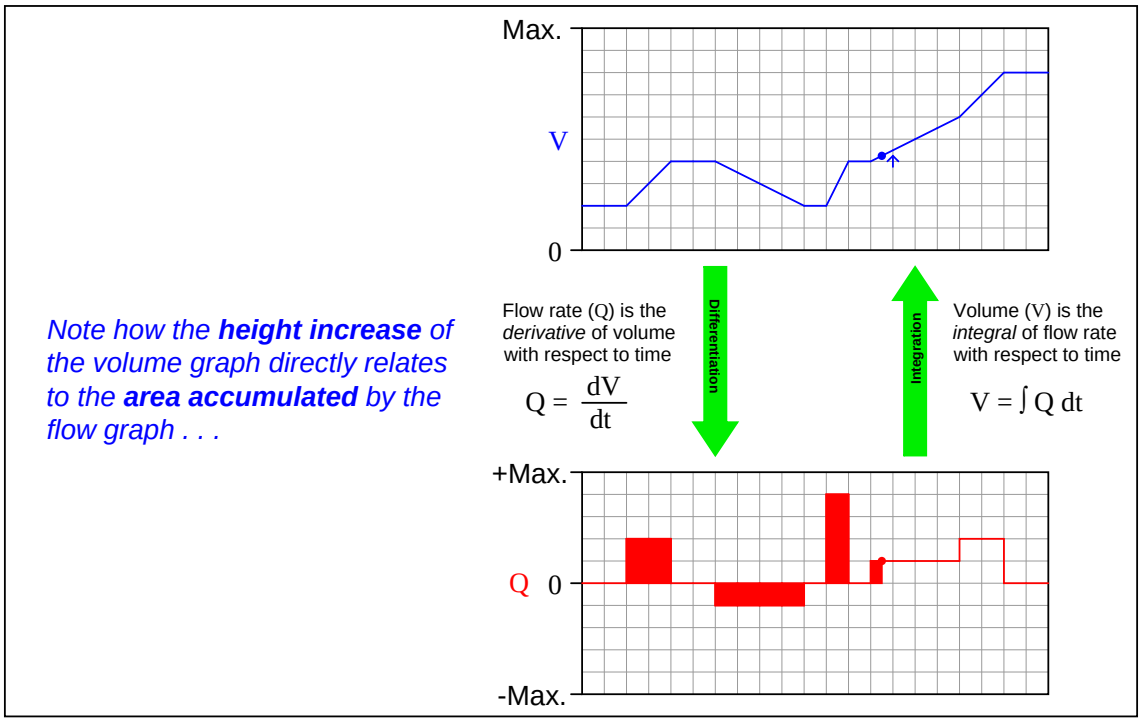

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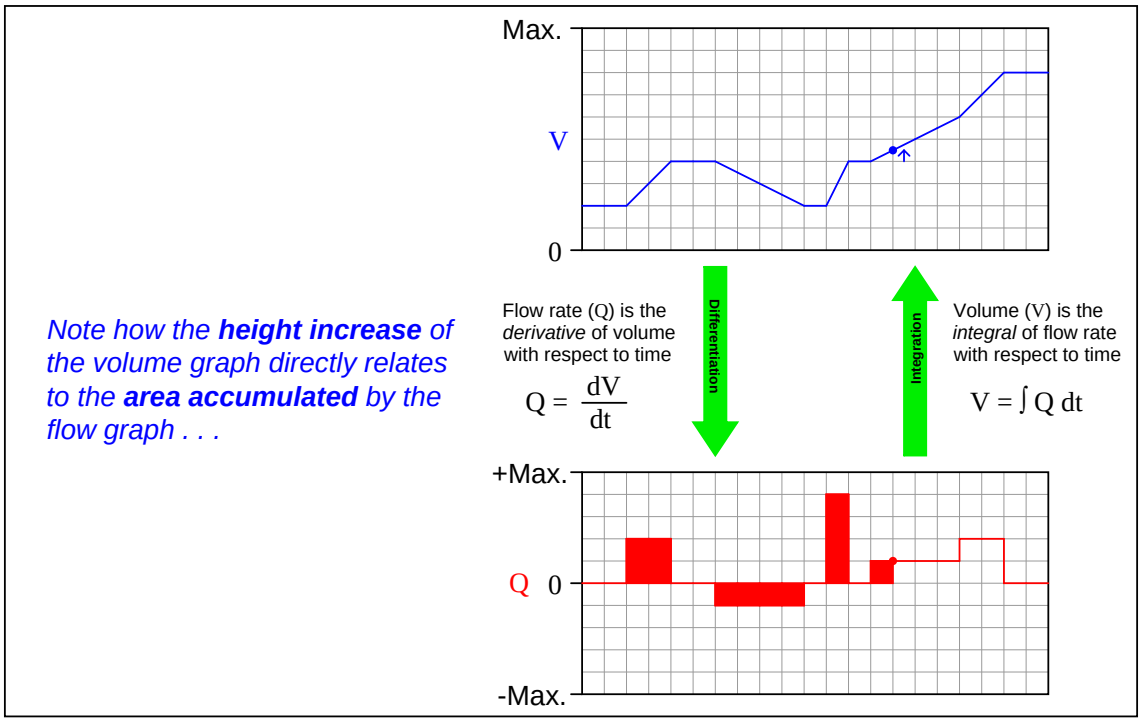

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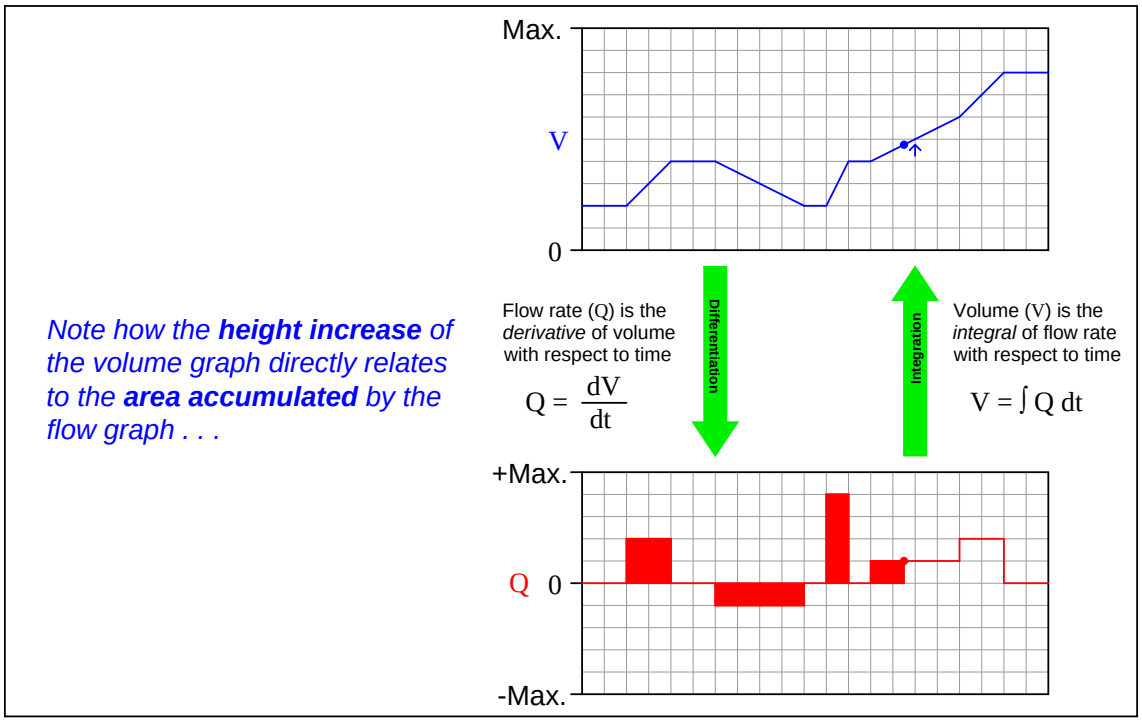

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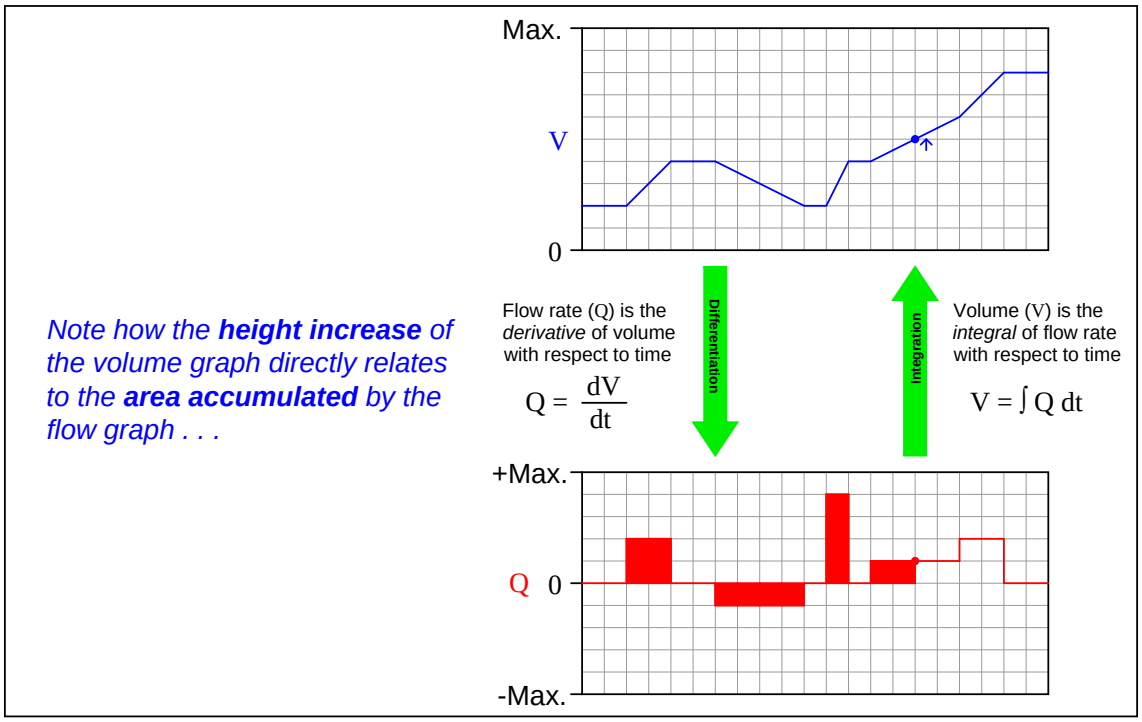

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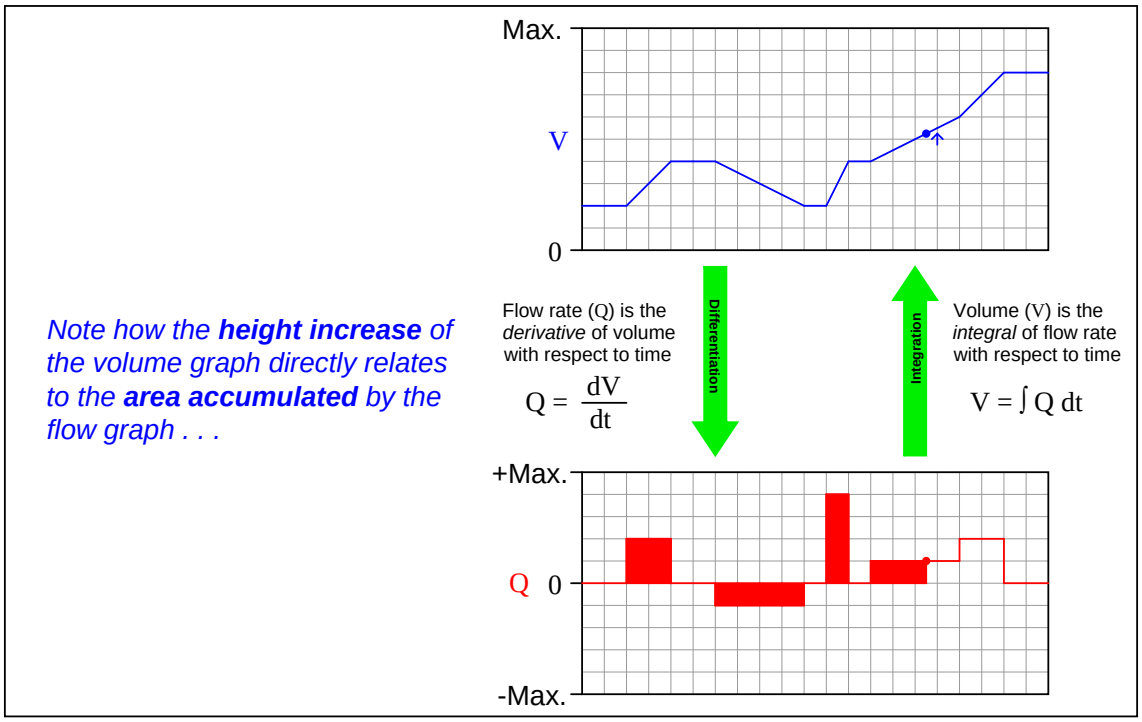

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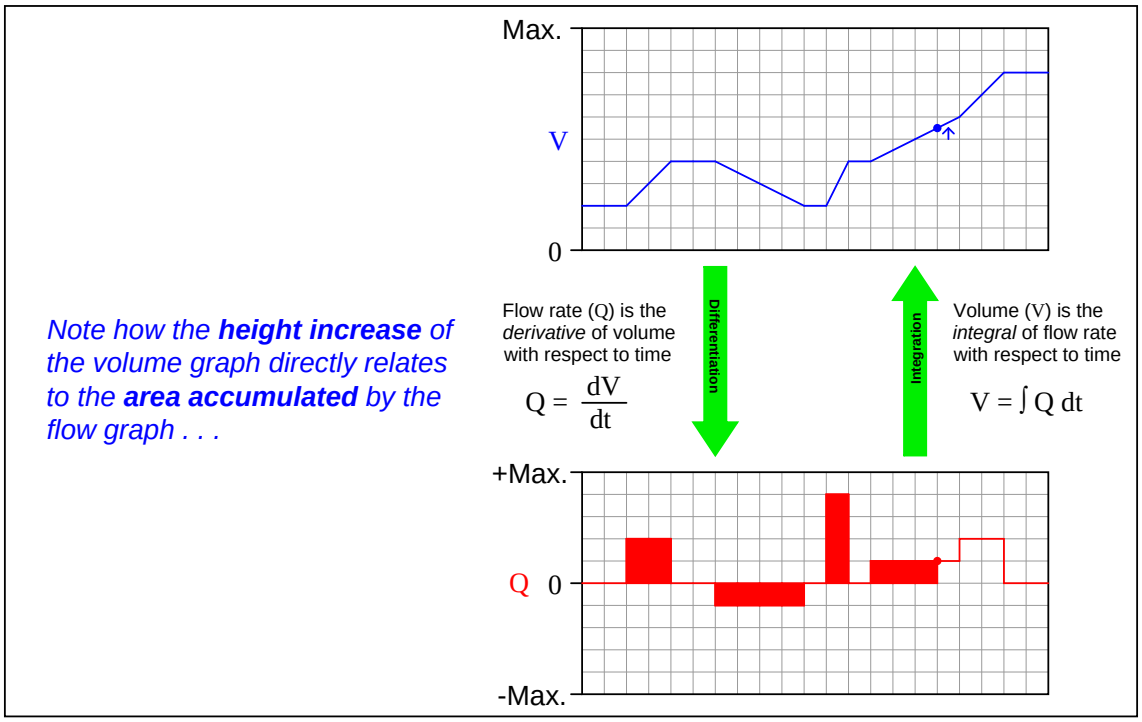

