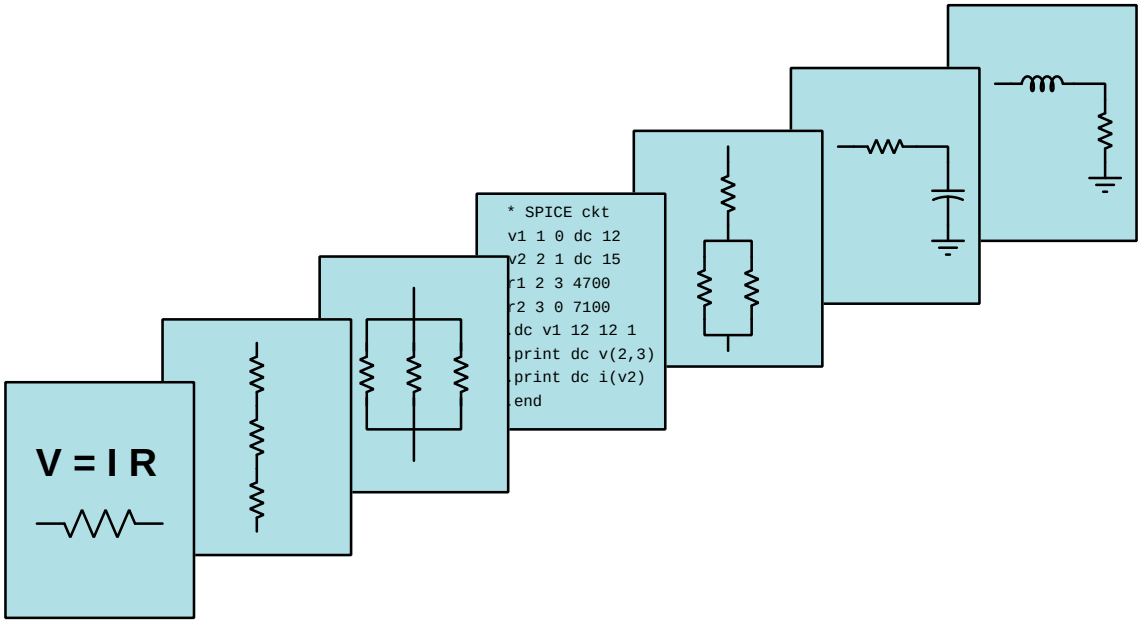
Modular Electronics Learning (ModEL) project



Resistance, Reactance, and Impedance

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Chapter 1

Introduction

1.1 Recommendations for students

Resistance (R) is, simply put, friction encountered by electric charge carriers while in a state of motion. It is one the foundational concepts of electric circuits, and in fact is given precise definition through Ohm's Law ($R = \frac{V}{I}$). Generally speaking, the concept of resistance typically does not pose a great challenge to students as they first encounter it. *Reactance* (X) and *impedance* (Z), which are the closest equivalent to resistance for components such as capacitors and inductors in AC circuit, often strike students as being vague and confusing. Resistance and reactance and impedance are all expressed in the same unit of measurement (the Ohm), but that is where the similarity ends. The mathematical strategies applied to the study of X and Z usually take the form of right-triangle trigonometry, and while this itself is not particularly difficult to execute, the logic of why right triangles are used to relate R, X, and Z together is not always presented, or presented well. As a consequence of this confusion, students have a tendency to view AC circuit analysis as wholly different from DC circuit analysis, with its own unique formulae and strategies. This is a shame, because when properly understood, the concepts of reactance and impedance actually demonstrate how AC circuit analysis is really the same as DC circuit analysis (just with a different type of mathematics), rather than being entirely different in principle. The following tutorials seek to remedy this sad state of affairs.

Important concepts related to these AC circuit parameters include **Ohm's Law**, **DC** versus **AC** electricity, **Joule's Law**, **energy exchange**, **sources** versus **loads**, the **Conservation of Energy**, **inductance**, **capacitance**, **rates of change**, **true** versus **reactive** versus **apparent** power, **phase shift**, **parasitic** properties, **induction** versus **dielectric** heating, **electromagnetic waves**, **phasor quantities**, **filter** networks, the **unit phasor**, and *phasor diagrams*.

This module contains several example calculations showing R, X, and Z applied to a range of AC circuits. An excellent active reading strategy is to attempt to perform all those calculations yourself, using the text as a guide to check your results. It is easy to deceive yourself by reading or listening to a thorough presentation on some challenging concept and think you understand it merely because it made sense to you at that time. The real test, however, is whether or not you can perform the same analysis without assistance, and so you should look at computational examples in

any textbook as opportunities to try and to practice.

It is also instructive in the example calculations to closely inspect the phasor diagrams for impedance, voltage, and/or current to see if you are able to properly match each vector with its respective complex-number quantity, as well as the sum. Again, do not be satisfied merely with reading the text's presentation, but prove to yourself that the numbers actually fit the diagrams!

Here are some good questions to ask of yourself while studying this subject:

- How might an experiment be designed and conducted to demonstrate the phenomenon of reactance? What hypothesis (i.e. prediction) might you pose for that experiment, and what result(s) would either support or disprove that hypothesis?
- How might an experiment be designed and conducted to test whether a component was resistive or reactive in nature? What hypothesis (i.e. prediction) might you pose for that experiment, and what result(s) would either support or disprove that hypothesis?
- What does it mean to say that energy is *conserved*?
- What does it mean when we say that a component *dissipates* energy?
- What are some of the different ways in which energy dissipates?
- How does reactance differ from resistance?
- How does impedance relate to resistance and reactance?
- How may we tell from an inspection of voltage and current waveforms in a resistive circuit that power dissipation will always be positive?
- How may we tell from an inspection of voltage and current waveforms in a reactive circuit that power dissipation will alternate between positive and negative?
- How may we tell from an inspection of voltage and current waveforms in a capacitive circuit that $I = C\frac{dV}{dt}$?
- How may we tell from an inspection of voltage and current waveforms in an inductive circuit that $V = L\frac{dI}{dt}$?
- What is a "parasitic" effect in an electrical component?
- What is the function of a radio transmitting antenna?
- What do the "real" and "imaginary" portions of a rectangular-format complex number represent for an AC quantity?
- What do the "magnitude" and "angle" portions of a polar-format complex number represent for an AC quantity?
- What practical benefit do complex numbers lend to the analysis of AC circuits?
- Do the properties of series networks still hold true for AC as they do for DC?

- Do the properties of parallel networks still hold true for AC as they do for DC?
- What is the purpose of a *filter* network?
- Of what consequence is reactive power in an AC power system?
- What trigonometric functions are used to convert a polar-form complex number into a rectangular-form complex number?
- What trigonometric functions are used to convert a rectangular-form complex number into a polar-form complex number?

1.2 Challenging concepts related to resitance, reactance, and impedance

The following list cites concepts related to this module's topic that are easily misunderstood, along with suggestions for properly understanding them:

- **Phase shift** – like voltage, phase shift is a relative quantity. A single AC waveform, without another waveform to compare to, cannot have a phase value at all. Only when two waves of the same frequency are compared against one another may we intelligently state a phase shift, or time displacement, between the two. Properly reading phase shifts from an oscillograph can also be challenging for students, and requires reminding that time goes from left to right on an oscillograph: i.e. the wave reaching its peak first (left) compared to the other is leading, while the other is lagging. Another confusing aspect of phase shift is that it may be expressed in multiple ways: e.g. $+90^o$ phase shift is equivalent to -270^o phase shift, since waves repeat their patterns every 360 degrees.

- **Phasors representing AC amplitudes and phase shifts** – a powerful tool used for understanding the operation of AC circuits is the *phasor diagram*, consisting of arrows pointing in different directions: the length of each arrow representing the amplitude of some AC quantity (voltage, current, or impedance), and the angle of each arrow representing the shift in phase relative to the other arrows. By representing each AC quantity thusly, we may more easily calculate their relationships to one another, with the phasors showing us how to apply trigonometry (Pythagorean Theorem, sine, cosine, and tangent functions) to the various calculations. An analytical parallel to the graphic tool of phasor diagrams is *complex numbers*, where we represent each phasor (arrow) by a pair of numbers: either a magnitude and angle (polar notation), or by "real" and "imaginary" magnitudes (rectangular notation). Where phasor diagrams are helpful is in applications where their respective AC quantities *add*: the resultant of two or more phasors stacked tip-to-tail being the mathematical sum of the phasors. Complex numbers, on the other hand, may be added, subtracted, multiplied, and divided; the last two operations being difficult to graphically represent with arrows.

- **Complex numbers in calculators** – while the ability of certain scientific calculators to perform complex-number arithmetic is an enormously helpful tool for students first learning to analyze AC circuits, some of these calculators prove to be finicky in their handling and entry of these quantities. Advice proven to be sound for all complex-number calculators is to save each and every complex-valued quantity into a memory location and then perform arithmetic operations on those stored variables rather than enter the complex numbers directly into the computation. For example, storing `3 - j4` into memory location `A` and `25` $\angle$ `30`^{o} into memory location `B`, then multiplying `A` $\times$ `B` rather than entering `3 - j4` $\times$ `25` $\angle$ `30`^{o}. Storing values into calculator memory and then retrieving them as needed for calculations is actually sound advice for many reasons, but many students resist taking these "extra" steps and as a result incur all the risks of hand-entering values (e.g. rounding errors due to truncating, keystroke errors when the same value must be used more than once, crowded displays where you cannot see the whole calculation, order-of-operations errors when complex numbers aren't enclosed in parentheses, etc.). Complex-number calculations reward good practices through consistently good results!

- **Complex numbers in measurement** – complex numbers may be expressed in either *rectangular* or *polar* form, either one of these being perfectly valid. However, measurement instruments such as multimeters only provide the magnitude of the polar form of the voltage or current in question. For example, if a component's voltage is 4.8 Volts RMS $\angle$ 35^o with the circuit's source voltage being the phase reference (0^o), an voltmeter reading that voltage will simply register 4.8 Volts RMS. An oscilloscope simultaneously measuring that component voltage on one channel and the source voltage on another will show the 35^o shift on the horizontal axis between the two waveforms.

- **Resistance versus Reactance versus Impedance** – these three terms represent different forms of opposition to electric current. Despite the fact that they are measured in the same unit (ohms: Ω), they are not the same concept. Resistance is best thought of as electrical *friction*, whereas reactance is best thought of as electrical *inertia*. Whereas resistance creates a voltage drop by dissipating energy, reactance creates a voltage drop by *storing* and *releasing* energy. Impedance is a term encompassing both resistance and reactance, usually a combination of both.

1.3 Recommendations for instructors

This section lists realistic student learning outcomes supported by the content of the module as well as suggested means of assessing (measuring) student learning. The outcomes state what learners should be able to do, and the assessments are specific challenges to prove students have learned.

- **Outcome** – Demonstrate effective technical reading and writing

  Assessment – Students present their outlines of this module's instructional chapters (e.g. Case Tutorial, Tutorial, Historical References, etc.) ideally as an entry to a larger Journal document chronicling their learning. These outlines should exhibit good-faith effort at summarizing major concepts explained in the text.

  Assessment – Students show how quantitative results were obtained by the author in the Tutorial chapter's examples.