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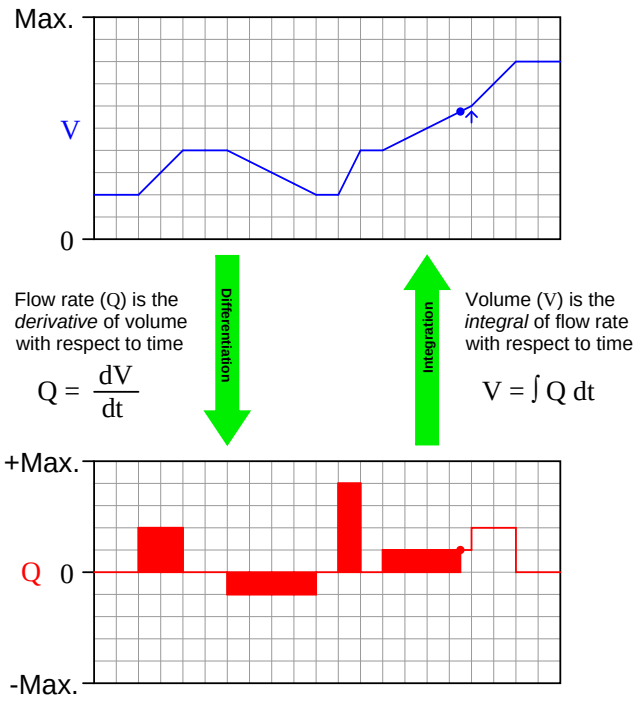

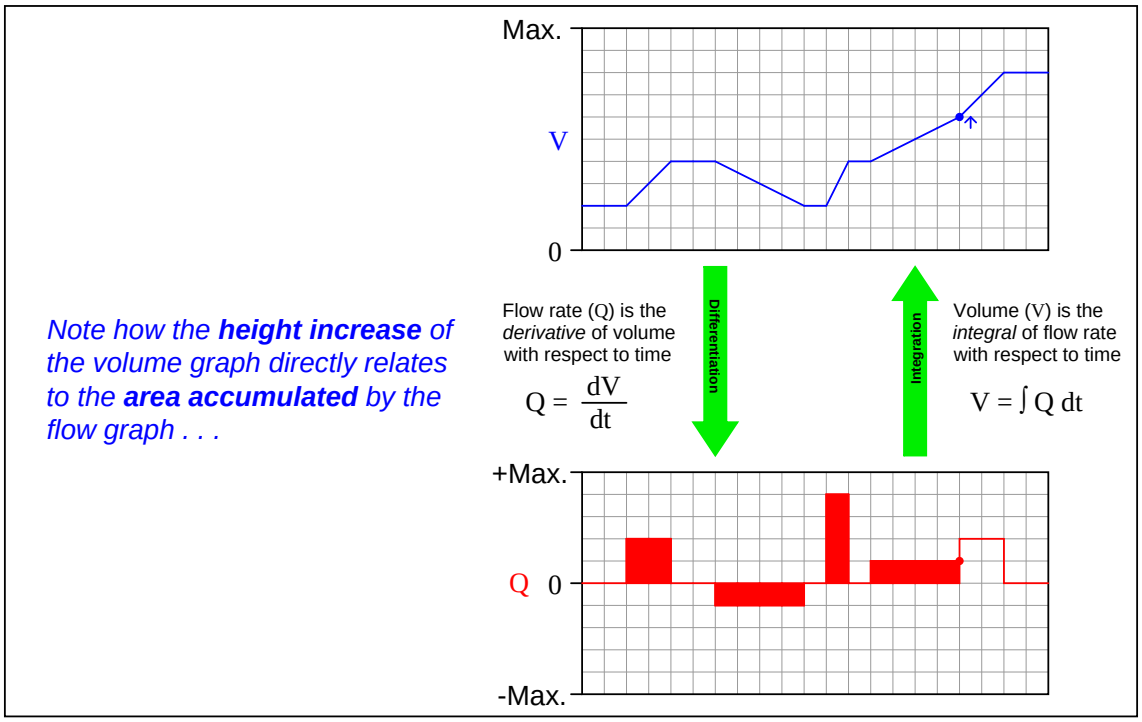

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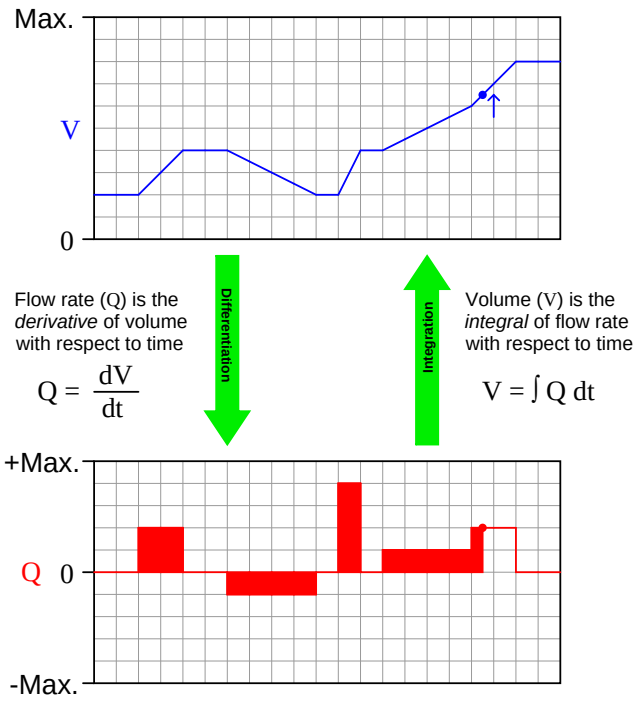

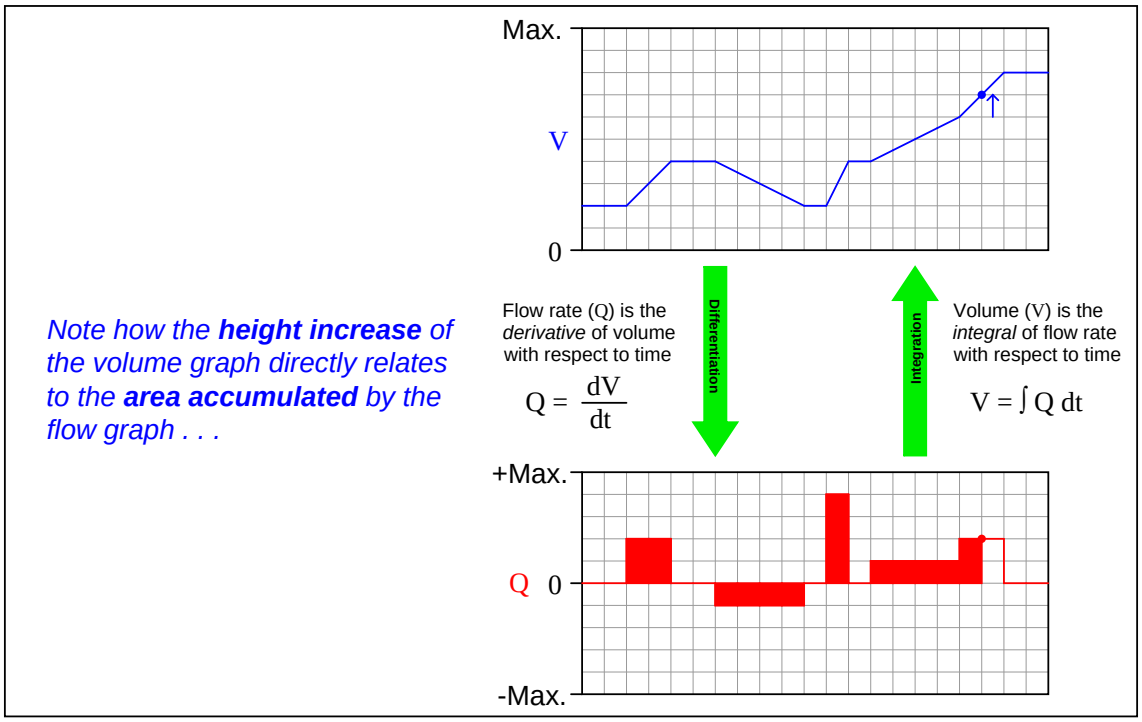

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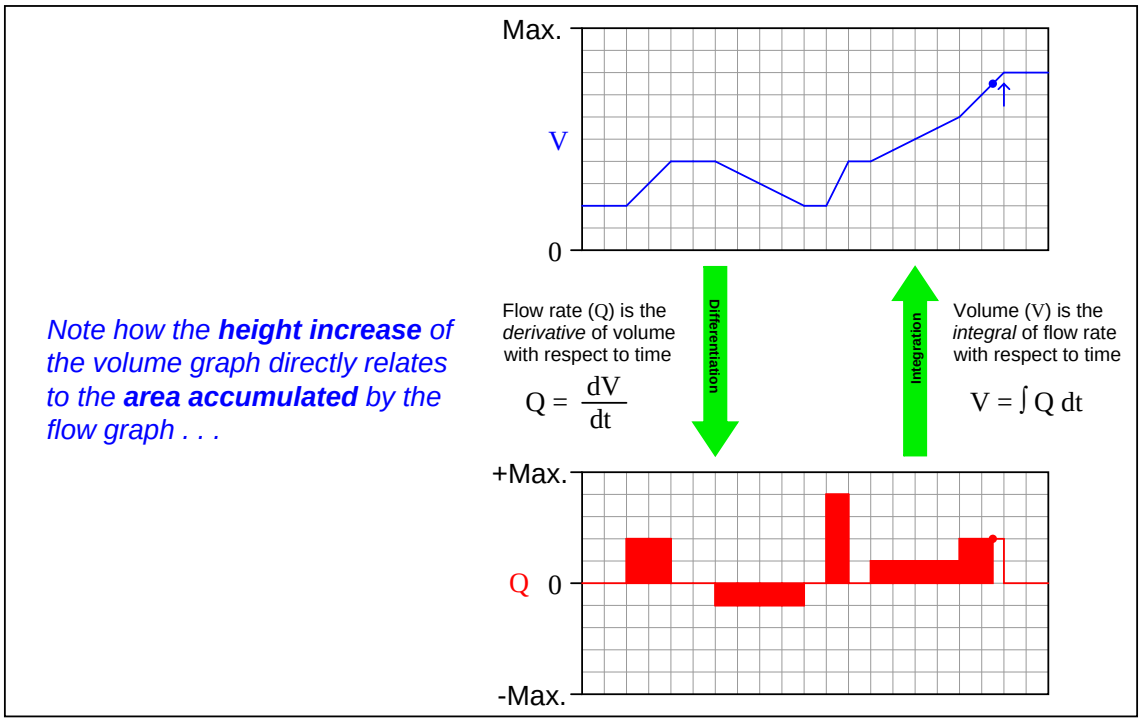

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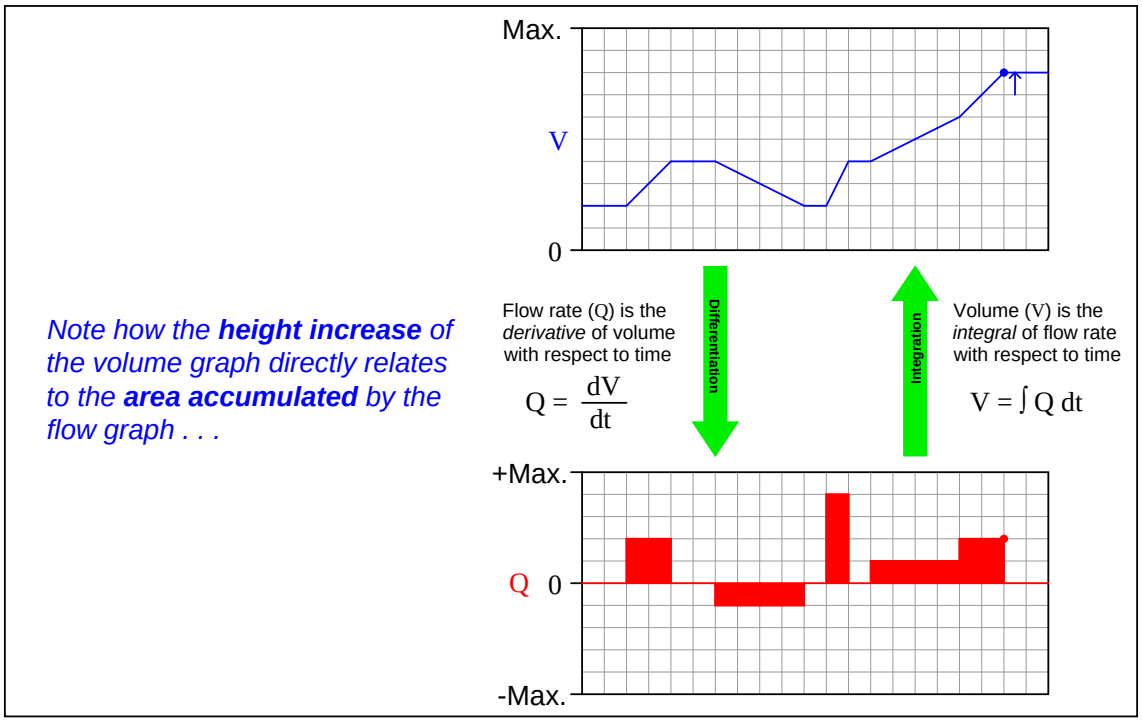
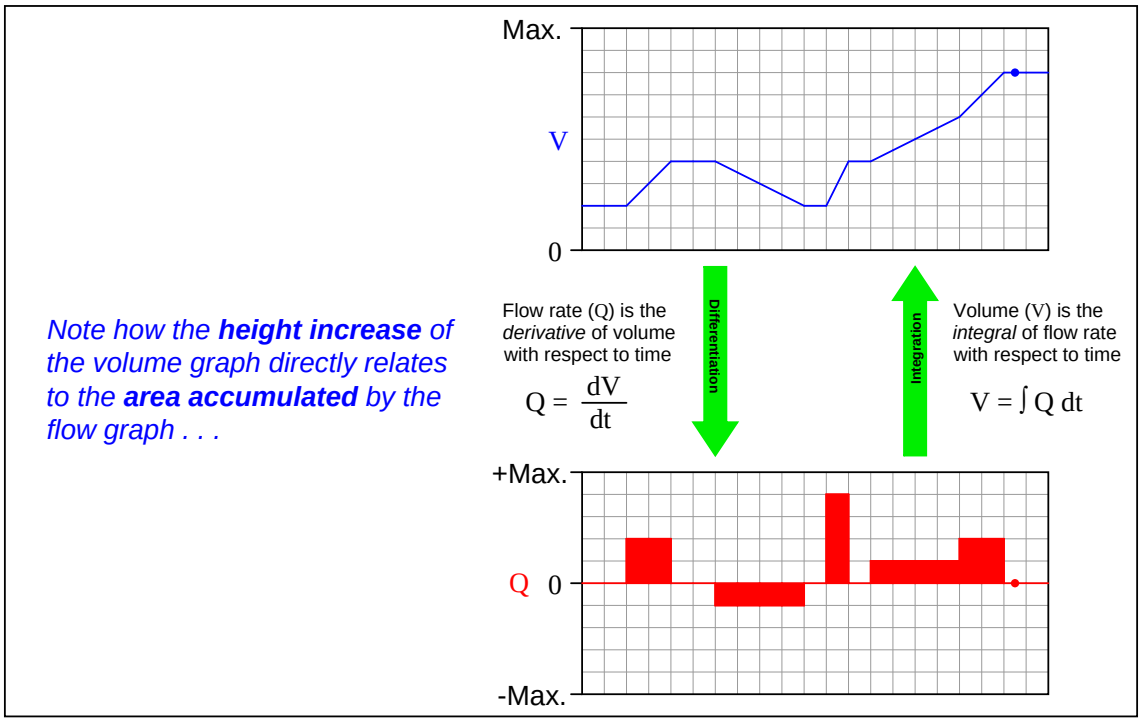
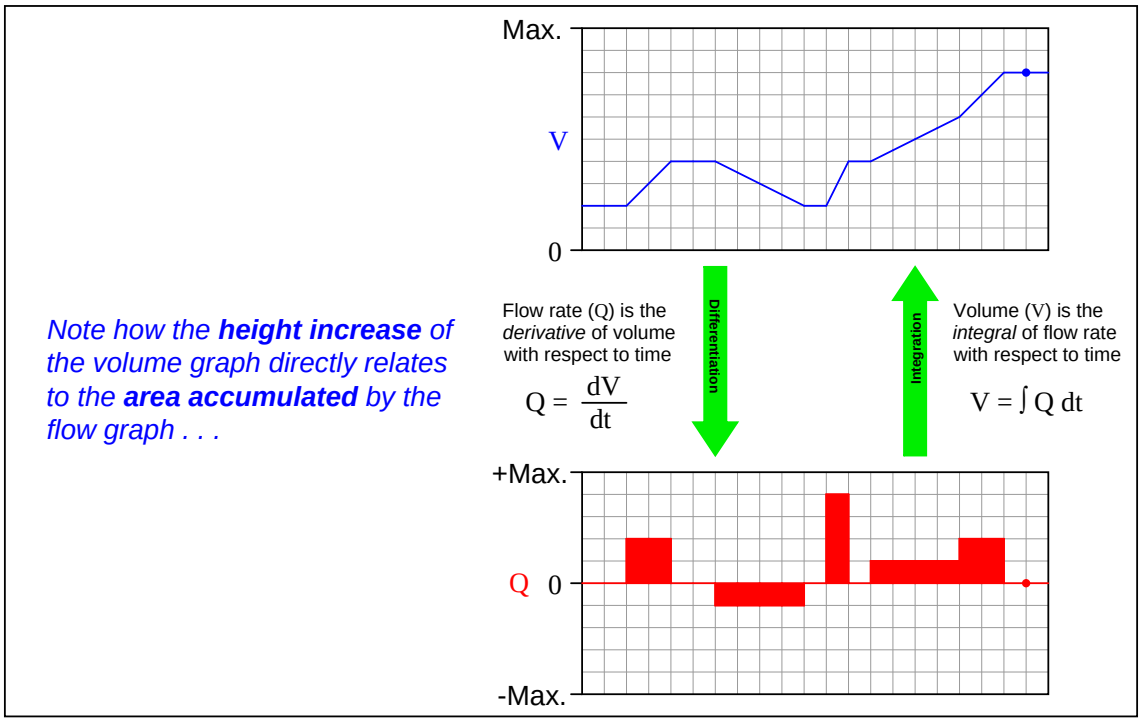


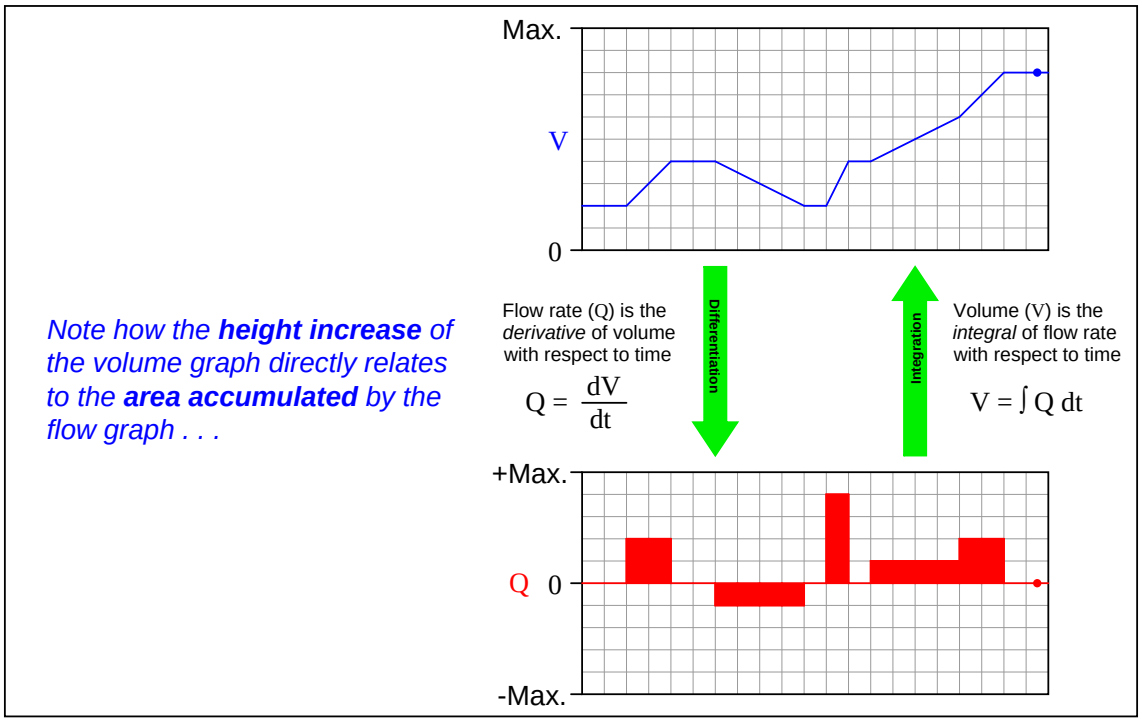
*Note how the **height increase** of the volume graph directly relates to the **area accumulated** by the flow graph . . .*



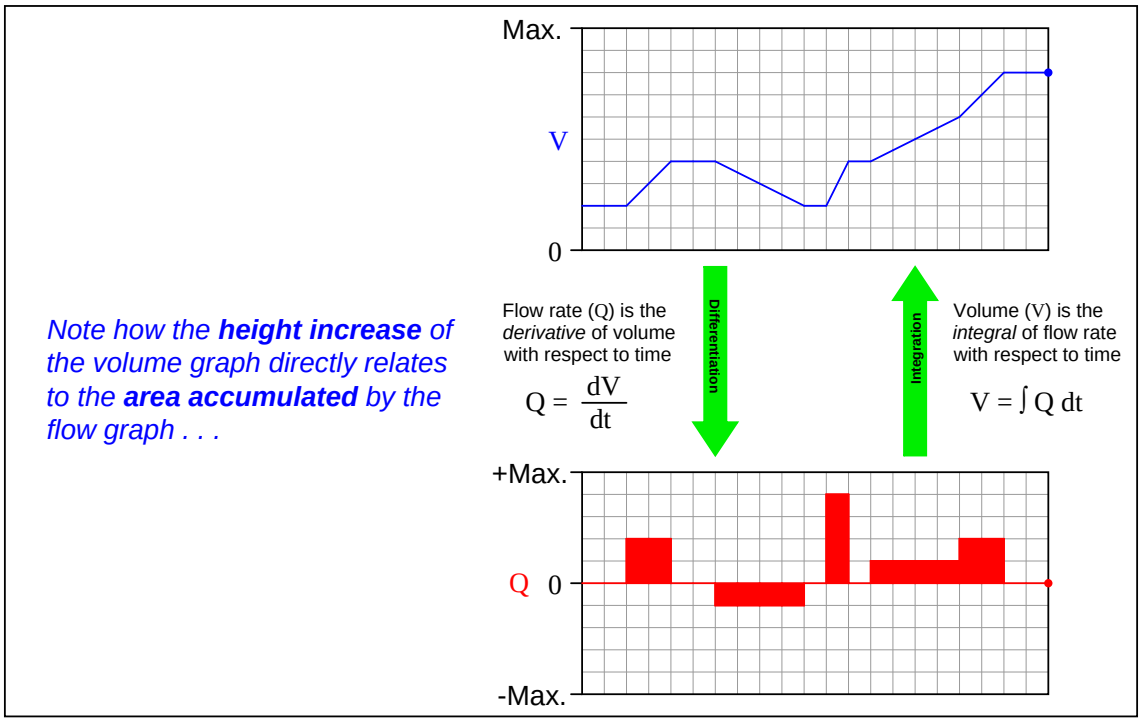
*Note how the **height increase** of the volume graph directly relates to the **area accumulated** by the flow graph . . .*



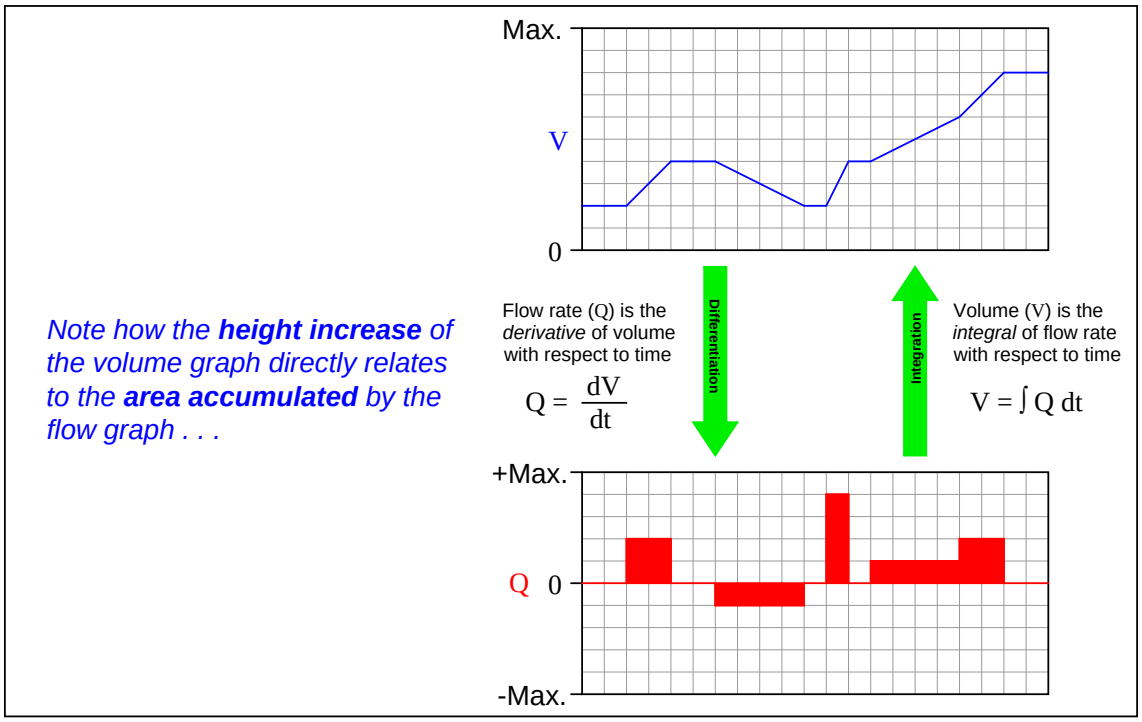
*Note how the **height increase** of the volume graph directly relates to the **area accumulated** by the flow graph . . .*



*Note how the **height increase** of the volume graph directly relates to the **area accumulated** by the flow graph . . .*



*Note how the **height increase** of the volume graph directly relates to the **area accumulated** by the flow graph . . .*



*Note how the **height increase** of the volume graph directly relates to the **area accumulated** by the flow graph . . .*

Chapter 5

Questions

This learning module, along with all others in the ModEL collection, is designed to be used in an inverted instructional environment where students independently read[1] the tutorials and attempt to answer questions on their own *prior* to the instructor's interaction with them. In place of lecture[2], the instructor engages with students in Socratic-style dialogue, probing and challenging their understanding of the subject matter through inquiry.

Answers are not provided for questions within this chapter, and this is by design. Solved problems may be found in the Tutorial and Derivation chapters, instead. The goal here is *independence*, and this requires students to be challenged in ways where others cannot think for them. Remember that you always have the tools of *experimentation* and *computer simulation* (e.g. SPICE) to explore concepts!

The following lists contain ideas for Socratic-style questions and challenges. Upon inspection, one will notice a strong theme of *metacognition* within these statements: they are designed to foster a regular habit of examining one's own thoughts as a means toward clearer thinking. As such these sample questions are useful both for instructor-led discussions as well as for self-study.

[1] Technical reading is an essential academic skill for any technical practitioner to possess for the simple reason that the most comprehensive, accurate, and useful information to be found for developing technical competence is in textual form. Technical careers in general are characterized by the need for continuous learning to remain current with standards and technology, and therefore any technical practitioner who cannot read well is handicapped in their professional development. An excellent resource for educators on improving students' reading prowess through intentional effort and strategy is the book textitReading For Understanding – How Reading Apprenticeship Improves Disciplinary Learning in Secondary and College Classrooms by Ruth Schoenbach, Cynthia Greenleaf, and Lynn Murphy.

[2] Lecture is popular as a teaching method because it is easy to implement: any reasonably articulate subject matter expert can talk to students, even with little preparation. However, it is also quite problematic. A good lecture always makes complicated concepts seem easier than they are, which is bad for students because it instills a false sense of confidence in their own understanding; reading and re-articulation requires more cognitive effort and serves to verify comprehension. A culture of teaching-by-lecture fosters a debilitating dependence upon direct personal instruction, whereas the challenges of modern life demand independent and critical thought made possible only by gathering information and perspectives from afar. Information presented in a lecture is ephemeral, easily lost to failures of memory and dictation; text is forever, and may be referenced at any time.