  Assessment – Students present their "Conceptual Toolbox" Theory Journal entries listing important concepts applied in the Tutorial text(s). A "Conceptual Toolbox" serves as inspiration when solving problems, each concept suggesting a potential problem-solving action. Challenge students to collaboratively generate a Conceptual Toolbox for the day's topic during the class session and then use it when solving problems together, thus fostering the habit of reasoning from principles rather than recalling procedures!

- **Outcome** – Apply the concepts of reactance and frequency to component impedance

  Assessment – Calculate component impedances in complex-number form given frequency values; e.g. pose problems in the form of the "Resistances and reactances expressed as impedances" Quantitative Reasoning question.

- **Outcome** – Apply Ohm's Law to simple AC circuits using complex number values

  Assessment – Calculate circuit currents and voltages given source and frequency values; e.g. pose problems in the form of the "Simple AC circuits" Quantitative Reasoning question.

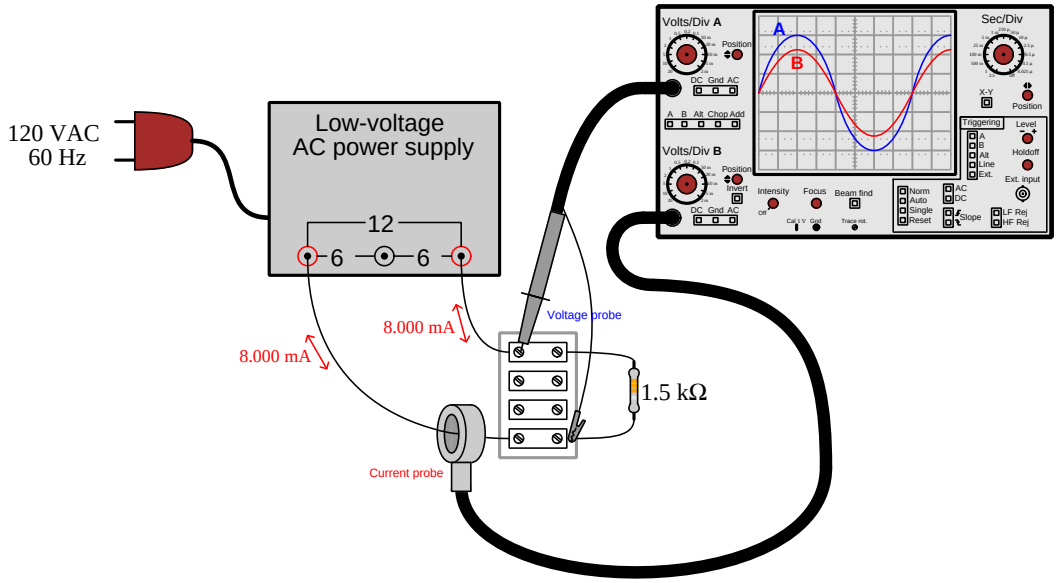
Chapter 2

Case Tutorial

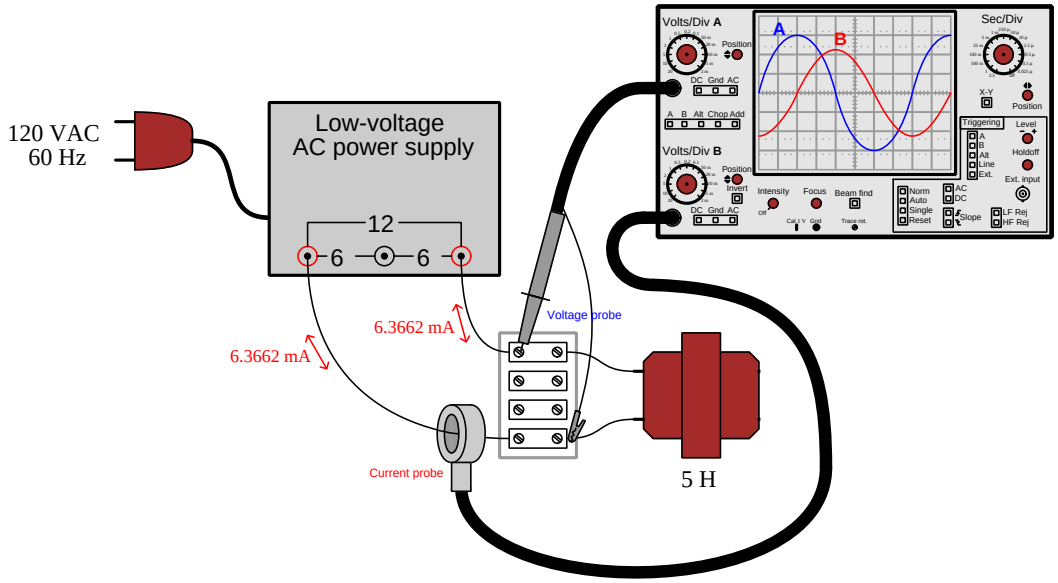
The idea behind a *Case Tutorial* is to explore new concepts by way of example. In this chapter you will read less presentation of theory compared to other Tutorial chapters, but by close observation and comparison of the given examples be able to discern patterns and principles much the same way as a scientific experimenter. Hopefully you will find these cases illuminating, and a good supplement to text-based tutorials.

These examples also serve well as challenges following your reading of the other Tutorial(s) in this module – can you explain *why* the circuits behave as they do?

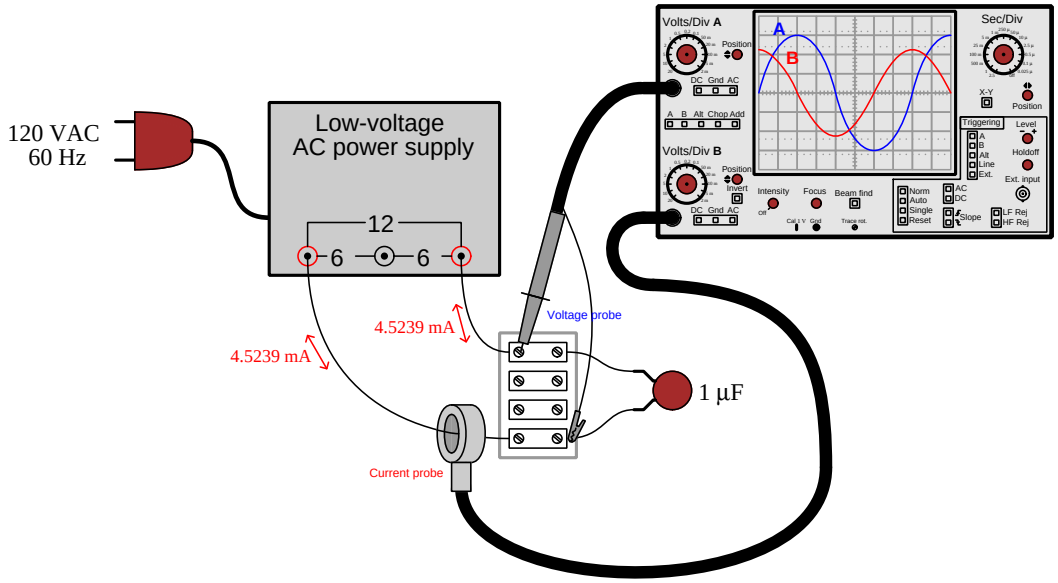
2.1 Example: AC source and resistor



2.2 Example: AC source and inductor



2.3 Example: AC source and capacitor



2.4 Example: custom inductor winding and testing

In this example we see a custom inductor made by winding 19 turns of 23-gauge enamel-coated copper wire around a Micrometals T50-66C powdered iron toroidal core. This core's datasheet specifies an inductance factor (A_L) of 51 nanoHenrys per turn-squared, so the inductor's value should be:

$$L = A_L N^2$$

$$L = (51 \times 10^{-9})(19^2) = 18.411\ \mu\text{H}$$

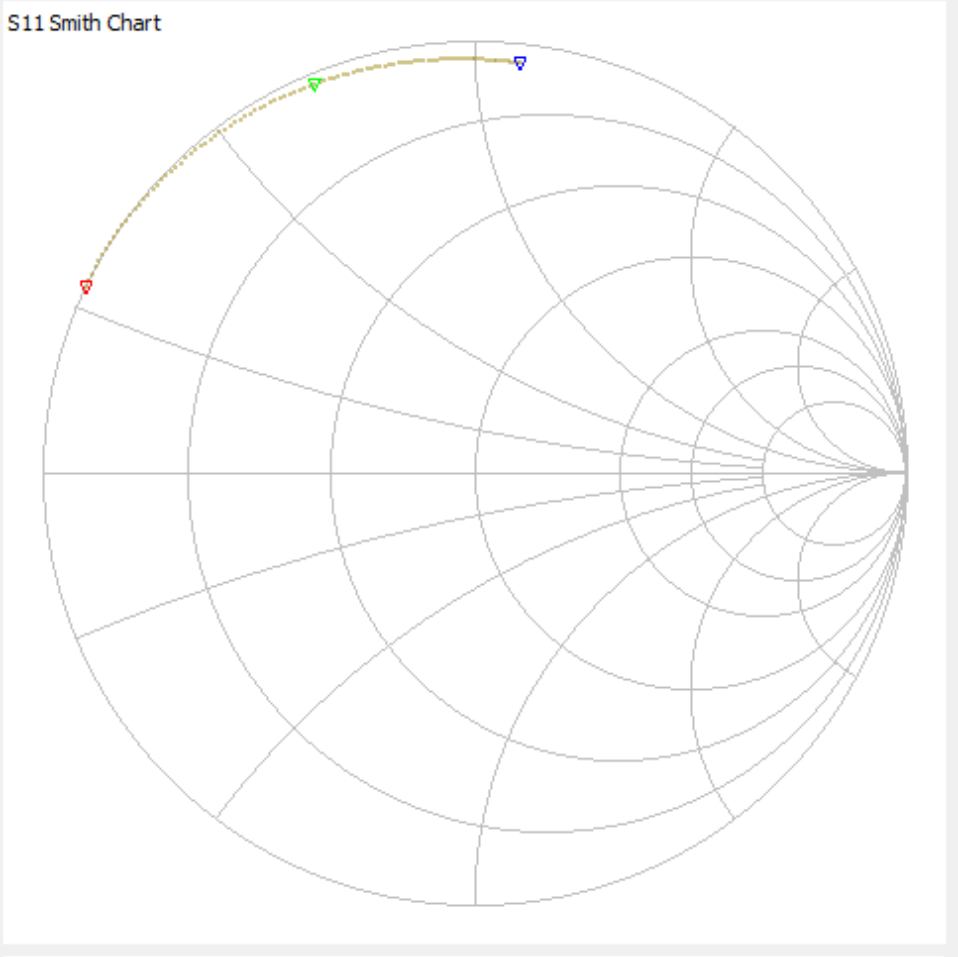
Micrometal's general catalog of core materials lists the type 66 as being "well-suited from 100 kHz to 500 kHz" and so a test was conducted using a NanoVNA vector network analyzer over that same range of signal frequencies:

Analysis from the VNA shows the measured inductance to be quite close to the predicted value of 18.411 microHenrys over the "swept" signal range of 100 kHz to 500 kHz:

Frequency	**Impedance**	**Series L**	**Deviation from predicted**
100 kHz	0.148 + j11.3 Ω	17.974 μH	−2.37%
300 kHz	1.11 + j33.5 Ω	17.782 μH	−3.42%
500 kHz	2.76 + j56.1 Ω	17.849 μH	−3.05%

The "real" portion of the complex impedance value represents series equivalent resistance in Ohms, equal in this case to the resistance of the wire in the inductor's coil as well as any connection resistances between the inductor and the VNA. This value rises slightly with signal frequency due to the *skin effect*, where electric current at very high frequency becomes excluded from traveling in the wire's center and therefore decreases the effective cross-sectional area of the conductor. The "imaginary" portion of the complex impedance value is the inductive reactance (X_L) in Ohms, relating to inductance by the formula $X_L = 2\pi f L$ where f is the signal frequency in Hertz.