General challenges following tutorial reading

- Summarize as much of the text as you can in one paragraph of your own words. A helpful strategy is to explain ideas as you would for an intelligent child: as simple as you can without compromising too much accuracy.
- Simplify a particular section of the text, for example a paragraph or even a single sentence, so as to capture the same fundamental idea in fewer words.
- Where did the text make the most sense to you? What was it about the text's presentation that made it clear?
- Identify where it might be easy for someone to misunderstand the text, and explain why you think it could be confusing.
- Identify any new concept(s) presented in the text, and explain in your own words.
- Identify any familiar concept(s) such as physical laws or principles applied or referenced in the text.
- Devise a proof of concept experiment demonstrating an important principle, physical law, or technical innovation represented in the text.
- Devise an experiment to disprove a plausible misconception.
- Did the text reveal any misconceptions you might have harbored? If so, describe the misconception(s) and the reason(s) why you now know them to be incorrect.
- Describe any useful problem-solving strategies applied in the text.
- Devise a question of your own to challenge a reader's comprehension of the text.

General follow-up challenges for assigned problems

- Identify where any fundamental laws or principles apply to the solution of this problem, especially before applying any mathematical techniques.
- Devise a thought experiment to explore the characteristics of the problem scenario, applying known laws and principles to mentally model its behavior.
- Describe in detail your own strategy for solving this problem. How did you identify and organized the given information? Did you sketch any diagrams to help frame the problem?
- Is there more than one way to solve this problem? Which method seems best to you?
- Show the work you did in solving this problem, even if the solution is incomplete or incorrect.
- What would you say was the most challenging part of this problem, and why was it so?
- Was any important information missing from the problem which you had to research or recall?
- Was there any extraneous information presented within this problem? If so, what was it and why did it not matter?
- Examine someone else's solution to identify where they applied fundamental laws or principles.
- Simplify the problem from its given form and show how to solve this simpler version of it. Examples include eliminating certain variables or conditions, altering values to simpler (usually whole) numbers, applying a limiting case (i.e. altering a variable to some extreme or ultimate value).
- For quantitative problems, identify the real-world meaning of all intermediate calculations: their units of measurement, where they fit into the scenario at hand. Annotate any diagrams or illustrations with these calculated values.
- For quantitative problems, try approaching it qualitatively instead, thinking in terms of "increase" and "decrease" rather than definite values.
- For qualitative problems, try approaching it quantitatively instead, proposing simple numerical values for the variables.
- Were there any assumptions you made while solving this problem? Would your solution change if one of those assumptions were altered?
- Identify where it would be easy for someone to go astray in attempting to solve this problem.
- Formulate your own problem based on what you learned solving this one.

General follow-up challenges for experiments or projects

- In what way(s) was this experiment or project easy to complete?
- Identify some of the challenges you faced in completing this experiment or project.

- Show how thorough documentation assisted in the completion of this experiment or project.
- Which fundamental laws or principles are key to this system's function?
- Identify any way(s) in which one might obtain false or otherwise misleading measurements from test equipment in this system.
- What will happen if (component X) fails (open/shorted/etc.)?
- What would have to occur to make this system unsafe?

5.1 Conceptual reasoning

These questions are designed to stimulate your analytic and synthetic thinking[3]. In a Socratic discussion with your instructor, the goal is for these questions to prompt an extended dialogue where assumptions are revealed, conclusions are tested, and understanding is sharpened. Your instructor may also pose additional questions based on those assigned, in order to further probe and refine your conceptual understanding.

Questions that follow are presented to challenge and probe your understanding of various concepts presented in the tutorial. These questions are intended to serve as a guide for the Socratic dialogue between yourself and the instructor. Your instructor's task is to ensure you have a sound grasp of these concepts, and the questions contained in this document are merely a means to this end. Your instructor may, at his or her discretion, alter or substitute questions for the benefit of tailoring the discussion to each student's needs. The only absolute requirement is that each student is challenged and assessed at a level equal to or greater than that represented by the documented questions.

It is far more important that you convey your *reasoning* than it is to simply convey a correct answer. For this reason, you should refrain from researching other information sources to answer questions. What matters here is that *you* are doing the thinking. If the answer is incorrect, your instructor will work with you to correct it through proper reasoning. A correct answer without an adequate explanation of how you derived that answer is unacceptable, as it does not aid the learning or assessment process.

You will note a conspicuous lack of answers given for these conceptual questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. The best way by far is to debate the answers with fellow students and also with the instructor during the Socratic dialogue sessions intended to be used with these learning modules. Reasoning through challenging questions with other people is an excellent tool for developing strong reasoning skills.

Another means of checking your conceptual answers, where applicable, is to use circuit simulation software to explore the effects of changes made to circuits. For example, if one of these conceptual questions challenges you to predict the effects of altering some component parameter in a circuit, you may check the validity of your work by simulating that same parameter change within software and seeing if the results agree.

[3] *Analytical* thinking involves the "disassembly" of an idea into its constituent parts, analogous to dissection. *Synthetic* thinking involves the "assembly" of a new idea comprised of multiple concepts, analogous to construction. Both activities are high-level cognitive skills, extremely important for effective problem-solving, necessitating frequent challenge and regular practice to fully develop.

5.1.1 Reading outline and reflections

"Reading maketh a full man; conference a ready man; and writing an exact man" – Francis Bacon

Francis Bacon's advice is a blueprint for effective education: reading provides the learner with knowledge, writing focuses the learner's thoughts, and critical dialogue equips the learner to confidently communicate and apply their learning. Independent acquisition and application of knowledge is a powerful skill, well worth the effort to cultivate. To this end, students should read these educational resources closely, journal their own reflections on the reading, and discuss in detail their findings with classmates and instructor(s). You should be able to do all of the following after reading any instructional text:

☑ Briefly SUMMARIZE THE TEXT in the form of a journal entry documenting your learning as you progress through the course of study. Share this summary in dialogue with your classmates and instructor. Journaling is an excellent self-test of thorough reading because you cannot clearly express what you have not read or did not comprehend.

☑ Demonstrate ACTIVE READING STRATEGIES, including verbalizing your impressions as you read, simplifying long passages to convey the same ideas using fewer words, annotating text and illustrations with your own interpretations, working through mathematical examples shown in the text, cross-referencing passages with relevant illustrations and/or other passages, identifying problem-solving strategies applied by the author, etc. Technical reading is a special case of problem-solving, and so these strategies work precisely because they help solve any problem: paying attention to your own thoughts (metacognition), eliminating unnecessary complexities, identifying what makes sense, paying close attention to details, drawing connections between separated facts, and noting the successful strategies of others.

☑ Identify IMPORTANT THEMES, especially GENERAL LAWS and PRINCIPLES, expounded in the text and express them in the simplest of terms as though you were teaching an intelligent child. This emphasizes connections between related topics and develops your ability to communicate complex ideas to anyone.

☑ Form YOUR OWN QUESTIONS based on the reading, and then pose them to your instructor and classmates for their consideration. Anticipate both correct and incorrect answers, the incorrect answer(s) assuming one or more plausible misconceptions. This helps you view the subject from different perspectives to grasp it more fully.

☑ Devise EXPERIMENTS to test claims presented in the reading, or to disprove misconceptions. Predict possible outcomes of these experiments, and evaluate their meanings: what result(s) would confirm, and what would constitute disproof? Running mental simulations and evaluating results is essential to scientific and diagnostic reasoning.

☑ Specifically identify any points you found CONFUSING. The reason for doing this is to help diagnose misconceptions and overcome barriers to learning.

5.1.2 Foundational concepts

Correct analysis and diagnosis of electric circuits begins with a proper understanding of some basic concepts. The following is a list of some important concepts referenced in this module's full tutorial. Define each of them in your own words, and be prepared to illustrate each of these concepts with a description of a practical example and/or a live demonstration.

Energy

Conservation of Energy

Simplification as a problem-solving strategy

Thought experiments as a problem-solving strategy

Limiting cases as a problem-solving strategy

Annotating diagrams as a problem-solving strategy

Interpreting intermediate results as a problem-solving strategy

Graphing as a problem-solving strategy

Converting a qualitative problem into a quantitative problem

Converting a quantitative problem into a qualitative problem

Working "backwards" to validate calculated results

Reductio ad absurdum

Re-drawing schematics as a problem-solving strategy

Cut-and-try problem-solving strategy

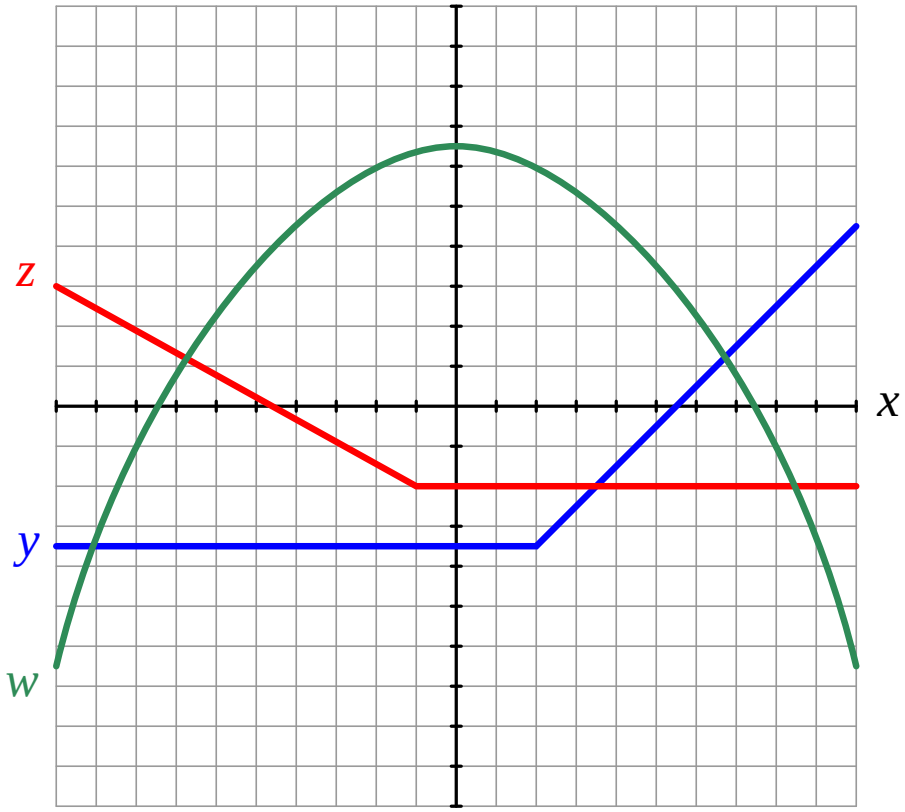
Algebraic substitution

???

5.1.3 Identifying integral areas

Graph #1

Shade the area on this graph representing the following integrals (assuming each horizontal and vertical division on the graph has an incremental value of 1):



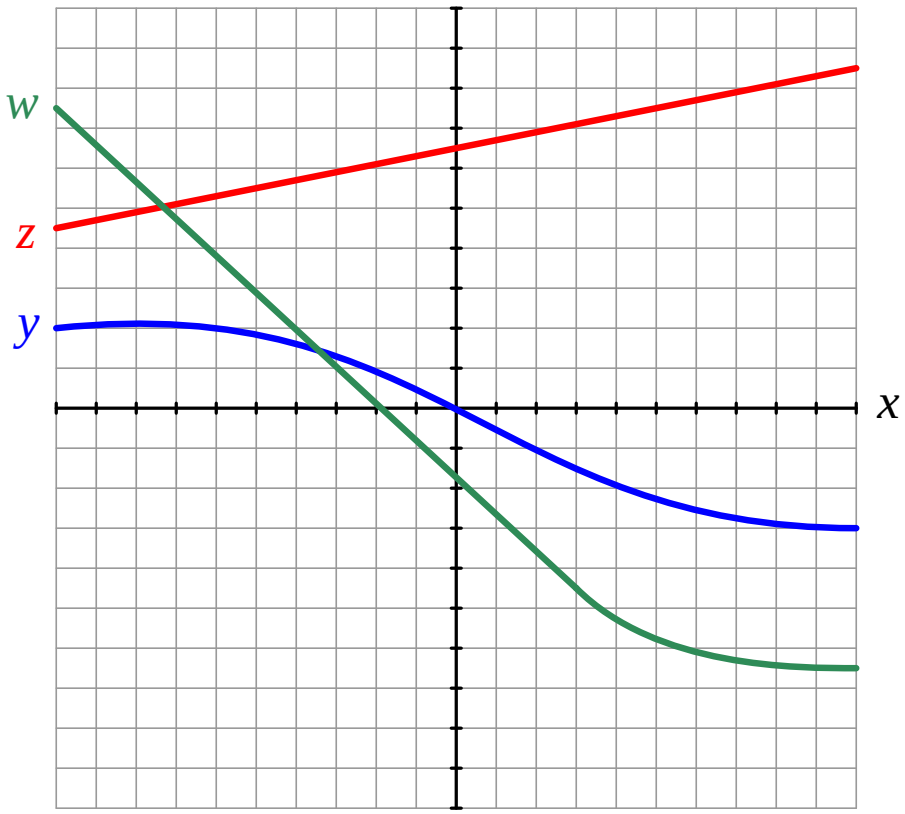
$$\int_{0}^{5} y\,dx$$

$$\int_{-1}^{-6} w\,dx$$

Also, determine whether the numerical values of these integrals are *positive* or *negative*.

Graph #2

Shade the area on this graph representing the following integral (assuming each horizontal and vertical division on the graph has an incremental value of 1):



$$\int_{-6}^{3} (w - y)\, dx$$

Also, determine whether the numerical value of this integral is *positive* or *negative*.

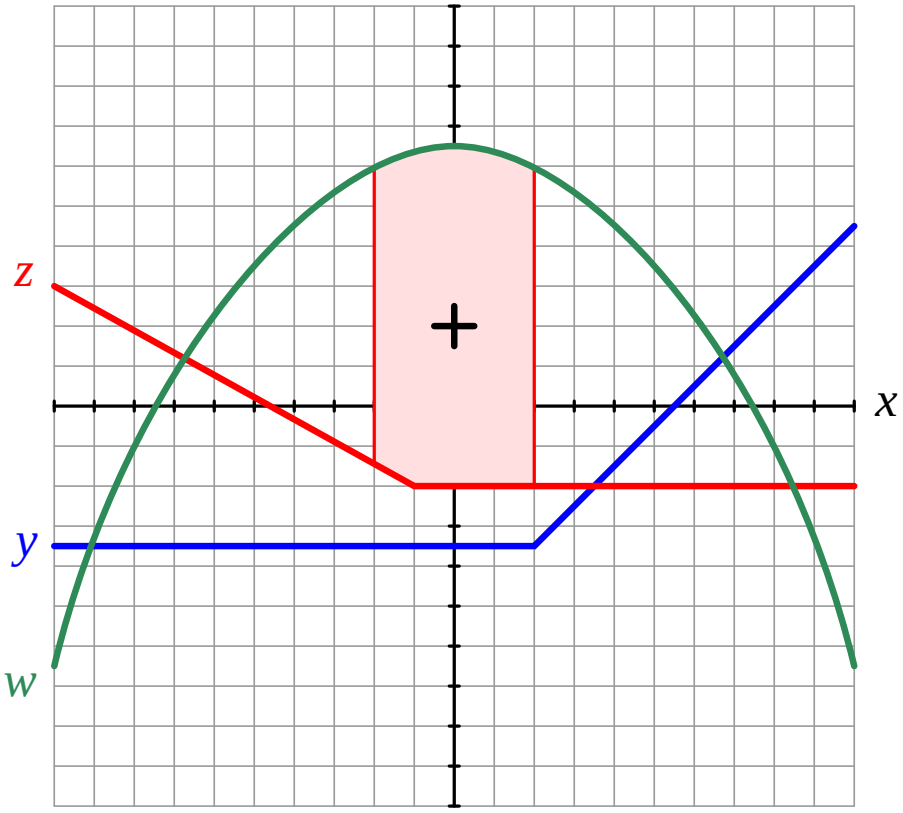
Challenges

- Identify intervals for each integral expression that would result in a net value of zero.