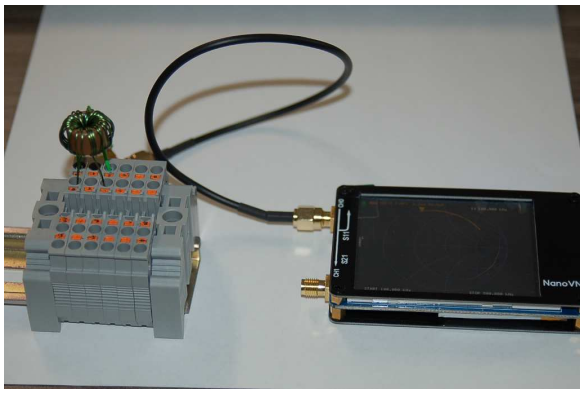
Like most vector network analyzers, the NanoVNA produces a Smith chart plot of complex impedance (normalized to 50 Ohms). As expected for a nearly-pure inductance, the path traced during the 100-500 kHz "sweep" closely follows the 0 Ohm resistive circle circumference while intersecting various inductive reactance circles as frequency rises. Here the red marker signifies 100 kHz (just above the +j0.2 circle denoting +j10 Ohms), the green marker 300 kHz (between the +j0.5 and +j1.0 circles), and the blue marker 500 kHz (just above the +j1.0 circle denoting +j50 Ohms):



Next we see the results of winding 30 turns of 23-gauge wire around another T50-66C powdered iron core with the same inductance factor of 51 nanoHenrys per turn-squared:

$$L = A_L N^2$$

$$L = (51 \times 10^{-9})(30^2) = 45.900\,\mu\text{H}$$



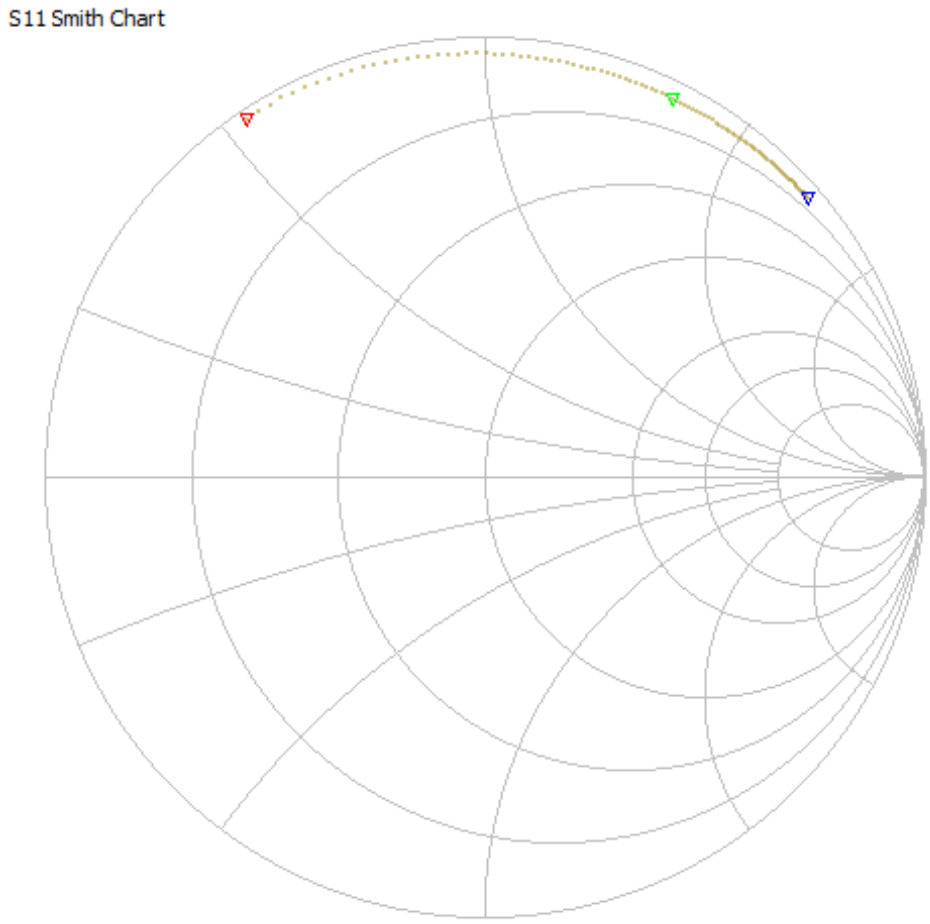
Analysis from the VNA shows the measured inductance to deviate more significantly from the predicted value of 45.9 microHenrys over the "swept" signal range of 100 kHz to 500 kHz than it did with the previous (19-turn) inductor:

Frequency	**Impedance**	**Series L**	**Deviation from predicted**
100 kHz	0.827 + j26.8 Ω	42.665 μH	−7.048%
300 kHz	3.86 + j80.6 Ω	42.771 μH	−6.817%
500 kHz	6.29 + j135 Ω	42.941 μH	−6.447%

A greater number of wire turns means an increase in two different parasitic properties: first, the wire will have greater total resistance because there is simply more of it for current to pass through; second, the greater number of turns packed onto the same size toroid means individual turns being closer together to one another than before which will increase the amount of inter-turn capacitance[1].

[1] Recall that capacitance between any two conductive surfaces is a direct function of the overlapping area of those surfaces, and an inverse function of separation distance. Here, with 30 turns of wire being packed closer together on the same toroid than 19 turns of wire, the inter-turn capacitance must be greater.

Here again we see the NanoVNA's Smith chart showing complex impedance normalized to 50 Ohms, this time for the 30-turn inductor:

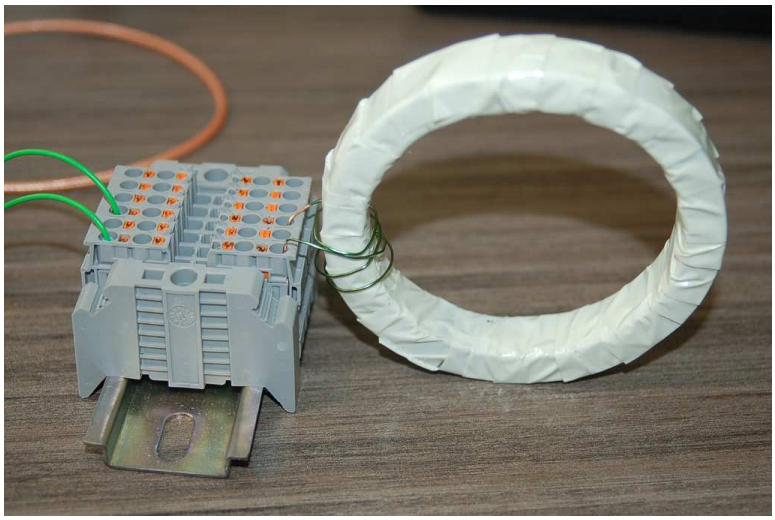


As one would expect, the curve lies farther to the right due to the increased inductive reactance of the larger inductance over the same frequency range. Increased resistance from the longer wire drives the curve closer to the middle of the Smith chart rather than remain on the periphery as before.

Using an entirely different toroidal core, I performed an experiment where I tried different numbers of turns of the same 23-gauge enameled wire and measured inductance using a different VNA (a PicoVNA model 106) sweeping from 300 kHz to 400 kHz:

Turns	**300 kHz**	**354 kHz**	**381 kHz**	**400 kHz**
2	6.06 μH	6.31 μH	6.43 μH	6.52 μH
4	22.79 μH	23.70 μH	24.13 μH	24.45 μH
5	36.36 μH	37.82 μH	38.51 μH	38.93 μH
10	149.63 μH	156.85 μH	159.24 μH	160.16 μH

The following photograph shows this iron core with four turns of wire wrapped around it, the bare iron core itself being wrapped in white electrical tape to prevent the enameled wire from chafing against the bare metal and possibly creating a coil-to-core shorted fault:



If you take any two inductance values at the same test frequency but for a different number of turns, the ratio of those two inductance values approximately equals the square of the ratio of the turns. For example, 160.16 microHenrys and 38.93 microHenrys at 400 kHz for 10 turns and 5 turns, respectively, makes an inductance ratio of nearly four to one (4.11405:1) for a turns ratio that is two to one (2:1). This confirms the general principle that inductance is proportional to the square of the number of wire turns wrapped around the core.

Computing the average A_L value at each test frequency based on the four measured inductance values and turns values ($\frac{L}{N^2}$) yields the following:

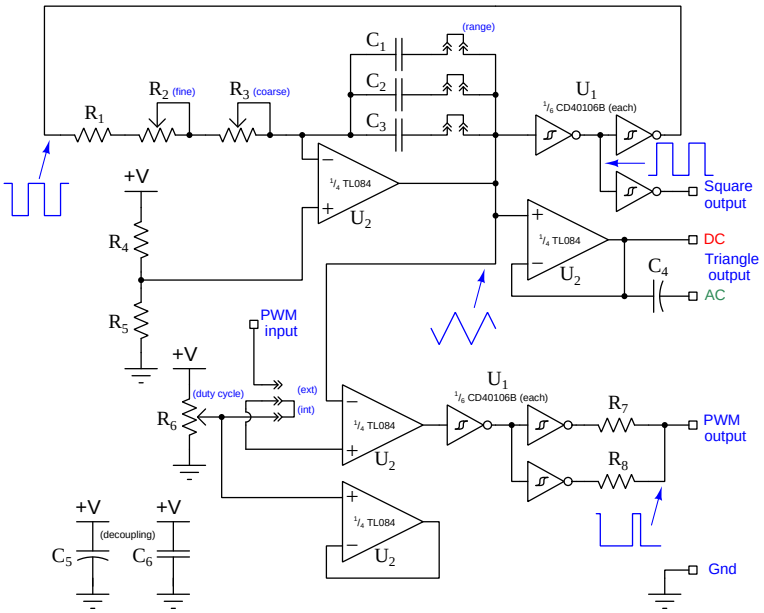
Frequency	**Average** A_L
300 kHz	1.473 μH per turn^2
354 kHz	1.535 μH per turn^2
381 kHz	1.562 μH per turn^2
400 kHz	1.579 μH per turn^2

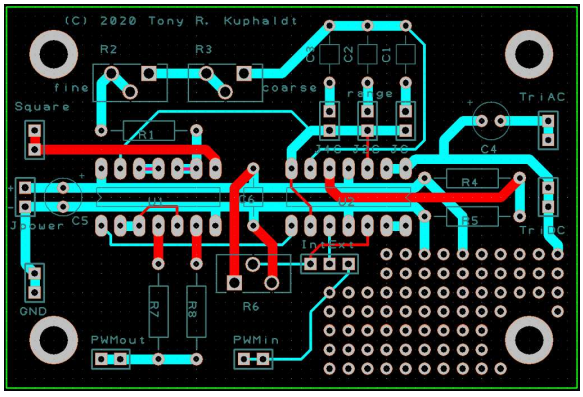
Averaging these four A_L values yields an aggregate A_L of 1.537 μH per turn-squared for this particular toroidal iron core. This figure, of course, will be useful for creating custom inductors based on this toroid as the core material.