5.1.4 Integral expressions for identified graph areas

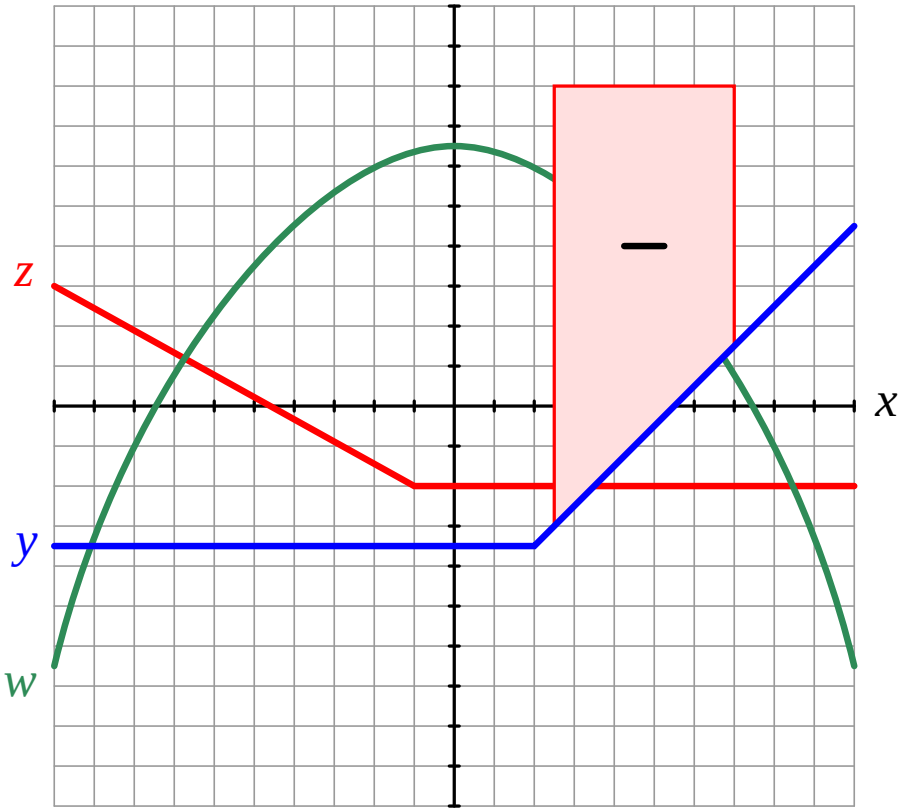
Graph #1

Write the integral expression represented by the shaded area on this graph (assuming each horizontal and vertical division on the graph has an incremental value of 1). The integral for this shaded area has a *positive* value:



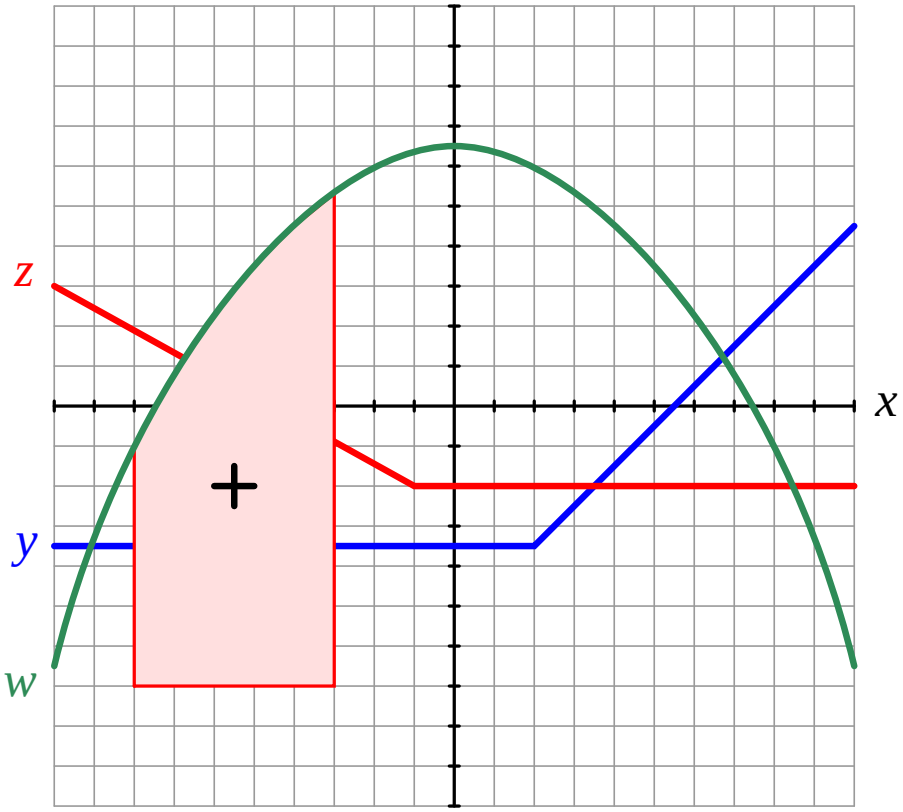
Graph #2

Write the integral expression represented by the shaded area on this graph (assuming each horizontal and vertical division on the graph has an incremental value of 1). The integral for this shaded area has a *positive* value:



Graph #3

Write the integral expression represented by the shaded area on this graph (assuming each horizontal and vertical division on the graph has an incremental value of 1). The integral for this shaded area has a *positive* value:



Challenges

- Identify intervals for each integral expression that would result in a net value of zero.

5.1.5 Second conceptual question

Challenges

- ???.
- ???.
- ???.

5.1.6 Applying foundational concepts to ???

Identify which foundational concept(s) apply to each of the declarations shown below regarding the following circuit. If a declaration is true, then identify it as such and note which concept supports that declaration; if a declaration is false, then identify it as such and note which concept is violated by that declaration:

(Under development)

- ???
- ???
- ???
- ???

Here is a list of foundational concepts for your reference: **Conservation of Energy**, **Conservation of Electric Charge**, **behavior of sources vs. loads**, **Ohm's Law**, **Joule's Law**, **effects of open faults**, **effect of shorted faults**, **properties of series networks**, **properties of parallel networks**, **Kirchhoff's Voltage Law**, **Kirchhoff's Current Law**. More than one of these concepts may apply to a declaration, and some concepts may not apply to any listed declaration at all. Also, feel free to include foundational concepts not listed here.

Challenges

- ???.
- ???.
- ???.

5.1.7 Explaining the meaning of calculations

Below is a quantitative problem where all the calculations have been performed for you, but all variable labels, units, and other identifying data are unrevealed. *Assign proper meaning* to each of the numerical values, identify the correct unit of measurement for each value as well as any appropriate metric prefix(es), explain the significance of each value by describing where it "fits" into the circuit being analyzed, and identify the general principle employed at each step:

Schematic diagram of the ??? circuit:

(Under development)

Calculations performed in order from first to last:

1. $x + y = z$
2. $x + y = z$
3. $x + y = z$
4. $x + y = z$
5. $x + y = z$
6. $x + y = z$

Challenges

- ???.
- ???.
- ???.

5.1.8 Explaining the meaning of code

Shown below is a schematic diagram for a ??? circuit, and after that a source-code listing of a computer program written in the ??? language simulating that circuit. Explain the purpose of each line of code relating to the circuit being simulated, identify the correct unit of measurement for each computed value, and identify all foundational concepts of electric circuits (e.g. Ohm's Law, Kirchhoff's Laws, etc.) employed in the program:

Schematic diagram of the ??? circuit:

(Under development)

Code listing:

```
#include <stdio.h>

int main (void)
{

  return 0;
}
```

Challenges

- ???.
- ???.
- ???.

5.2 Quantitative reasoning

These questions are designed to stimulate your computational thinking. In a Socratic discussion with your instructor, the goal is for these questions to reveal your mathematical approach(es) to problem-solving so that good technique and sound reasoning may be reinforced. Your instructor may also pose additional questions based on those assigned, in order to observe your problem-solving firsthand.

Mental arithmetic and estimations are strongly encouraged for all calculations, because without these abilities you will be unable to readily detect errors caused by calculator misuse (e.g. keystroke errors).

You will note a conspicuous lack of answers given for these quantitative questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. My advice is to use circuit simulation software such as SPICE to check the correctness of quantitative answers. Refer to those learning modules within this collection focusing on SPICE to see worked examples which you may use directly as practice problems for your own study, and/or as templates you may modify to run your own analyses and generate your own practice problems.

Completely worked example problems found in the Tutorial may also serve as "test cases[4]" for gaining proficiency in the use of circuit simulation software, and then once that proficiency is gained you will never need to rely[5] on an answer key!

[4] In other words, set up the circuit simulation software to analyze the same circuit examples found in the Tutorial. If the simulated results match the answers shown in the Tutorial, it confirms the simulation has properly run. If the simulated results disagree with the Tutorial's answers, something has been set up incorrectly in the simulation software. Using every Tutorial as practice in this way will quickly develop proficiency in the use of circuit simulation software.

[5] This approach is perfectly in keeping with the instructional philosophy of these learning modules: *teaching students to be self-sufficient thinkers*. Answer keys can be useful, but it is even more useful to your long-term success to have a set of tools on hand for checking your own work, because once you have left school and are on your own, there will no longer be "answer keys" available for the problems you will have to solve.

5.2.1 Miscellaneous physical constants

Note: constants shown in **bold** type are *exact*, not approximations. Values inside of parentheses show one standard deviation (σ) of uncertainty in the final digits: for example, the magnetic permeability of free space value given as $1.25663706212(19) \times 10^{-6}$ H/m represents a center value (i.e. the location parameter) of $1.25663706212 \times 10^{-6}$ Henrys per meter with one standard deviation of uncertainty equal to $0.0000000000019 \times 10^{-6}$ Henrys per meter.

Avogadro's number (N_A) = **6.02214076** $\times 10^{23}$ **per mole** (mol^{-1})

Boltzmann's constant (k) = **1.380649** $\times 10^{-23}$ **Joules per Kelvin** (J/K)

Electronic charge (e) = **1.602176634** $\times 10^{-19}$ **Coulomb** (C)

Faraday constant (F) = **96,485.33212...** $\times 10^{4}$ **Coulombs per mole** (C/mol)

Magnetic permeability of free space (μ_0) = $1.25663706212(19) \times 10^{-6}$ Henrys per meter (H/m)

Electric permittivity of free space (ϵ_0) = $8.8541878128(13) \times 10^{-12}$ Farads per meter (F/m)

Characteristic impedance of free space (Z_0) = 376.730313668(57) Ohms (Ω)

Gravitational constant (G) = $6.67430(15) \times 10^{-11}$ cubic meters per kilogram-seconds squared ($m^3/kg\text{-}s^2$)

Molar gas constant (R) = **8.314462618... Joules per mole-Kelvin** (J/mol-K) = 0.08205746(14) liters-atmospheres per mole-Kelvin

Planck constant (h) = **6.62607015** $\times 10^{-34}$ **joule-seconds** (J-s)

Stefan-Boltzmann constant (σ) = **5.670374419...** $\times 10^{-8}$ **Watts per square meter-Kelvin**^{4} ($W/m^2 \cdot K^4$)

Speed of light in a vacuum (c) = **299,792,458 meters per second** (m/s) = 186282.4 miles per second (mi/s)

Note: All constants taken from NIST data "Fundamental Physical Constants – Complete Listing", from `http://physics.nist.gov/constants`, National Institute of Standards and Technology (NIST), 2018 CODATA Adjustment.

5.2.2 Introduction to spreadsheets

A powerful computational tool you are encouraged to use in your work is a *spreadsheet*. Available on most personal computers (e.g. Microsoft Excel), *spreadsheet* software performs numerical calculations based on number values and formulae entered into cells of a grid. This grid is typically arranged as lettered columns and numbered rows, with each cell of the grid identified by its column/row coordinates (e.g. cell `B3`, cell `A8`). Each cell may contain a string of text, a number value, or a mathematical formula. The spreadsheet automatically updates the results of all mathematical formulae whenever the entered number values are changed. This means it is possible to set up a spreadsheet to perform a series of calculations on entered data, and those calculations will be re-done by the computer any time the data points are edited in any way.

For example, the following spreadsheet calculates average speed based on entered values of distance traveled and time elapsed:

	A	B	C	D
1	Distance traveled	46.9	Kilometers	
2	Time elapsed	1.18	Hours	
3	Average speed	= B1 / B2	km/h	
4				
5				

Text labels contained in cells `A1` through `A3` and cells `C1` through `C3` exist solely for readability and are not involved in any calculations. Cell `B1` contains a sample distance value while cell `B2` contains a sample time value. The formula for computing speed is contained in cell `B3`. Note how this formula begins with an "equals" symbol (`=`), references the values for distance and speed by lettered column and numbered row coordinates (`B1` and `B2`), and uses a forward slash symbol for division (`/`). The coordinates `B1` and `B2` function as *variables*[6] would in an algebraic formula.

When this spreadsheet is executed, the numerical value `39.74576` will appear in cell `B3` rather than the formula `= B1 / B2`, because 39.74576 is the computed speed value given 46.9 kilometers traveled over a period of 1.18 hours. If a different numerical value for distance is entered into cell `B1` or a different value for time is entered into cell `B2`, cell `B3`'s value will automatically update. All you need to do is set up the given values and any formulae into the spreadsheet, and the computer will do all the calculations for you.

Cell `B3` may be referenced by other formulae in the spreadsheet if desired, since it is a variable just like the given values contained in `B1` and `B2`. This means it is possible to set up an entire chain of calculations, one dependent on the result of another, in order to arrive at a final value. The arrangement of the given data and formulae need not follow any pattern on the grid, which means you may place them anywhere.

[6] Spreadsheets may also provide means to attach text labels to cells for use as variable names (Microsoft Excel simply calls these labels "names"), but for simple spreadsheets such as those shown here it's usually easier just to use the standard coordinate naming for each cell.

Common[7] arithmetic operations available for your use in a spreadsheet include the following:

- Addition (`+`)
- Subtraction (`-`)
- Multiplication (`*`)
- Division (`/`)
- Powers (`^`)
- Square roots (`sqrt()`)
- Logarithms (`ln()` , `log10()`)

Parentheses may be used to ensure[8] proper order of operations within a complex formula. Consider this example of a spreadsheet implementing the *quadratic formula*, used to solve for roots of a polynomial expression in the form of $ax^2 + bx + c$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

	A	B
1	x_1	= (-B4 + sqrt((B4^2) - (4*B3*B5))) / (2*B3)
2	x_2	= (-B4 - sqrt((B4^2) - (4*B3*B5))) / (2*B3)
3	a =	9
4	b =	5
5	c =	-2

This example is configured to compute roots[9] of the polynomial $9x^2 + 5x - 2$ because the values of 9, 5, and -2 have been inserted into cells `B3`, `B4`, and `B5`, respectively. Once this spreadsheet has been built, though, it may be used to calculate the roots of *any* second-degree polynomial expression simply by entering the new a, b, and c coefficients into cells `B3` through `B5`. The numerical values appearing in cells `B1` and `B2` will be automatically updated by the computer immediately following any changes made to the coefficients.

[7] Modern spreadsheet software offers a bewildering array of mathematical functions you may use in your computations. I recommend you consult the documentation for your particular spreadsheet for information on operations other than those listed here.

[8] Spreadsheet programs, like text-based programming languages, are designed to follow standard order of operations by default. However, my personal preference is to use parentheses even where strictly unnecessary just to make it clear to any other person viewing the formula what the intended order of operations is.

[9] Reviewing some algebra here, a *root* is a value for x that yields an overall value of zero for the polynomial. For this polynomial ($9x^2 + 5x - 2$) the two roots happen to be $x = 0.269381$ and $x = -0.82494$, with these values displayed in cells `B1` and `B2`, respectively upon execution of the spreadsheet.

Alternatively, one could break up the long quadratic formula into smaller pieces like this:

$$y = \sqrt{b^2 - 4ac} \qquad z = 2a$$

$$x = \frac{-b \pm y}{z}$$

	A	B	C
1	x_1	= (-B4 + C1) / C2	= sqrt((B4^2) - (4*B3*B5))
2	x_2	= (-B4 - C1) / C2	= 2*B3
3	a =	9	
4	b =	5	
5	c =	-2	

Note how the square-root term (y) is calculated in cell C1, and the denominator term (z) in cell C2. This makes the two final formulae (in cells B1 and B2) simpler to interpret. The positioning of all these cells on the grid is completely arbitrary[10] – all that matters is that they properly reference each other in the formulae.

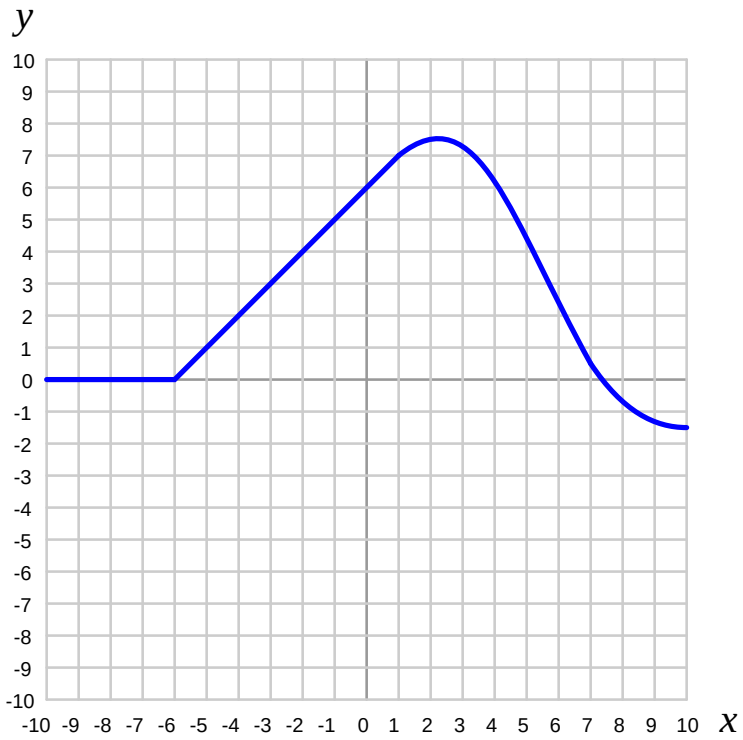
Spreadsheets are particularly useful for situations where the same set of calculations representing a circuit or other system must be repeated for different initial conditions. The power of a spreadsheet is that it automates what would otherwise be a tedious set of calculations. One specific application of this is to simulate the effects of various components within a circuit failing with abnormal values (e.g. a shorted resistor simulated by making its value nearly zero; an open resistor simulated by making its value extremely large). Another application is analyzing the behavior of a circuit design given new components that are out of specification, and/or aging components experiencing drift over time.

[10] My personal preference is to locate all the "given" data in the upper-left cells of the spreadsheet grid (each data point flanked by a sensible name in the cell to the left and units of measurement in the cell to the right as illustrated in the first distance/time spreadsheet example), sometimes coloring them in order to clearly distinguish which cells contain entered data versus which cells contain computed results from formulae. I like to place all formulae in cells below the given data, and try to arrange them in logical order so that anyone examining my spreadsheet will be able to figure out *how* I constructed a solution. This is a general principle I believe all computer programmers should follow: *document and arrange your code to make it easy for other people to learn from it.*

5.2.3 Determining derivative values from graphed functions

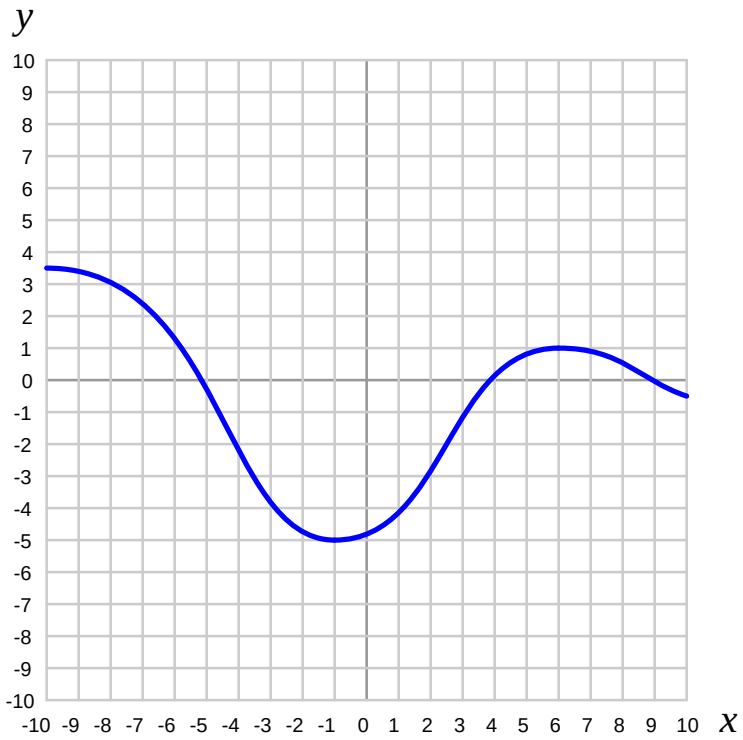
Determine approximate *derivative* ($\frac{dy}{dx}$) values for the following functions at the specified points:

Function #1



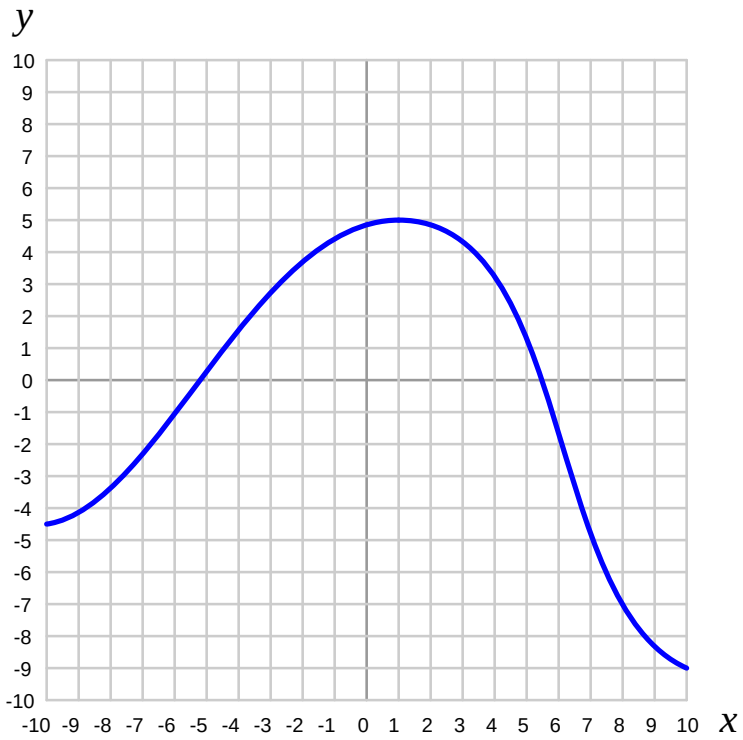
- $\frac{dy}{dx}$ at $x = +5$
- $\frac{dy}{dx}$ at $x = +10$
- $\frac{dy}{dx}$ at $x = -2$

Function #2



- $\frac{dy}{dx}$ at $x = +6$
- $\frac{dy}{dx}$ at $x = +9$
- $\frac{dy}{dx}$ at $x = +3$

Function #3



- $\frac{dy}{dx}$ at $x = -9$
- $\frac{dy}{dx}$ at $x = +1$
- $\frac{dy}{dx}$ at $x = +3$

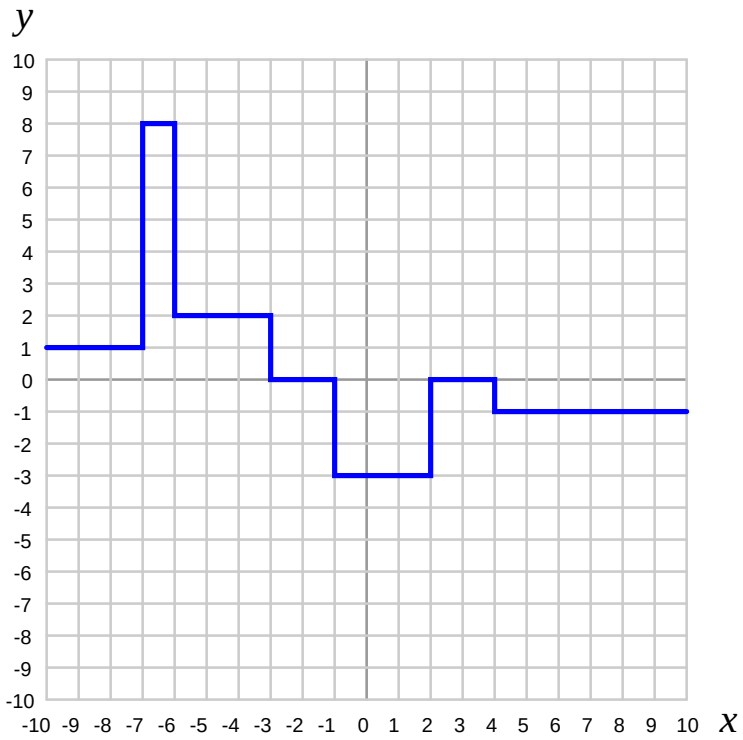
Challenges

- Identify any sources of error in your estimations of these derivative values.

5.2.4 Determining integral values from graphed functions

Determine approximate *integral* values for the following functions over the specified intervals:

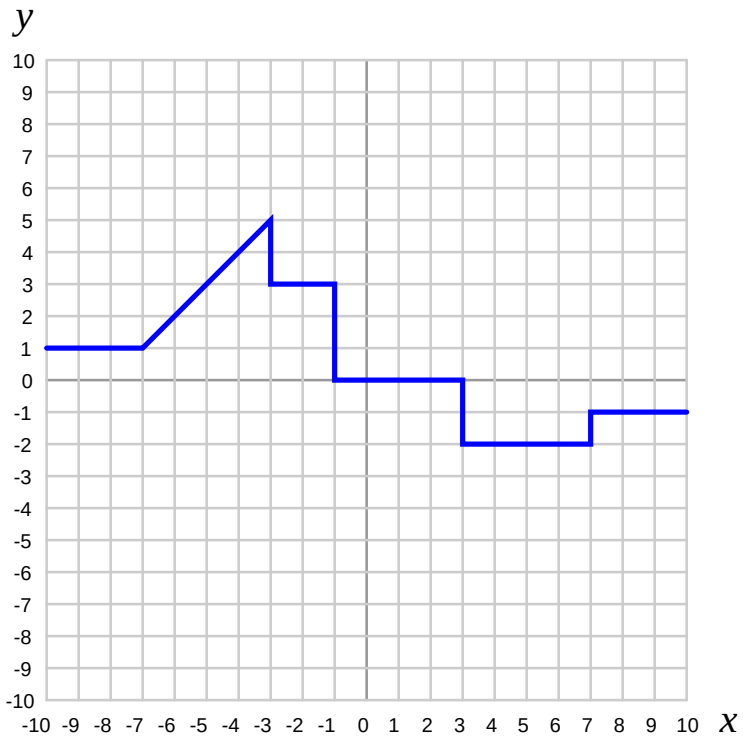
Function #1



$$\int_{-8}^{-4} f(x)\,dx$$

$$\int_{-1}^{+2} f(x)\,dx$$

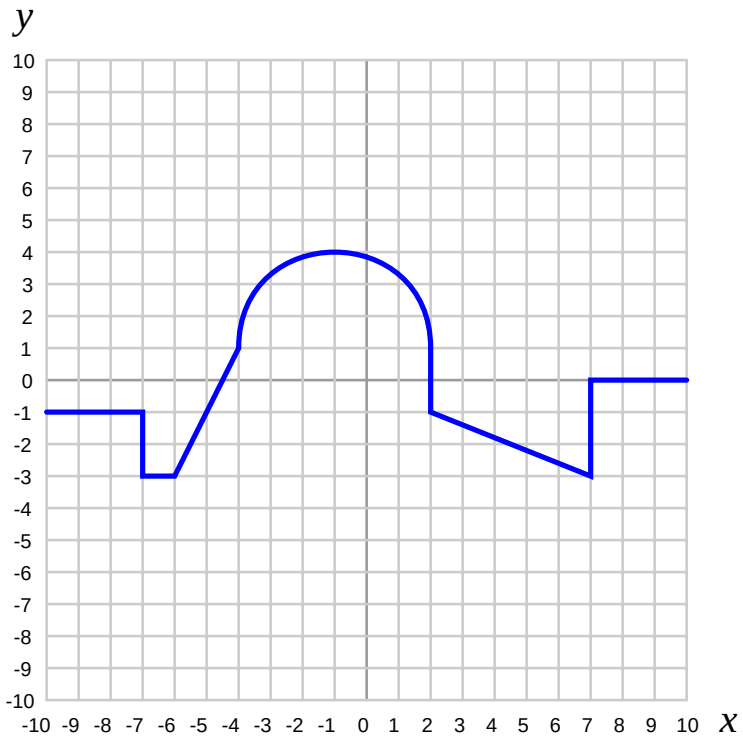
Function #2



$$\int_{-3}^{+10} f(x)\,dx$$

$$\int_{-10}^{0} f(x)\,dx$$

Function #3



$$\int_{-1}^{+2} f(x)\,dx$$

$$\int_{8}^{-4} f(x)\,dx$$

Challenges

- Identify any sources of error in your estimations of these derivative values.