2.5 Example: sine versus non-sine AC sources

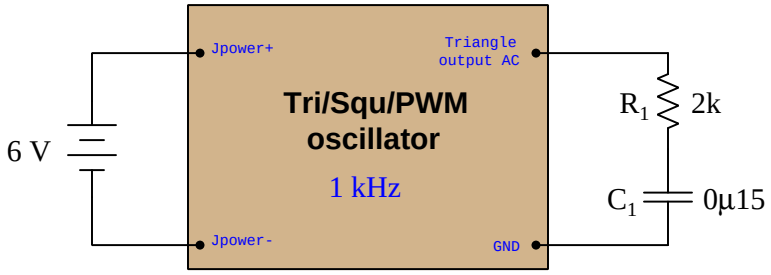
Students learning to analyze RLC networks powered by AC voltage sources typically rely on expensive signal generators to produce the pure sine-wave AC excitation voltage necessary for voltmeter measurements to closely match predictions. However, robust triangle-wave oscillator circuits are much less complicated to design and build than sine-wave oscillator circuits, so if students wish to build their own signal generators for these introductory AC experiments it is good to know that triangle-wave excitation yields results very close to sine-wave excitation.

A simple and versatile signal generator circuit appears below, outputting triangle, square, and PWM (pulse-width-modulated) signals. The first image is the schematic diagram, followed by a PCB layout:





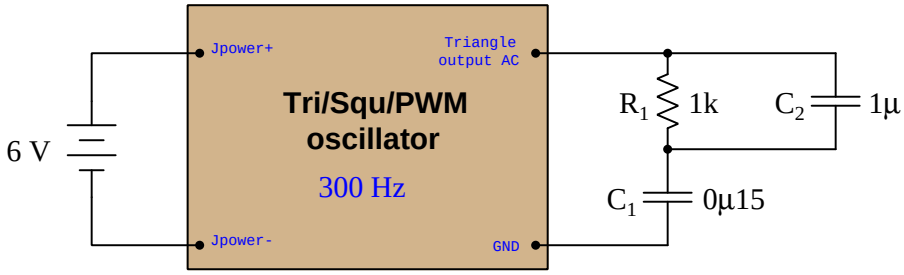
Here are some test results on simple RC networks:



Connected to the RC network, frequency was measured using a Fluke model 87-III multimeter and adjusted to 1 kHz, and then total voltage measured across the series $R_1 \leftrightarrow C_1$ combination as 233.0 mVAC.

Parameter	**Measured** (triangle-wave)	**Predicted** (sine-wave)
V_{R1}	205.7 mVAC	205.8 mVAC
V_{C1}	109.3 mVAC	109.2 mVAC

Testing a slightly more complex circuit at a frequency of 300 Hz, the loaded voltage output of the oscillator being 231.6 mVAC this time:



Parameter	**Measured** (triangle-wave)	**Predicted** (sine-wave)
V_{R1}	23.7 mVAC	27.43 mVAC
V_{C1}	207.5 mVAC	207.0 mVAC
V_{C2}	23.7 mVAC	27.43 mVAC

In both applications, the greatest error between measured voltage and predicted voltage as a percentage of total voltage was in the second circuit across $R_1||C_1$ (23.7 milliVolts rather than 27.43 milliVolts), and this is only -1.61% of the source voltage which is considerably less than the $\pm$ 5% tolerance of the resistor and capacitors!

If we compare the Fourier series for a sine wave and a triangle wave (both having unity peak values and a frequency of ω) we see that the first harmonic of the triangle wave function is identical to the sine wave, and that all the other harmonics in the triangle wave are significantly smaller-amplitude than the fundamental:

Sine wave

$$\cos \omega t$$

Triangle wave

$$\cos \omega t + \frac{1}{9} \cos 3\omega t + \frac{1}{25} \cos 5\omega t + \frac{1}{49} \cos 7\omega t + \cdots + \frac{1}{n^2} \cos n\omega t$$

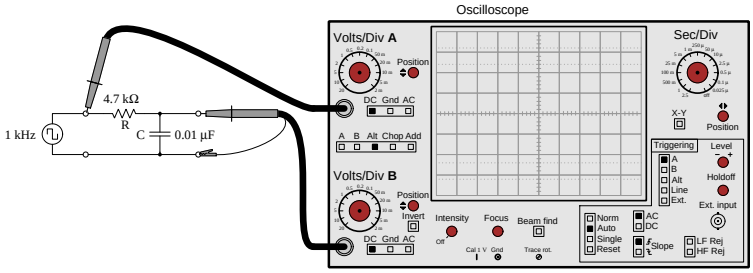
This tells us any deviations between the measured (triangle-wave) and predicted (sine-wave) voltage values are likely to be minimal, the third harmonic being only 11.1% of the fundamental's amplitude, the fifth harmonic being only 4% of the fundamental's amplitude, etc. The effects of higher-order harmonics are truly negligible due to their vastly smaller amplitudes as well as due to the fact that most digital multimeters suffer "cut off" in the audio-frequency range and therefore cannot measure signal components in the tens of thousands of Hertz.

With access to a digital oscilloscope having FFT capability[2] to show precise voltage values for each harmonic of a measured waveform, we have an even better solution for obtaining voltage measurements in agreement with predicted values when not using perfectly sinusoidal signal generators. Since the oscilloscope's FFT algorithm separates and displays each of the sinusoidal harmonics apart from one another in any non-sinusoidal waveform, if we simply pay attention to the magnitudes of a common harmonic frequency within each voltage measurement we will essentially take circuit measurements on purely sinusoidal voltages of the same frequency. For example, we could measure the fundamental (i.e. the first harmonic)[3] amplitude of source voltage, then the fundamental amplitudes of each of the other components' voltages, and check to see that these measured voltage values match well with our predictions at that frequency. This technique, in effect, lets us measure the effects of a purely sinusoidal signal even when the real signal is not sinusoidal at all, by taking measurements only on a common harmonic of the measured voltages!

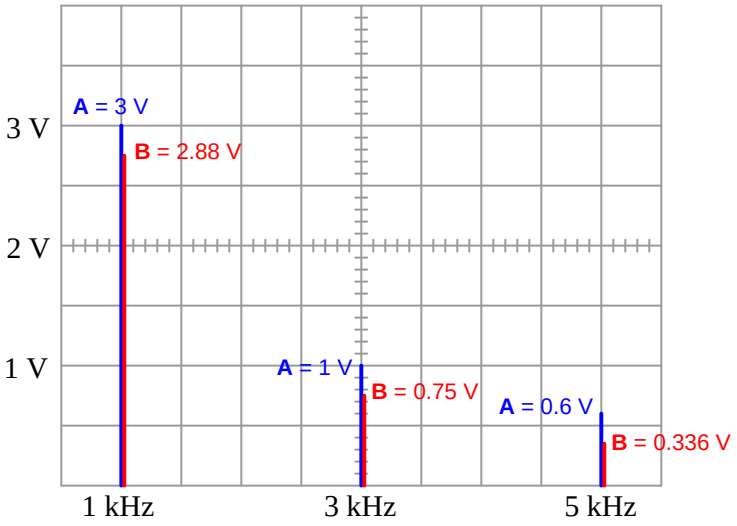
[2] At the time of this writing (2022) some inexpensive oscilloscopes may be found with rather poor FFT resolution, resulting in wide spectral peaks with uncertain height (voltage) values. You know you are working with a sufficiently precise instrument when the harmonic peaks show as thin lines rather than exaggerated bell-curves.

[3] There is no particular reason why we might choose the first harmonic over any of the others, other than the fact that with triangle and square waves this fundamental will be vastly stronger than any of the other harmonics.

This testing technique deserves some elaboration, and so we shall explore it by example. Consider the following test circuit where a signal generator configured to output a square-wave AC signal at 1000 Hz energizes a simple RC network consisting of a 4.7 kΩ resistor and a 0.01 μF capacitor:

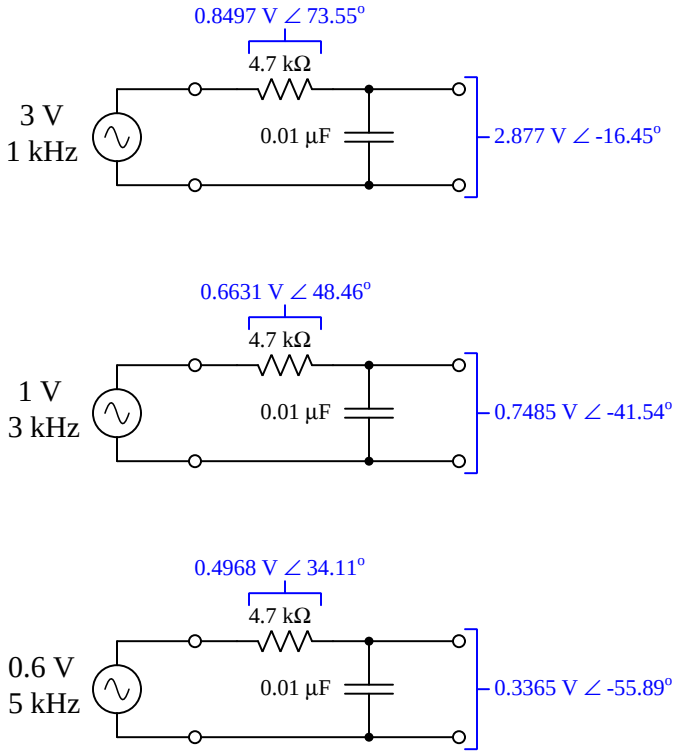


We know from Fourier analysis that a square wave is actually equivalent to a sine wave at the same fundamental frequency added to another sine wave one-third the amplitude at three times that frequency (3rd harmonic) added to another sine wave one-fifth the amplitude of the fundamental at five times that frequency (5th harmonic), and so on. If we examine the frequency-domain plots of the signal generator's output (channel A) versus the capacitor's voltage drop (channel B), we see the circuit's response to pure sine waves at each of those frequencies:



The relative peak heights of the channel A signal (3 Volts, 1 Volt, 0.6 Volts) are simply the result of the Fourier series for a square wave and has nothing to do with the RC network. The ratios between peak heights of channel A and channel B at each harmonic frequency, however, are unique to the 4.7 kΩ and 0.01 μF RC network because those voltage pairs represent the attenuation of this particular network at each of those sinusoidal frequencies.

If we mathematically analyze this same RC network for each of the square wave's harmonic amplitudes and frequencies used in the test circuit, we should obtain results verifiable by using the oscilloscope in FFT mode:



For students with access to oscilloscopes having fine-resolution FFT capability, this not only means it is unnecessary to secure a signal generator with pure sinusoidal output, but it also means the ability to energize any AC circuit with *any* waveshape and test its response at multiple sinusoidal frequencies *simultaneously*!

2.6 Examples: complex-number arithmetic

These complex-number arithmetic problems are presented to you, complete with answers (shown in **bold**), for the purpose of practice, since nearly all AC circuit calculations will need to be performed using complex numbers. Use these practice calculations to check your ability either to perform these calculations "by hand" (using trigonometric functions) or your ability to use your calculator's complex-number functionality.

Note: electronic hand calculators and computer-based calculation programs use the proper mathematical notation i to represent imaginary numbers rather than j. The letter j is used in electrical engineering work in order to avoid confusion with i being misinterpreted as "current". Also note that calculators and software programs usually default to *radians* for angle measurement rather than *degrees*, and will have to be configured (or converted) for degrees in order to handle the polar-form complex quantities shown here. Check the "mode" options of your hand calculator to ensure angles are in the correct unit and also that it will display in either rectangular or polar.

Addition and subtraction:

$(5 + j6) + (2 - j1) = \mathbf{7 + j5}$
$(10 - j8) + (4 - j3) = \mathbf{14 - j11}$
$(-3 + j0) + (9 - j12) = \mathbf{6 - j12}$

$(3 + j5) - (0 - j9) = \mathbf{3 + j14}$
$(25 - j84) - (4 - j3) = \mathbf{21 - j81}$
$(-1500 + j40) + (299 - j128) = \mathbf{-1201 - j88}$

$(25\angle 15^o) + (10\angle 74^o) = \mathbf{31.35\angle 30.87^o}$
$(1000\angle 43^o) + (1200\angle -20^o) = \mathbf{1878.7\angle 8.311^o}$
$(522\angle 71^o) - (85\angle 30^o) = \mathbf{461.23\angle 77.94^o}$

Multiplication and division:

$(25\angle 15^o) \times (12\angle 10^o) = \mathbf{300\angle 25^o}$
$(1\angle 25^o) \times (500\angle -30^o) = \mathbf{500\angle -5^o}$
$(522\angle 71^o) \times (33\angle 9^o) = \mathbf{17226\angle 80^o}$

$\frac{10\angle -80^o}{1\angle 0^o} = \mathbf{10\angle -80^o}$

$\frac{25\angle 120^o}{3.5\angle -55^o} = \mathbf{7.142\angle 175^o}$

$\frac{-66\angle 67^o}{8\angle -42^o} = \mathbf{8.25\angle -71^o}$

$(3 + j5) \times (2 - j1) = \mathbf{11 + j7}$
$(10 - j8) \times (4 - j3) = \mathbf{16 - j62}$

$\frac{(3+j4)}{(12-j2)} = \mathbf{0.1892 + j0.3649}$

Reciprocation:

$$\frac{1}{(15\angle 60^o)} = \mathbf{0.0667\angle -60^o}$$

$$\frac{1}{(750\angle -38^o)} = \mathbf{0.00133\angle 38^o}$$

$$\frac{1}{(10+j3)} = \mathbf{0.0917 - j0.0275}$$

$$\frac{1}{\frac{1}{15\angle 45^o} + \frac{1}{92\angle -25^o}} = \mathbf{14.06\angle 36.74^o}$$

$$\frac{1}{\frac{1}{1200\angle 73^o} + \frac{1}{574\angle 21^o}} = \mathbf{425.7\angle 37.23^o}$$

$$\frac{1}{\frac{1}{23k\angle -67^o} + \frac{1}{10k\angle -81^o}} = \mathbf{7.013k\angle -76.77^o}$$

$$\frac{1}{\frac{1}{110\angle -34^o} + \frac{1}{80\angle 19^o} + \frac{1}{70\angle 10}} = \mathbf{29.89\angle 2.513^o}$$

$$\frac{1}{\frac{1}{89k\angle -5^o} + \frac{1}{15k\angle 33^o} + \frac{1}{9.35k\angle 45}} = \mathbf{5.531k\angle 37.86^o}$$

$$\frac{1}{\frac{1}{512\angle 34^o} + \frac{1}{1k\angle -25^o} + \frac{1}{942\angle -20} + \frac{1}{2.2k\angle 44^o}} = \mathbf{256.4\angle 9.181^o}$$

Sign reversal:

$-(45\angle 70^o) = \mathbf{45\angle -110^o}$

$-(90\angle -20^o) = \mathbf{90\angle 160^o}$

$-(5 + j8) = \mathbf{-5 - j8}$

$-(-3 + j9) = \mathbf{3 - j9}$

$-(10 - j15) = \mathbf{-10 + j15}$

Practical suggestions for using your calculator to perform these operations:

- Surround each complex-number quantity with parentheses when setting up arithmetic operations; e.g., `(3 + j5) * (4 - j2)` instead of `3 + j5 * 4 - j2`. This habit will guarantee your calculator executes the desired order of operations rather than assert its own. For instance, in the example given here the calculator may choose to multiply $j5$ by 4 and then add on 3 and $-j2$ since multiplication typically precedes addition, if the two complex numbers are not encapsulated in their own sets of parentheses.

- Store all calculated results in memory and then recall from memory when re-using those values, rather than re-entering previously-calculated values by hand or sampling previously-calculated values from the multi-line display. Manually re-entering values invites rounding errors and keystroke errors in all cases, and I've found certain calculators (I'm looking at you, *TI*!) fail to properly enter complex-number values when sampled from their multi-line displays. Getting in the habit of using your calculator's memory locations is an all-around good habit that will serve you very well!

Some Texas Instruments brand calculators such as the TI-84 offer an exponential key and imaginary (i) key which allows you to enter numbers in complex exponential form (i.e. $e^{i\theta}$). With the TI-84, for example, the complex number $10 - j8$ may be entered in either of the two following forms:

`(10 - i8)` or `(10 - 8i)`

The result may be displayed in either rectangular or polar forms according to the complex-number display *mode* the TI-84 calculator has been set to. In rectangular mode the displayed result for $10 - j8$ will be `10 - 8i`, whereas in polar mode the displayed result will be `12.806` $\mathtt{e}^{-38.66i}$. Note how the TI-84 uses exponential notation for polar display, where the angle (-38.66 degrees, in this example) is an imaginary power of e.

If you wish to enter a complex number in polar form on a TI-84, you must unfortunately express the angle in units of *radians* (even though the calculator is able to display the result in *degrees*). For example, to enter the number $25\angle 15^o$ into a TI-84 calculator, you must type:

`25` $\mathtt{e}^{i15\pi/180}$

The fraction $\pi/180$ is the conversion factor from degrees to radians, since there are 2π radians to a full circle, or π radians to every 180 degrees. Thus, writing $15\pi/180$ multiplies the desired angle (15 degrees) by the conversion factor $\pi/180$ to yield a power in radians. The obligatory i simply makes this power an imaginary quantity, which is mathematically necessary with exponential notation for describing a complex number. It should be noted that the order of entry for the power matters little. $i15\pi/180$ works just as well as $15i\pi/180$ or $15\pi/180i$.

A time-saving step some students find useful is to save the imaginary quantity $i\pi/180$ to a memory location in the TI-84 such as Z. That way, they can recall that imaginary factor from memory instead of typing the whole thing by hand every time they wish to enter a polar-form complex number. Supposing the memory location Z contains $i\pi/180$, entering the number $25\angle 15^o$ becomes as simple as:

25 e^{15Z} or 25 e^{Z15}

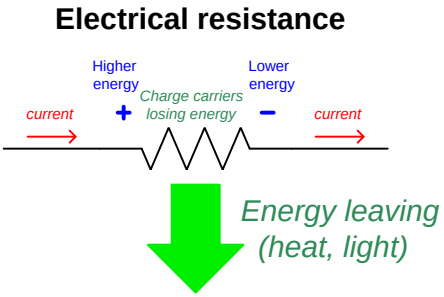
It should be understood that *any* memory location in your calculator is suitable for storing $i\pi/180$, not just Z. The TI-84 calculator even provides a Θ memory location (<Alpha>-<3>) that you may use and find easy to remember because of its common association with angles. It should also be understood that this imaginary quantity is not the same as i or j, which the calculator already provides a dedicated function for. The imaginary quantity we're storing in memory for the purpose of entering polar-notation angles contains not only i but also the $\pi/180$ conversion factor necessary for translating your *degree* entry into *radians*.

Some calculators provide easier means of entering and displaying complex numbers in polar form. Both the Texas Instruments TI-36X Pro and TI-89 offer an angle symbol ($\angle$) for this purpose, the TI-36X Pro being a much less expensive and less complex calculator than the TI-89. Entering the number $25 \angle 15^o$ into one of these calculators is as easy as typing 25 $\angle$ 15, and likewise the result will be displayed in this same form when the calculator is set to "polar" complex mode.

Chapter 3

Simplified Tutorial

Electrical *resistance* is essentially the friction experienced by electric charge carriers as they travel from one location to another, and like all forms of friction any attempt at motion results in energy dissipation (i.e. energy leaving the system and never returning). Charge carriers exit an electrical resistance at a lower energy level than they enter, as is the case with all electrical loads:

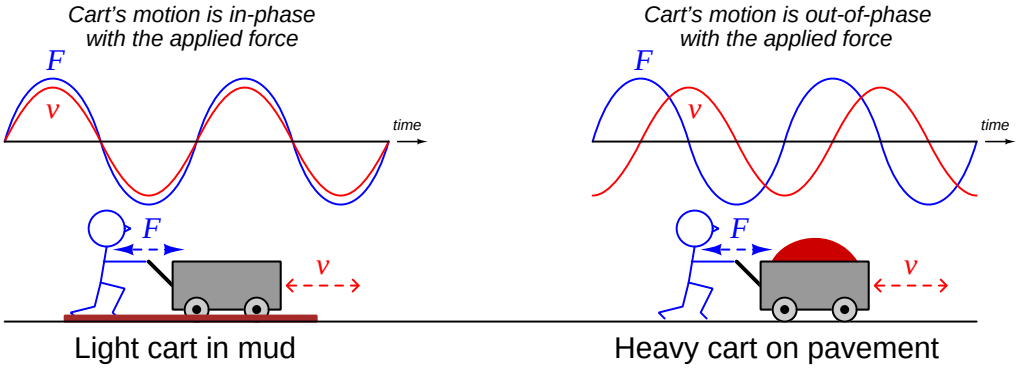


Resistance is not the only form of opposition to the flow of electric charge carriers. Some electrical components are able to alternately store and release energy, and they too oppose electric charge carrier motion. Opposition to charge carrier motion based on energy storage and release is called *reactance* rather than resistance. This distinction may be clearly understood by analogy to a wheeled push-cart:

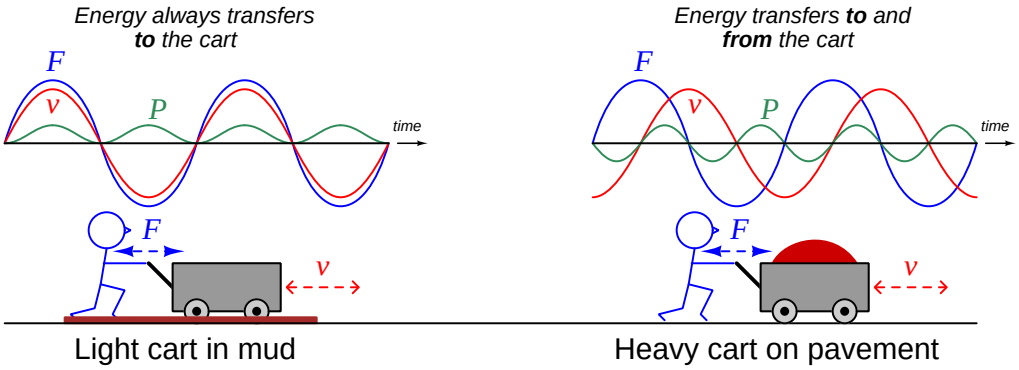
- Resistance is akin to a light-weight cart on a muddy road: moving the cart requires an expenditure of energy to overcome the rolling resistance of the wheels turning through mud. As soon as you stop pushing, the cart comes to a halt. *Friction* dominates the cart's behavior.