5.2.5 Determining vehicle speed from distance and time

A bored child is traveling in a car with his parents, and decides to pass the time by writing mileage values displayed by the odometer at different times, and then noting those times next to the distances:

Odometer reading (km)	**Time** (hour:minute)
60,344.1	2:14
60,346.3	2:17
60,347.1	2:18
60,351.7	2:25
60,353.9	2:27
60,357.4	2:30
60,359.5	2:35

Calculate the average speed of the car between the following times:

- Between 2:17 and 2:18, average speed =
- Between 2:18 and 2:25, average speed =
- Between 2:25 and 2:27, average speed =
- Between 2:17 and 2:27, average speed =

Then, compare the average speeds taken in the first three intervals with the average speed over the sum of those intervals (2:17 to 2:27). What does this tell us about the calculation of speed based on distance and time measurements.

Challenges

- What could this bored child to in order to arrive at more precise estimations of speed?

5.2.6 Determining vehicle travel distance from speed and time

A bored child is traveling in a car with his parents, and decides to pass the time by writing speed values displayed by the speedometer at different times, and then noting those times next to the distances:

Speedometer reading (km)	**Time** (hour:minute)
55	2:14
57	2:17
60	2:18
61	2:25
58	2:27
55	2:30
60	2:35

Describe how you may calculate the distance traveled by this car between 2:14 and 2:35 based on speed values from this table.

Challenges

- What could this bored child to in order to arrive at more precise estimations of speed?

5.2.7 Power rule for derivatives

The "Power Rule" in differential calculus tells us how to find a formula for the derivative of a function consisting of an integer power of the independent variable:

Using the Power Rule, differentiate the following functions:

$$y = x \qquad \frac{dy}{dx} =$$

$$y = x^2 \qquad \frac{dy}{dx} =$$

$$y = x^3 \qquad \frac{dy}{dx} =$$

$$y = x^4 \qquad \frac{dy}{dx} =$$

$$y = x^5 \qquad \frac{dy}{dx} =$$

$$y = x^{-1} \qquad \frac{dy}{dx} =$$

$$y = \sqrt{x} \qquad \frac{dy}{dx} =$$

$$y = 11 \qquad \frac{dy}{dx} =$$

Challenges

- ???.

- ???.
- ???.

5.2.8 ??? simulation program

Write a text-based computer program (e.g. C, C++, Python) to calculate ???

Challenges

- ???.
- ???.
- ???.

5.3 Diagnostic reasoning

These questions are designed to stimulate your deductive and inductive thinking, where you must apply general principles to specific scenarios (deductive) and also derive conclusions about the failed circuit from specific details (inductive). In a Socratic discussion with your instructor, the goal is for these questions to reinforce your recall and use of general circuit principles and also challenge your ability to integrate multiple symptoms into a sensible explanation of what's wrong in a circuit. Your instructor may also pose additional questions based on those assigned, in order to further challenge and sharpen your diagnostic abilities.

As always, your goal is to fully *explain* your analysis of each problem. Simply obtaining a correct answer is not good enough – you must also demonstrate sound reasoning in order to successfully complete the assignment. Your instructor's responsibility is to probe and challenge your understanding of the relevant principles and analytical processes in order to ensure you have a strong foundation upon which to build further understanding.

You will note a conspicuous lack of answers given for these diagnostic questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. The best way by far is to debate the answers with fellow students and also with the instructor during the Socratic dialogue sessions intended to be used with these learning modules. Reasoning through challenging questions with other people is an excellent tool for developing strong reasoning skills.

Another means of checking your diagnostic answers, where applicable, is to use circuit simulation software to explore the effects of faults placed in circuits. For example, if one of these diagnostic questions requires that you predict the effect of an open or a short in a circuit, you may check the validity of your work by simulating that same fault (substituting a very high resistance in place of that component for an open, and substituting a very low resistance for a short) within software and seeing if the results agree.

5.3.1 First diagnostic scenario

Challenges

- ???.
- ???.
- ???.

5.3.2 Second diagnostic scenario

Challenges

- ???.
- ???.
- ???.

Appendix A

Problem-Solving Strategies

The ability to solve complex problems is arguably one of the most valuable skills one can possess, and this skill is particularly important in any science-based discipline.

- Study principles, not procedures. Don't be satisfied with merely knowing how to compute solutions – learn *why* those solutions work.
- Identify what it is you need to solve, identify all relevant data, identify all units of measurement, identify any general principles or formulae linking the given information to the solution, and then identify any "missing pieces" to a solution. Annotate all diagrams with this data.
- Sketch a diagram to help visualize the problem. When building a real system, always devise a plan for that system and analyze its function *before* constructing it.
- Follow the units of measurement and meaning of every calculation. If you are ever performing mathematical calculations as part of a problem-solving procedure, and you find yourself unable to apply each and every intermediate result to some aspect of the problem, it means you don't understand what you are doing. Properly done, every mathematical result should have practical meaning for the problem, and not just be an abstract number. You should be able to identify the proper units of measurement for each and every calculated result, and show where that result fits into the problem.
- Perform "thought experiments" to explore the effects of different conditions for theoretical problems. When troubleshooting real systems, perform *diagnostic tests* rather than visually inspecting for faults, the best diagnostic test being the one giving you the most information about the nature and/or location of the fault with the fewest steps.
- Simplify the problem until the solution becomes obvious, and then use that obvious case as a model to follow in solving the more complex version of the problem.
- Check for exceptions to see if your solution is incorrect or incomplete. A good solution will work for *all* known conditions and criteria. A good example of this is the process of testing scientific hypotheses: the task of a scientist is not to find support for a new idea, but rather to *challenge* that new idea to see if it holds up under a battery of tests. The philosophical

principle of *reductio ad absurdum* (i.e. disproving a general idea by finding a specific case where it fails) is useful here.

- Work "backward" from a hypothetical solution to a new set of given conditions.
- Add quantities to problems that are qualitative in nature, because sometimes a little math helps illuminate the scenario.
- Sketch graphs illustrating how variables relate to each other. These may be quantitative (i.e. with realistic number values) or qualitative (i.e. simply showing increases and decreases).
- Treat quantitative problems as qualitative in order to discern the relative magnitudes and/or directions of change of the relevant variables. For example, try determining what happens if a certain variable were to increase or decrease before attempting to precisely calculate quantities: how will each of the dependent variables respond, by increasing, decreasing, or remaining the same as before?
- Consider limiting cases. This works especially well for qualitative problems where you need to determine which direction a variable will change. Take the given condition and magnify that condition to an extreme degree as a way of simplifying the direction of the system's response.
- Check your work. This means regularly testing your conclusions to see if they make sense. This does *not* mean repeating the same steps originally used to obtain the conclusion(s), but rather to use some other means to check validity. Simply repeating procedures often leads to *repeating the same errors* if any were made, which is why alternative paths are better.

Appendix B

Instructional philosophy

B.1 First principles of learning

- **Anyone can learn anything** given appropriate time, effort, resources, challenges, encouragement, and expectations. Dedicating time and investing effort are the student's responsibility; providing resources, challenges, and encouragement are the teacher's responsibility; high expectations are a responsibility shared by both student and teacher.

- **Transfer is not automatic.** The human mind has a natural tendency to compartmentalize information, which means the process of taking knowledge learned in one context and applying it to another usually does not come easy and therefore should never be taken for granted.

- **Learning is iterative.** The human mind rarely learns anything perfectly on the first attempt. Anticipate mistakes and plan for multiple tries to achieve full understanding, using the lessons of those mistakes as feedback to guide future attempts.

- **Information is absorbed, but understanding is created.** Facts and procedures may be memorized easily enough by repeated exposure, but the ability to reliably apply principles to novel scenarios only comes through intense personal effort. This effort is fundamentally creative in nature: casting new concepts in one's own words, running mental experiments to perceive patterns and trends, etc. These acts of making knowledge "one's own" need not be perfect in order to be effective, as the value lies in the activity and not necessarily the finished product. Depth of understanding and permanence of memory are both directly related to mental effort during the learning process – i.e. *easy learning is shallow and fleeting.*

- **Education trumps training.** There is no such thing as an entirely isolated subject, as all fields of knowledge are connected. Training is narrowly-focused and task-oriented. Education is broad-based and principle-oriented. When preparing for a life-long technical career, education beats training every time.

- **Character matters.** Poor habits are more destructive than deficits of knowledge or skill. This is especially true in collective endeavors, where a team's ability to function depends on trust between its members. Simply put, no one wants an untrustworthy person on their team. An essential component of education then, is character development.

- **People learn to be responsible by bearing responsibility.** An irresponsible person is someone who has never *had* to be responsible for anything that mattered enough to them. Just as anyone can learn anything, anyone can become responsible if the personal cost of irresponsibility becomes high enough.

- **What gets measured, gets done.** Accurate and relevant assessment of learning is key to ensuring all students learn. Therefore, it is imperative to measure what matters.

- **Failure is nothing to fear.** Every human being fails, and fails in multiple ways at multiple times. Eventual success only happens when we don't stop trying.

B.2 Proven strategies for instructors

- Assume every student is capable of learning anything they desire given the proper conditions. Treat them as capable adults by granting real responsibility and avoiding artificial incentives such as merit or demerit points.
- Create a consistent culture of high expectations across the entire program of study. Demonstrate and encourage patience, persistence, and a healthy sense of self-skepticism. Anticipate and de-stigmatize error. Teach respect for the capabilities of others as well as respect for one's own fallibility.
- Replace lecture with "inverted" instruction, where students first encounter new concepts through reading and then spend class time in Socratic dialogue with the instructor exploring those concepts and solving problems individually. There is a world of difference between observing someone solve a problem versus actually solving a problem yourself, and so the point of this form of instruction is to place students in a position where they *cannot* passively observe.
- Require students to read extensively, write about what they learn, and dialogue with you and their peers to sharpen their understanding. Apply Francis Bacon's advice that "reading maketh a full man; conference a ready man; and writing an exact man". These are complementary activities helping students expand their confidence and abilities.
- Use artificial intelligence (AI) to challenge student understanding rather than merely provide information. Find productive ways for AI to critique students' clarity of thought and of expression, for example by employing AI as a Socratic-style interlocutor or as a reviewer of students' journals. Properly applied, AI has the ability to expand student access to critical review well outside the bounds of their instructor's reach.
- Build frequent and rapid feedback into the learning process so that students know at all times how well they are learning, to identify problems early and fix them before they grow. Model the intellectual habit of self-assessing and self-correcting your own understanding (i.e. a cognitive *feedback loop*), encouraging students to do the same.
- Use "mastery" as the standard for every assessment, which means the exam or experiment or project must be done with 100% competence in order to pass. Provide students with multiple opportunity for re-tries (different versions of the assessment every time).
- Require students to devise their own hypotheses and procedures on all experiments, so that the process is truly a scientific one. Have students assess their proposed experimental procedures for risk and devise mitigations for those risks. Let nothing be pre-designed about students' experiments other than a stated task (i.e. what principle the experiment shall test) at the start and a set of demonstrable knowledge and skill objectives at the end.
- Have students build as much of their lab equipment as possible: building power sources, building test assemblies[1], and building complete working systems (no kits!). In order to provide

[1] In the program I teach, every student builds their own "Development Board" consisting of a metal chassis with DIN rail, terminal blocks, and an AC-DC power supply of their own making which functions as a portable lab environment they can use at school as well as take home.

this same "ground-up" experience for every new student, this means either previous students take their creations with them, or the systems get disassembled in preparation for the new students, or the systems grow and evolve with each new student group.

- Incorporate external accountability for you and for your students, continuously improving the curriculum and your instructional methods based on proven results. Have students regularly network with active professionals through participation in advisory committee meetings, service projects, tours, jobshadows, internships, etc. Practical suggestions include requiring students to design and build projects for external clients (e.g. community groups, businesses, different departments within the institution), and also requiring students attend all technical advisory committee meetings and to dialogue with the industry representatives at those meetings.
- Repeatedly explore difficult-to-learn concepts across multiple courses, so that students have multiple opportunities to build their understanding.
- Relate all new concepts, whenever possible, to previous concepts and to relevant physical laws. Challenge each and every student, every day, to *reason* from concept to concept and to explain the logical connections between. Challenge students to verify their conclusions by multiple approaches (e.g. double-checking their work using different methods). Ask *"Why?"* often.
- Maintain detailed records on each student's performance and share these records privately with them. These records should include academic performance as well as professionally relevant behavioral tendencies.
- Hold mandatory "check-in" meetings between all program faculty and each new student during their first term. Offer these to all other students as an option, except for any students continuing to manifest unprofessional behaviors, poor academic performance, or who have some other need for a face-to-face meeting with faculty.
- Address problems while they are small, before they grow larger. This is equally true for tutoring technical concepts as it is for helping students build professional habits.
- Build rigorous quality control into the curriculum to ensure every student masters every important concept, and that the mastery is retained over time. This includes (1) review questions added to every exam to re-assess knowledge taught in previous terms, (2) cumulative exams at the end of every term to re-assess all important concepts back to the very beginning of the program, and (3) review assessments in practical (hands-on) coursework to ensure critically-important skills were indeed taught and are still retained. What you will find by doing this is that it actually boosts retention of students by ensuring that important knowledge gets taught and is retained over long spans of time. In the absence of such quality control, student learning and retention tends to be spotty and this contributes to drop-out and failure rates later in their education.
- Finally, *never rush learning.* Education is not a race. Give your students ample time to digest complex ideas, as you continually remind yourself of just how long it took you to achieve mastery! Long-term retention and the consistently correct application of concepts are always the result of *focused effort over long periods of time* which means there are no shortcuts to learning.