- Reactance is akin to a heavy-weight cart on a hard road: accelerating the cart requires an investment of energy that is returned upon deceleration. The cart continues to coast when you stop pushing, and acts as an energy source (i.e. pulling you along with it) when you try to slow it back down. *Inertia* dominates the cart's behavior.

Energizing an electrical component with an *alternating* ("AC") voltage is akin to alternately pushing and pulling on a hand cart, but the timing of the cart's motion versus the applied force depends entirely on whether friction or inertia dominates. The light-weight cart mired in mud moves in time with the applied force; the heavy-weight cart rolling freely on hard pavement moves out-of-sync with the applied force. The following graphs of force (F) and velocity (v) compare these two carts' behavior:



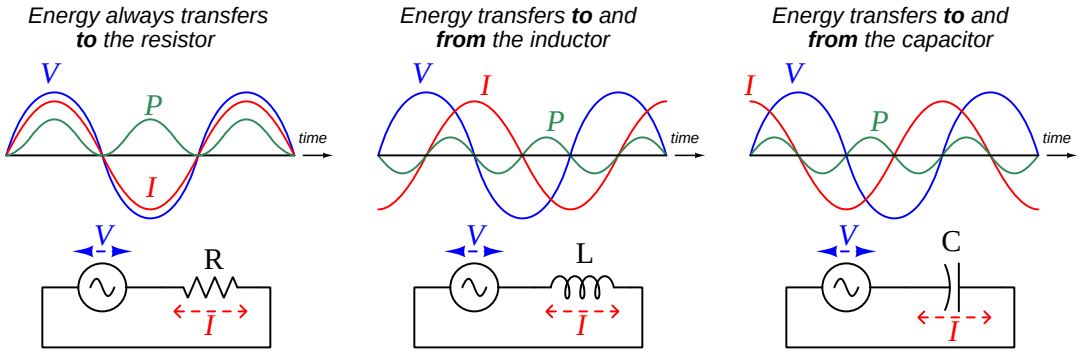
Exploring this concept in more detail, we may graph the *power* (P) transferred to and from these two carts. When applied force and cart motion are in the same direction, the person is doing work on the cart (i.e. transferring energy to the cart) and power is a positive quantity. When force and motion are opposed, the cart is doing work on the person (i.e. releasing energy) and power is negative:



The light-weight cart mired in mud always requires work done by the person because applied force and motion are always in the same directions: the person does work pushing the cart through the mud, and also does work pulling the cart through the mud. That energy gets dissipated in the mud, ultimately as heat. The heavy-weight cart rolling freely on hard pavement, however, alternately requires work from the person and does work on the person: the person does work on the cart when force and motion are aligned, but the cart does work on the person when force and motion are opposite. Energy is exchanged back and forth, but never dissipated.

The difference between resistance and reactance (friction and inertia) is fundamentally a matter of energy transfer. Resistance (friction) always results in energy dissipation, while reactance (inertia) results in energy absorption and return. Resistance always functions as a *load*, while reactance alternately functions as a *load* and then a *source*. Resistance "spends" energy while reactance "borrows" and "returns" energy.

Returning to electrical circuits, we may graph voltage (V), current (I), and power (P) for three different types of components: a *resistor* (R), an *inductor* (L), and a *capacitor* (C). Resistors are dissipative devices, with voltage drop proportional to current (Ohm's Law, $V = IR$). Inductors are reactive devices, with voltage proportional to rate-of-change of current ($V = L\frac{dI}{dt}$). Capacitors are also reactive devices, with current proportional to rate-of-change of voltage ($I = C\frac{dV}{dt}$).



The resistor circuit is analogous to the light-weight cart mired in mud: it only ever dissipates energy, usually in the form of heat. Both the inductor and capacitor circuits are analogous to the heavy-weight cart rolling on hard pavement, each in their own way: energy transfers to and from these reactive components but never leaves the circuit. With inductors the voltage waveform leads 90^o ahead of the current waveform; for capacitors the current waveform leads 90^o ahead of the voltage waveform.

Resistance and reactance both relate current to voltage, and so both quantities are expressed in the unit of the *Ohm* (Ω). To distinguish resistance from reactance, we use different mathematical symbols: R for resistance and X for reactance. We may further distinguish types of reactance using subscripts: X_L for inductive reactance and X_C for capacitive reactance.

Resistance is unaffected by frequency (f), but reactance varies linearly with frequency:

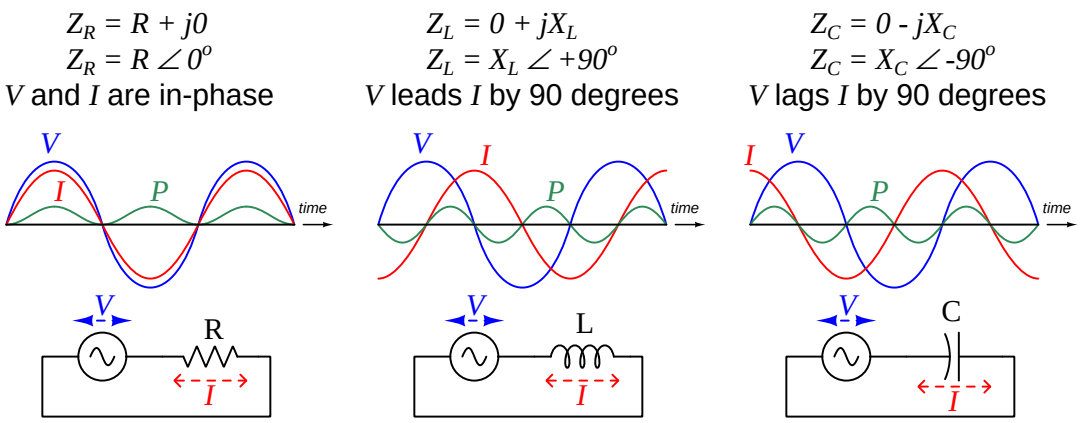
- Inductive reactance is directly proportional to frequency: $X_L = 2\pi f L$
- Capacitive reactance is inversely proportional to frequency: $X_C = \frac{1}{2\pi f C}$

Impedance (Z) is the combined effect of resistance and reactance, usually expressed as a *complex number* where resistance is the "real" portion and reactance is the "imaginary[1]" portion.

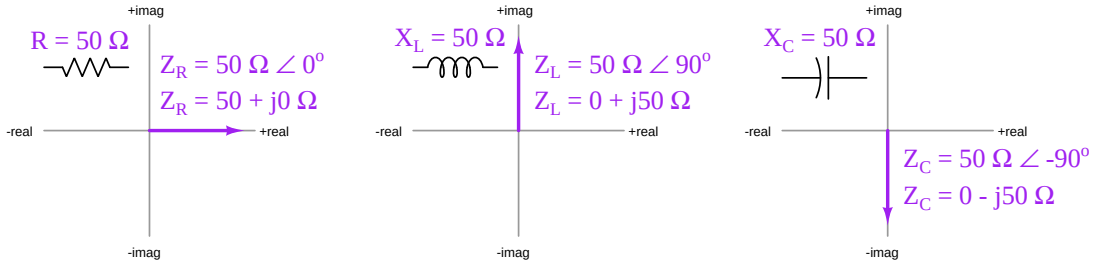
[1] In mathematics, an *imaginary* number is the square root of any negative real number. The symbol i or j is

Combinations of resistance and inductive reactance are expressed as $Z = R + jX_L$, while combinations of resistance and capacitive reactance are expressed as $Z = R - jX_C$. Impedance is typically expressed in *polar form* ($Z \angle \theta$) rather than rectangular form ($R \pm jX$).

Impedance is a more universal expression of opposition to charge carrier motion than either resistance or reactance. In fact, pure resistance and pure reactance alike are just special cases of impedances. Resistance is impedance with a phase angle of 0^o because with a resistor there is zero phase shift between voltage and current. Inductive reactance is impedance with a phase angle of $+90^o$ because with an inductor voltage leads current by $+90^o$. Capacitive impedance is reactance with a phase angle of -90^o because voltage lags current by -90^o for a capacitor:



Graphical expressions of complex-number resistance and reactance values may be shown in *phasor diagrams*, where the length of a vector arrow represents the phasor's magnitude and its angle represents phase shift, the horizontal axis representing the real number line and the vertical axis representing the imaginary number (j) line. Three such diagrams illustrate the difference between a 50 Ω resistor versus an inductor having 50 Ω of reactance and a capacitor also having 50 Ω of reactance:



used to express $\sqrt{-1}$. This abstract concept may seem far removed from the practicalities of AC circuits, but it is quite relevant. Specifically, imaginary numbers are key to simplifying what would otherwise be an enormously tedious process of regarding every AC quantity as a sinusoidal (trigonometric) function. *Euler's Relation* ($e^{jx} = \cos x + j \sin x$) is the link joining imaginary numbers to sinusoids, and is the mathematical foundation of phasors.

Some examples[2] of component impedances expressed as complex quantities are shown here:

- A 33 kΩ resistor at any frequency will have an impedance of:

  33 kΩ $\angle$ 0^o *(polar form)*

  33 k + j0 Ω *(rectangular form)*

- A 250 mH inductor at a frequency of 6 kHz will have an impedance of:

  9.425 kΩ $\angle$ $+90^o$ *(polar form)*

  0 + j9.425 kΩ *(rectangular form)*

- A 2.2 μF capacitor at a frequency of 150 Hz will have an impedance of:

  482.3 Ω $\angle$ -90^o *(polar form)*

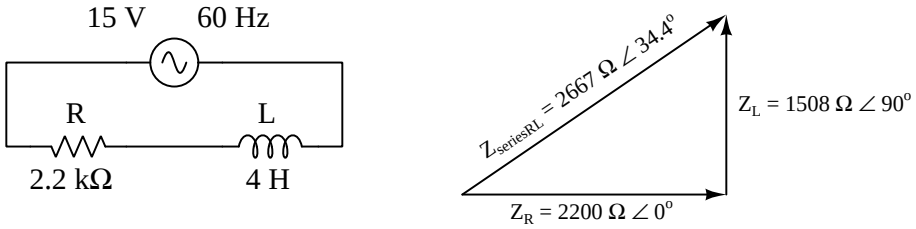
  0 − j482.3 Ω *(rectangular form)*

With all passive component values expressed as complex-number impedances and all voltage and current quantities also expressed as complex numbers, all the fundamental rules regarding voltage and current learned in DC circuit theory still apply to AC circuits: all the properties of series and parallel networks remain true, as well as Ohm's Law ($V = IZ$, using impedance rather than resistance), Kirchhoff's Laws, and network theorems. Several example circuits are analyzed in the Full Tutorial, and you are encouraged to work through those examples yourself (using a calculator capable of performing arithmetic functions on complex numbers) to gain familiarity with the procedures.