B.3 Proven strategies for students

The single most important piece of advice I have for any student of any subject is to take responsibility for your own development in all areas of life including mental development. Expecting others in your life to entirely guide your own development is a recipe for disappointment. This is just as true for students enrolled in formal learning institutions as it is for auto-didacts pursuing learning entirely on their own. Learning to think in new ways is key to being able to gainfully use information, to make informed decisions about your life, and to best serve those you care about. With this in mind, I offer the following advice to students:

- **Approach all learning as valuable**. No matter what course you take, no matter who you learn from, no matter the subject, there is something useful in every learning experience. If you don't see the value of every new experience, you are not looking closely enough!
- **Continually challenge yourself**. Let other people take shortcuts and find easy answers to easy problems. The purpose of education is to stretch your mind, in order to shape it into a more powerful tool. This doesn't come by taking the path of least resistance. An excellent analogy for an empowering education is productive physical exercise: becoming stronger, more flexible, and more persistent only comes through intense personal effort.
- **Master the use of language**. This includes reading extensively, writing every day, listening closely, and speaking articulately. To a great extent language channels and empowers thought, so the better you are at wielding language the better you will be at grasping abstract concepts and articulating them not only for your benefit but for others as well.
- **Do not limit yourself to the resources given to you**. Read books that are not on the reading list. Run experiments that aren't assigned to you. Form study groups outside of class. Take an entrepreneurial approach to your own education, as though it were a business you were building for your future benefit.
- **Express and share what you learn**. Take every opportunity to teach what you have learned to others, as this will not only help them but will also strengthen your own understanding[2].
- Realize that **no one can give you understanding**, just as no one can give you physical fitness. These both must be *built*.
- **Above all, recognize that learning is hard work, and that a certain level of frustration is unavoidable**. There are times when you will struggle to grasp some of these concepts, and that struggle is a natural thing. Take heart that it will yield with persistent and varied[3] effort, and never give up! That concepts don't immediately come to you is not a sign of something wrong, but rather of something right: that you have found a worthy challenge!

[2] On a personal note, I was surprised to learn just how much my own understanding of electronics and related subjects was strengthened by becoming a teacher. When you are tasked every day with helping other people grasp complex topics, it catalyzes your own learning by giving you powerful incentives to study, to articulate your thoughts, and to reflect deeply on the process of learning.

[3] As the old saying goes, "Insanity is trying the same thing over and over again, expecting different results." If you find yourself stumped by something in the text, you should attempt a different approach. Alter the thought experiment, change the mathematical parameters, do whatever you can to see the problem in a slightly different light, and then the solution will often present itself more readily.

B.4 Design of these learning modules

"The unexamined circuit is not worth energizing" – Socrates (if he had taught electricity)

These learning modules, although useful for self-study, were designed to be used in a formal learning environment where a subject-matter expert challenges students to digest the content and exercise their critical thinking abilities in the answering of questions and in the construction and testing of working circuits. Every effort has been made to embed the following instructional and assessment philosophies within:

- The first goal of education is to enhance clear and independent thought, in order that every student reach their fullest potential in a highly complex and inter-dependent world. Robust reasoning is *always* more important than particulars of any subject matter, because its application is universal.
- Literacy is fundamental to independent learning and thought because text continues to be the most efficient way to communicate complex ideas over space and time. Those who cannot read with ease are limited in their ability to acquire knowledge and perspective.
- Articulate communication is fundamental to work that is complex and interdisciplinary.
- Faulty assumptions and poor reasoning are best corrected through challenge, not presentation. The rhetorical technique of *reductio ad absurdum* (disproving an assertion by exposing an absurdity) works well to discipline student's minds, not only to correct the problem at hand but also to learn how to detect and correct future errors.
- Important principles should be repeatedly explored and widely applied throughout a course of study, not only to reinforce their importance and help ensure their mastery, but also to showcase the interconnectedness and utility of knowledge.

These learning modules were expressly designed to be used in an "inverted" teaching environment[4] where students first read the introductory and tutorial chapters on their own, then individually attempt to answer the questions and construct working circuits according to the experiment and project guidelines. The instructor never lectures, but instead meets regularly with each individual student to review their progress, answer questions, identify misconceptions, and challenge the student to new depths of understanding through further questioning. Regular meetings between instructor and student should resemble a Socratic[5] dialogue, where questions serve as scalpels to dissect topics and expose assumptions. The student passes each module only after consistently demonstrating their ability to logically analyze and correctly apply all major concepts in each question or project/experiment. The instructor must be vigilant in probing each student's understanding to ensure they are truly *reasoning* and not just *memorizing*. This is why "Challenge" points appear throughout, as prompts for students to think deeper about topics and as starting points for instructor queries. Sometimes these challenge points require additional knowledge that hasn't been covered in the series to answer in full. This is okay, as the major purpose of the Challenges is to stimulate analysis and synthesis on the part of each student.

The instructor must possess enough mastery of the subject matter and awareness of students' reasoning to generate their own follow-up questions to practically any student response. Even completely correct answers given by the student should be challenged by the instructor for the purpose of having students practice articulating their thoughts and defending their reasoning. Conceptual errors committed by the student should be exposed and corrected not by direct instruction, but rather by reducing the errors to an absurdity[6] through well-chosen questions and thought experiments posed by the instructor. Becoming proficient at this style of instruction requires time and dedication, but the positive effects on critical thinking for both student and instructor are spectacular.

An inspection of these learning modules reveals certain unique characteristics. One of these is a bias toward thorough explanations in the tutorial chapters. Without a live instructor to explain concepts and applications to students, the text itself must fulfill this role. This philosophy results in lengthier explanations than what you might typically find in a textbook, each step of the reasoning process fully explained, including footnotes addressing common questions and concerns students raise while learning these concepts. Each tutorial seeks to not only explain each major concept in sufficient detail, but also to explain the logic of each concept and how each may be developed

[4] In a traditional teaching environment, students first encounter new information via *lecture* from an expert, and then independently apply that information via *homework*. In an "inverted" course of study, students first encounter new information via *homework*, and then independently apply that information under the scrutiny of an expert. The expert's role in lecture is to simply *explain*, but the expert's role in an inverted session is to *challenge*, *critique*, and if necessary *explain* where gaps in understanding still exist.

[5] Socrates is a figure in ancient Greek philosophy famous for his unflinching style of questioning. Although he authored no texts, he appears as a character in Plato's many writings. The essence of Socratic philosophy is to leave no question unexamined and no point of view unchallenged. While purists may argue a topic such as electric circuits is too narrow for a true Socratic-style dialogue, I would argue that the essential thought processes involved with scientific reasoning on *any* topic are not far removed from the Socratic ideal, and that students of electricity and electronics would do very well to challenge assumptions, pose thought experiments, identify fallacies, and otherwise employ the arsenal of critical thinking skills modeled by Socrates.

[6] This rhetorical technique is known by the Latin phrase *reductio ad absurdum*. The concept is to expose errors by counter-example, since only one solid counter-example is necessary to disprove a universal claim. As an example of this, consider the common misconception among beginning students of electricity that voltage cannot exist without current. One way to apply *reductio ad absurdum* to this statement is to ask how much current passes through a fully-charged battery connected to nothing (i.e. a clear example of voltage existing without current).

from "first principles". Again, this reflects the goal of developing clear and independent thought in students' minds, by showing how clear and logical thought was used to forge each concept. Students benefit from witnessing a model of clear thinking in action, and these tutorials strive to be just that.

Another feature of these learning modules is that they do not treat topics in isolation. Rather, important concepts are introduced early in the series, and appear repeatedly as stepping-stones toward other concepts in subsequent modules. This helps to avoid the "compartmentalization" of knowledge, demonstrating the inter-connectedness of concepts and simultaneously reinforcing them. Each module is fairly complete in itself, reserving the beginning of its tutorial to a review of foundational concepts.

To high standards of education,

Tony R. Kuphaldt

Appendix C

Tools used

I am indebted to the developers of many open-source software applications in the creation of these learning modules. The following is a list of these applications with some commentary on each.

You will notice a theme common to many of these applications: a bias toward *code*. Although I am by no means an expert programmer in any computer language, I understand and appreciate the flexibility offered by code-based applications where the user (you) enters commands into a plain ASCII text file, which the software then reads and processes to create the final output. Code-based computer applications are by their very nature *extensible*, while WYSIWYG (What You See Is What You Get) applications are generally limited to whatever user interface the developer makes for you.

The `GNU/Linux` computer operating system

> There is so much to be said about Linus Torvalds' `Linux` and Richard Stallman's `GNU` project. First, to credit just these two individuals is to fail to do justice to the *mob* of passionate volunteers who contributed to make this amazing software a reality. I first learned of `Linux` back in 1996, and have been using this operating system on my personal computers almost exclusively since then. It is *free*, it is completely *configurable*, and it permits the continued use of highly efficient `Unix` applications and scripting languages (e.g. shell scripts, Makefiles, `sed`, `awk`) developed over many decades. `Linux` not only provided me with a powerful computing platform, but its open design served to inspire my life's work of creating open-source educational resources.

Bram Moolenaar's `Vim` text editor

> Writing code for any code-based computer application requires a *text editor*, which may be thought of as a word processor strictly limited to outputting plain-ASCII text files. Many good text editors exist, and one's choice of text editor seems to be a deeply personal matter within the programming world. I prefer `Vim` because it operates very similarly to `vi` which is ubiquitous on `Unix`/`Linux` operating systems, and because it may be entirely operated via keyboard (i.e. no mouse required) which makes it fast to use.

Donald Knuth's TEX typesetting system

Developed in the late 1970's and early 1980's by computer scientist extraordinaire Donald Knuth to typeset his multi-volume magnum opus *The Art of Computer Programming*, this software allows the production of formatted text for screen-viewing or paper printing, all by writing plain-text code to describe how the formatted text is supposed to appear. TEX is not just a markup language for documents, but it is also a Turing-complete programming language in and of itself, allowing useful algorithms to be created to control the production of documents. Simply put, TEX *is a programmer's approach to word processing*. Since TEX is controlled by code written in a plain-text file, this means anyone may read that plain-text file to see exactly how the document was created. This openness afforded by the code-based nature of TEX makes it relatively easy to learn how other people have created their own TEX documents. By contrast, examining a beautiful document created in a conventional WYSIWYG word processor such as Microsoft `Word` suggests nothing to the reader about *how* that document was created, or what the user might do to create something similar. As Mr. Knuth himself once quipped, conventional word processing applications should be called WYSIAYG (What You See Is *All* You Get).

Leslie Lamport's LATEX extensions to TEX

Like all true programming languages, TEX is inherently extensible. So, years after the release of TEX to the public, Leslie Lamport decided to create a massive extension allowing easier compilation of book-length documents. The result was LATEX, which is the markup language used to create all ModEL module documents. You could say that TEX is to LATEX as `C` is to `C++`. This means it is permissible to use any and all TEX commands within LATEX source code, and it all still works. Some of the features offered by LATEX that would be challenging to implement in TEX include automatic index and table-of-content creation.

Tim Edwards' `Xcircuit` drafting program

This wonderful program is what I use to create all the schematic diagrams and illustrations (but not photographic images or mathematical plots) throughout the ModEL project. It natively outputs PostScript format which is a true vector graphic format (this is why the images do not pixellate when you zoom in for a closer view), and it is so simple to use that I have never had to read the manual! Object libraries are easy to create for `Xcircuit`, being plain-text files using PostScript programming conventions. Over the years I have collected a large set of object libraries useful for drawing electrical and electronic schematics, pictorial diagrams, and other technical illustrations.

`Gimp` graphic image manipulation program

Essentially an open-source clone of Adobe's `PhotoShop`, I use `Gimp` to resize, crop, and convert file formats for all of the photographic images appearing in the ModEL modules. Although `Gimp` does offer its own scripting language (called `Script-Fu`), I have never had occasion to use it. Thus, my utilization of `Gimp` to merely crop, resize, and convert graphic images is akin to using a sword to slice bread.

`SPICE` circuit simulation program

`SPICE` is to circuit analysis as TEX is to document creation: it is a form of markup language designed to describe a certain object to be processed in plain-ASCII text. When the plain-text "source file" is compiled by the software, it outputs the final result. More modern circuit analysis tools certainly exist, but I prefer `SPICE` for the following reasons: it is *free*, it is *fast*, it is *reliable*, and it is a fantastic tool for *teaching* students of electricity and electronics how to write simple code. I happen to use rather old versions of `SPICE`, version 2g6 being my "go to" application when I only require text-based output. `NGSPICE` (version 26), which is based on Berkeley `SPICE` version 3f5, is used when I require graphical output for such things as time-domain waveforms and Bode plots. In all `SPICE` example netlists I strive to use coding conventions compatible with all `SPICE` versions.

Andrew D. Hwang's `ePiX` mathematical visualization programming library

This amazing project is a `C++` library you may link to any `C/C++` code for the purpose of generating PostScript graphic images of mathematical functions. As a completely free and open-source project, it does all the plotting I would otherwise use a Computer Algebra System (CAS) such as `Mathematica` or `Maple` to do. It should be said that `ePiX` is *not* a Computer Algebra System like `Mathematica` or `Maple`, but merely a mathematical *visualization* tool. In other words, it won't determine integrals for you (you'll have to implement that in your own `C/C++` code!), but it can graph the results, and it does so beautifully. What I really admire about `ePiX` is that it is a `C++` programming library, which means it builds on the existing power and toolset available with that programming language. Mr. Hwang could have probably developed his own stand-alone application for mathematical plotting, but by creating a `C++` library to do the same thing he accomplished something much greater.

`gnuplot` mathematical visualization software

Another open-source tool for mathematical visualization is `gnuplot`. Interestingly, this tool is *not* part of Richard Stallman's GNU project, its name being a coincidence. For this reason the authors prefer "gnu" *not* be capitalized at all to avoid confusion. This is a much "lighter-weight" alternative to a spreadsheet for plotting tabular data, and the fact that it easily outputs directly to an X11 console or a file in a number of different graphical formats (including PostScript) is very helpful. I typically set my `gnuplot` output format to default (X11 on my Linux PC) for quick viewing while I'm developing a visualization, then switch to PostScript file export once the visual is ready to include in the document(s) I'm writing. As with my use of `Gimp` to do rudimentary image editing, my use of `gnuplot` only scratches the surface of its capabilities, but the important points are that it's *free* and that it *works well*.

`Python` programming language

Both Python and C++ find extensive use in these modules as instructional aids and exercises, but I'm listing Python here as a *tool* for myself because I use it almost daily as a *calculator*. If you open a Python interpreter console and type `from math import *` you can type mathematical expressions and have it return results just as you would on a hand calculator. Complex-number (i.e. *phasor*) arithmetic is similarly supported if you include the complex-math library (`from cmath import *`). Examples of this are shown in the Programming References chapter (if included) in each module. Of course, being a fully-featured programming language, Python also supports conditionals, loops, and other structures useful for calculation of quantities. Also, running in a console environment where all entries and returned values show as text in a chronologically-ordered list makes it easy to copy-and-paste those calculations to document exactly how they were performed.

Appendix D

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Appendix E

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Appendix F

Version history

This is a list showing all significant additions, corrections, and other edits made to this learning module. Each entry is referenced by calendar date in reverse chronological order (newest version first), which appears on the front cover of every learning module for easy reference. Any contributors to this open-source document are listed here as well.

13-14 February 2025 – document first created, sampling content from my *Lessons In Industrial Instrumentation* textbook and modifying that content somewhat. Also added questions from the Socratic Instrumentation project.

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