[2] Try calculating these impedance values from the given component values, to check your understanding of how reactances related to impedances. This is a good learning strategy to apply when reading any mathematical text: work through the presented examples on your own to see if you achieve the same results! Please note that when you apply either the $X_L = 2\pi fL$ formula or the $X_C = \frac{1}{2\pi fC}$ formula using your calculator to compute reactance, the result will *only* be a reactance value and not a (complex) impedance value. In order to attach the desired phase angle to your computed reactance value, you will have to perform the additional step of multiplying that reactance by a *unit phasor* which is nothing more than the quantity of 1 with the correct phase angle. For example, an inductive reactance of 9.425 kΩ would be multiplied by 1 $\angle$ 90^o to yield an inductive *impedance* of 9.425 kΩ $\angle$ 90^o.

When the AC electrical quantities in question are known to *add*, we may graphically represent that sum by stacking their representative vectors tip-to-tail, the sum being the resultant vector from the first vector's tail (starting point) to the last vector's tip (ending point). The impedances of resistive and reactive impedances in a series AC circuit, for example, form a right triangle when summed. Since resistive impedance has a phase angle of zero degrees its vector points to the right, while inductive impedance with its 90^o phase angle points up and capacitive impedance with its -90^o phase angle points down.

To illustrate, we will compute the total impedance for a series network consisting of a single resistor and a single inductor:



We see how the 2.2 kΩ resistor forms a vector pointed to the right, and the 4 H inductor forms a vector (with length equal to 1508 Ω from the $X_L = 2\pi fL$ formula) pointed up, together forming a total impedance of 2667 Ω having a phase angle of 34.4^o. If the reactance happened to be capacitive rather than inductive, the only graphical difference it would make is that the capacitor's impedance vector would be pointed down rather than up as for the inductor.

The total impedance's magnitude may be computed using the Pythagorean Theorem ($Z = \sqrt{R + X}$) and the angle using the arc-tangent function $\theta = \arctan\left(\frac{X}{R}\right)$. Using a scientific calculator with the additional capability of complex-number arithmetic, however, makes this computation trivial: just enter Z_R and Z_L as complex quantities, and then directly add them together[3].

Current in this circuit follows Ohm's Law, where $I = \frac{V}{Z_{total}}$. In this case, the current will be $\frac{15\ \text{V}}{2667\ \Omega} = 5.624$ mA. This current produces voltage drops across the resistor and inductor as predicted by Ohm's Law ($V = IZ$), 12.37 Volts across the resistor and 8.481 Volts across the inductor.

Again, it is important to emphasize that this use of the Pythagorean Theorem to stack two vectors together at right angles to each other in order to find the resultant vector length only applies to circuit quantities known to *add*. For a series circuit such as this, representing each component's impedance as a vector and stacking them tip-to-tail to find total impedance works because we know impedance (just like resistance) adds in series. Voltages also add in series, and so we could just as appropriately do the same for two vectors representing component voltages to find total (source) voltage. If the circuit were parallel, however, our use of the Pythagorean Theorem to find resultant vector length would be limited to *current* vectors rather than voltage or impedance, because while branch currents do indeed add to equal total source current in parallel circuits, voltages and impedances do not.

If we have access to a calculator capable of complex-number arithmetic, trigonometric functions and the Pythagorean Theorem no longer are necessary to perform this addition, and in fact we won't be limited to just addition either!

[3]Such a calculator will be able to display the sum in either polar ($2667\angle 34.4^o$) or rectangular ($2200 + j1508$) format.

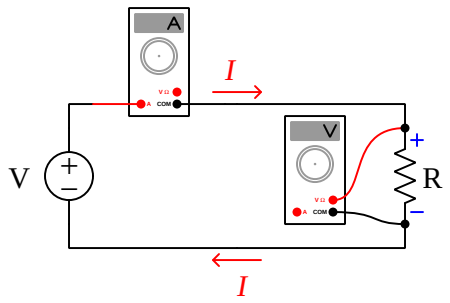
Chapter 4

Full Tutorial

Ohm's Law is one of the foundational principles applied in the analysis of DC (Direct Current) circuits, and for good reason: it gives *resistance* a mathematical definition, showing it to be the ratio of voltage to current, $R = \frac{V}{I}$. In this tutorial we will expand upon the concept of resistance to include the relationships of voltage to current for other phenomena such as *inductance* and *capacitance*. This conceptual expansion will equip us with mathematical tools for analyzing AC (Alternating Current) circuits where capacitance and inductance find many practical applications.

4.1 Purely resistive circuit

To begin, we will review the behavior of electrical resistance in a very simple DC circuit. Suppose we connect a constant-voltage DC generator to a resistor, a voltmeter, and an ammeter as shown:

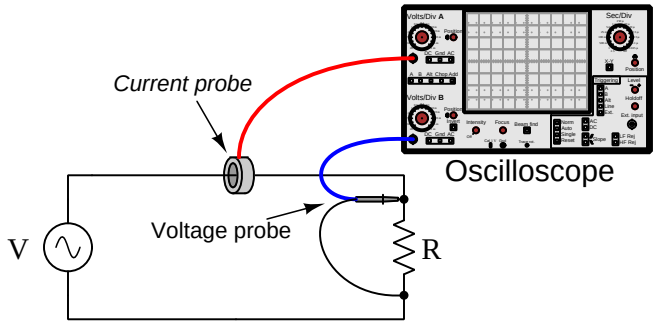


Being a complete circuit, we would expect current to flow in a clockwise direction (exiting the generator's positive terminal and entering the resistor's positive terminal) at a rate predicted by Ohm's Law ($I = \frac{V}{R}$). Electric charge carriers passing upward through the generator gain potential energy, which is why the entering terminal of the generator is labeled with a negative symbol ($-$) and the exiting terminal of the generator is labeled with a positive terminal ($+$). Those same charge carriers passing downward through the resistor lose potential energy, giving up that energy in the form of heat, which is why they enter on the ($+$) terminal at high energy and exit on the ($-$) terminal

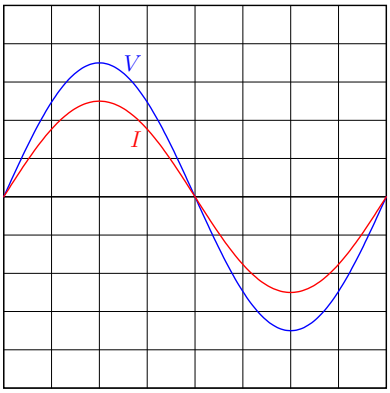
at low energy. This is why we classify the resistor as a *load*: it drawn energy from passing charge carriers and converts that energy into some other form where it exits the circuit never to return. The generator, by contrast, is classified as a *source*: it infuses energy into the passing charges from the external energy source (e.g. mechanical driver).

Joule's Law predicts that the rate of energy transfer at the resistive load, from electric charge potential energy into heat energy, will be the mathematical product of voltage and current: $P = IV$.

This is simple enough in the case of simple DC (Direct Current) circuit such as this, where quantities of voltage and current remain relatively stable over time. What happens if we replace the DC (Direct Current) generator with an AC (Alternating Current) generator? The following diagram shows such a circuit, using a dual-trace oscilloscope equipped with a current probe[1] to measure voltage and current:



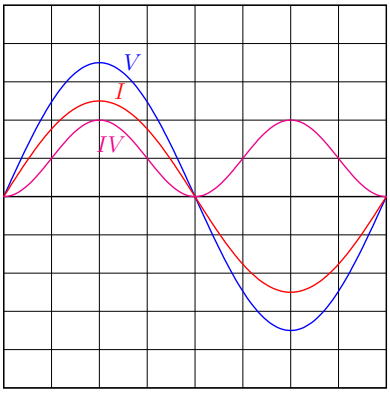
The oscilloscope displays a graphical representation of resistor voltage and resistor current over time. If the generator's output voltage is sinusoidal, both voltage and current waveforms will appear as sine waves on the oscilloscope's screen:



[1] Oscilloscopes are essentially graphical *voltmeters*, and so they do not naturally accept *current* signals as inputs. Therefore, some form of current probe is necessary to detect current and translate that into a voltage signal the oscilloscope can interpret. The current probe shown in this illustration works by sensing the magnetic field produced around the wire by current, outputting a voltage signal to the oscilloscope proportional to the strength of that magnetic field.

Note how both the voltage (V) and current (I) waveforms are completely in-phase with each other: crossing zero at the same points of time and reaching their peak values at the same points in time. Positive values on this graph refer to voltage polarity with + on top and − on bottom, current flowing clockwise. Negative values refer to voltage polarity with − on top and + on bottom, current counter-clockwise. Ohm's Law ($I = \frac{V}{R}$) applies to each and every moment.

Power (P) is a more challenging quantity to calculate, since voltage and current are both continuously changing over time. Joule's Law ($P = IV$) still holds true at any given instant, so we may illustrate power in this circuit by configuring the oscilloscope to plot the product of current and voltage (IV) as a third waveform[2]:



Note how the power waveform never dips down below the zero line. The amount of power in this circuit at any point in time is either power is positive or zero. This positive sign for power refers to the resistor's constant function as a *load*: always converting electric charge potential into heat. This must mean the generator constantly functions as a *source*: drawing energy from its mechanical driver to infuse into passing electric charge carriers. The fact that the generator outputs AC rather than DC has no effect on the roles of source and load in this circuit.

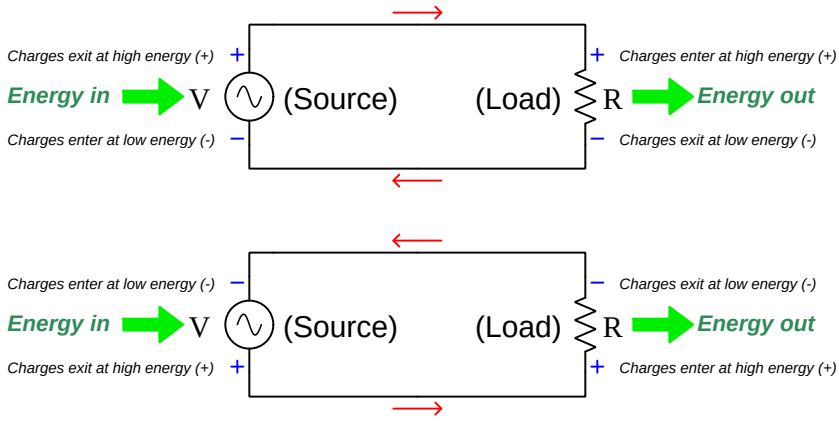
At this point, one might pose the question: "What would *negative* power mean, anyway?" This is easy to answer mathematically: the only way to obtain a negative product of two real numbers is to multiply a positive quantity by a negative quantity. This would mean current would have to flow "backwards" during a part of the cycle where voltage polarity was still "forwards", or vice-versa. In other words, the only way for a resistance to experience negative power would be it to somehow behave as a source: the resistor would have to have current exiting its positive terminal and entering its negative terminal. We know for a fact that resistance can never be a source, since it is essentially "friction"[3] to the movement of electrical charge carriers and therefore may only rob energy from

[2] Instead of using an oscilloscope to generate this plot, I used a computer to mathematically calculate the instantaneous product of two sine waves ($\sin t \times \sin t$). However, modern digital oscilloscopes typically offer "math" channels where you may configure the instrument to plot some mathematical function of the measured variables, for example the product of two input signals. Please note the lack of any *scaling* in these plots. Waveform amplitude is scaled only for esthetics, so that the waveforms do not cross each other unnecessarily and thereby hinder legibility.

[3] A more scientifically accurate description of electrical resistance is the conversion of charge carrier energy into heat by means of *collisions*. This is why voltage is proportional to current for any given resistance: the faster charge carriers try to move through a resistance, the more energetic their collisions and therefore the more energy each one loses (voltage) while passing through.

them, and never infuse energy into them. Consequently, negative power is an impossibility for any purely resistive component.

This also makes sense if we view the current directions and voltage polarities during each half-cycle of the generator's sine wave. During both half-cycles the generator acts as a source and the resistor acts as a load[4]:



No matter which direction charge carriers circulate, the resistor terminal they enter must be the positive (+) and the resistor terminal they exit must be the negative (−) because that properly expresses the energy lost by those charge carriers as they pass through the resistor and encounter friction.

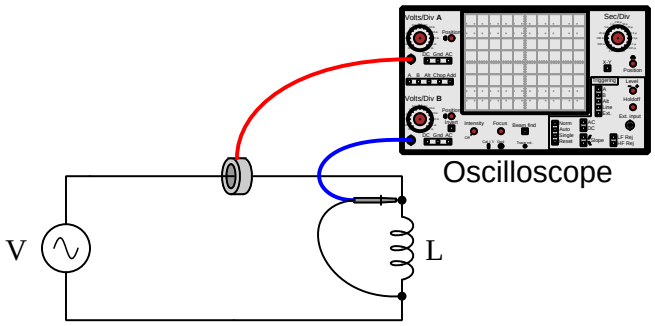
Whatever mechanism turns this generator's shaft experiences this unidirectional power flow as a constant mechanical burden. If the generator were turned by a hand crank, the person spinning the crank would definitely feel this burden as laborious, unrelenting work. Practically[5] all of the energy imparted to the generator by the person becomes converted into heat at the resistor. The Law of Energy Conservation tells us all the heat dissipated in this circuit must be matched by work done turning the generator shaft, for energy cannot appear from nowhere.

[4] Returning to the concept of positive and negative power, it is fair to say that an electrical load exhibits positive power while an electrical source exhibits negative power. One way to relate this mathematical sign to the real world is to see it as an indication of energy direction: positive power (load) refers to energy entering the outside world from the circuit, while negative power (source) refers to energy entering the circuit from the outside world. For example, from the perspective of an external observer, an electric lamp is a provider of *positive* power because it provides heat and light; an electromechanical generator, on the other hand, imposes a demand for power (*negative*) because something must do work to turn its shaft.

[5] Due to resistance intrinsic to the generator's wiring and to the wires connecting the generator to the resistor, some energy will be converted into heat at those locations as well.

4.2 Purely inductive circuit

Let us now consider the same circuit, but with a perfect[6] inductance replacing the resistance:



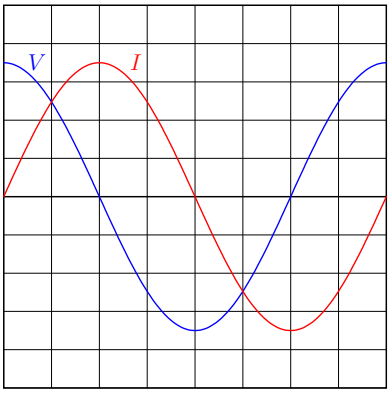
Unlike resistance where voltage is proportional to current, an inductance's voltage is proportional to how quickly current *varies* over time. Instead of Ohm's Law where $V = IR$ for a resistance, we have the following formula describing an inductance's relation of voltage to current:

$$V = L\frac{dI}{dt}$$

Where,

V = Voltage across the inductance's terminals (Volts)
L = Inductance (Henrys)
$\frac{dI}{dt}$ = Rate-of-change of current over time (Amperes per second)

An oscilloscope plot of voltage and current for this purely inductive circuit shows the relationship between voltage and the rate-of-change of current:

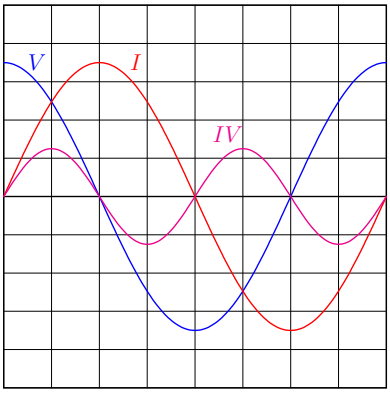


We see voltage (V) reaching its peak values when current (I) is at its maximum rate-of-change (steepness) crossing the zero line. We also see voltage at zero whenever current levels off (at the

[6] By "perfect" we mean an inductance with no energy losses whatsoever. Such an inductance would have superconducting wires to avoid losses there, as well as a core material lacking any magnetic hysteresis.

current waveform's peaks). For perfect sinusoidal waveshapes, the phase shift between voltage and current for an inductance will always be 90^o, with current lagging and voltage leading.

If we plot the product of current and voltage ($P = IV$) as a third waveform on the oscilloscope display, we see something very interesting:



Unlike the resistive circuit where power (P) varied between positive and zero values, this power waveform completely alternates between positive and negative values. This means the inductance alternately experiences positive power and negative power. As discussed previously, positive power is when the generator acts as a source and the other component acts as a load; negative power is when those roles reverse and the component sends energy back to the generator.

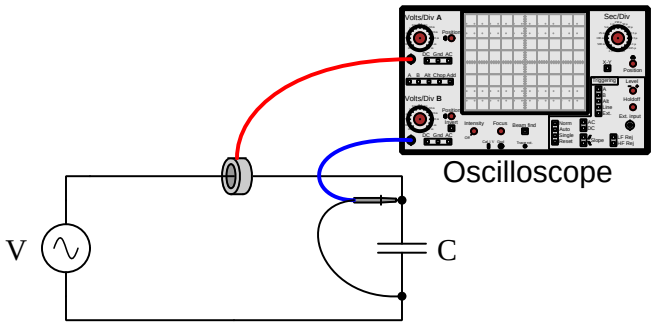
This should make sense to us based on our understanding of inductance. While resistance is simply a form of electrical "friction" dissipating energy in the form of heat, inductance stores and releases energy by means of a magnetic field. Those periods of positive power are when the inductance acts as a load, and those periods of negative power are when the inductance acts as a source.

Any person turning a crank to spin the generator's shaft would notice a profound difference between this and the resistive circuit. Continual effort was required in the resistive circuit because the resistor caused energy to leave the circuit in the form of heat, and this persistent out-flow of energy required a continuous in-flow of energy into the circuit from the person laboring to turn the crank. Here, though, the inductance alternately receives and gives energy in the circuit, which means part of each cycle the generator requires energy to turn and an equal duration of each cycle the generator runs like a motor on the energy returned from the inductance. The net effect of this perfectly balanced back-and-forth energy exchange is that the generator requires no effort to turn even though AC voltage and current are both present in the circuit. In other words, *the generator requires no work because the inductance does no work*. This is in agreement with the Law of Energy Conservation, since with no heat dissipated in this circuit or other electrical work being done, no work should be required to turn the generator shaft.

Power that leaves a circuit and does not return is called *true power*. Power that transfers back and forth between circuit components while never leaving is called *reactive power*. All power in the resistive circuit was true because it left the circuit never to return. All power in this inductive circuit is reactive, because it merely shuttles back and forth between the two components and never leaves the circuit.

4.3 Purely capacitive circuit

Now we will consider the same circuit, but with a perfect[7] capacitance replacing the inductance:



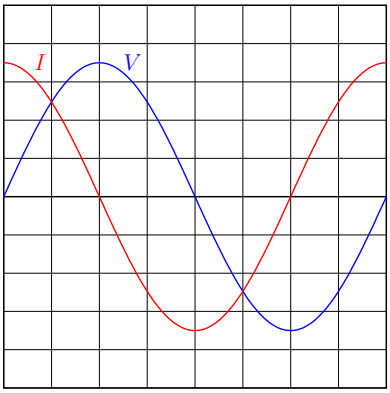
Unlike resistance where current is proportional to voltage, a capacitance's current is proportional to how quickly voltage *varies* over time. Instead of Ohm's Law where $I = \frac{V}{R}$ for a resistance, we have the following formula describing the relationship between voltage and current for a capacitance:

$$I = C\frac{dV}{dt}$$

Where,

I = Current through the capacitance's terminals (Amperes)
C = Capacitance (Farads)
$\frac{dV}{dt}$ = Rate-of-change of voltage over time (Volts per second)

An oscilloscope plot of voltage and current for this purely capacitive circuit shows the relationship between current and the rate-of-change of voltage:

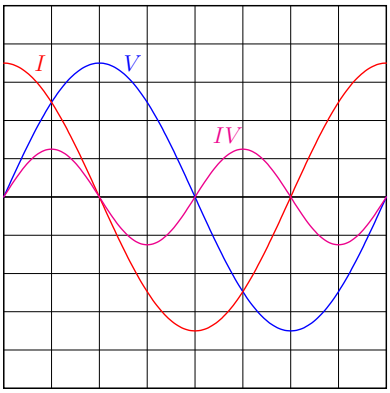


We see current (I) reaching its peak values when voltage (V) is at its maximum rate-of-change (steepness) crossing the zero line. We also see current at zero whenever voltage levels off (at the

[7] By "perfect" we mean a capacitance having no energy losses whatsoever. Such a capacitance would have superconducting conductors to avoid losses there, as well as a lossless (i.e. non-heating) dielectric material.

voltage waveform's peaks). For perfect sinusoidal waveshapes, the phase shift between voltage and current for a capacitance will always be -90^o, with voltage lagging and current leading.

As you might have already reasoned, if we plot the product of current and voltage ($P = IV$) as a third waveform on the oscilloscope display we see an effect similar to the inductive circuit:



The fact that it is current leading voltage this time rather than voltage leading current makes little difference for power – the 90^o phase shift is what makes power alternate equally between positive and negative values. As with the purely inductive circuit, this purely capacitive circuit performs no work and therefore all of its power is *reactive* (exchanged back and forth between components) rather than *real* (leaving the circuit). Similarly, no mechanical energy is required[8] to maintain the generator's shaft rotation (i.e. the generator is effortless to turn) because no energy ever leaves the capacitive circuit.

By this point, hopefully, the lesson is clear: resistive AC circuits exhibit no phase shift between voltage and current waveforms, and this results in power leaving the circuit never to return (i.e. *true* power); reactive AC circuits (either inductive or capacitive) exhibit a 90^o phase shift between voltage and current resulting in power exchanged equally back and forth between components and never leaving the circuit (i.e. *reactive* power). The fundamental difference between inductive and capacitive circuits is the leading or lagging nature of that 90^o phase shift: in an inductive circuit, voltage leads current by $+90^o$; in a capacitive circuit, voltage lags current by -90^o[9]

[8] For this purely capacitive circuit, as with the purely inductive circuit, there would actually be some energy required to initially set the generator shaft in motion and thereby invest energy in the reactive component, but once that energy is returned to the generator (forcing it to act as a motor, investing that energy kinetically in the generator's spinning mass) no additional work would be necessary. In either the inductive or the capacitive case, such perfect energy exchange would require zero resistance throughout the circuit, frictionless bearings in the generator, zero hysteresis losses in the magnetic core material of the generator, etc. In actual practice there would be *some* small amount of mechanical power required to maintain the generator's motion in order to satisfy these power losses, but this would certainly be far less than the amount of mechanical power necessary to drive the generator in a resistive circuit with the same AC voltage and current values.

[9] A simple memory trick often taught to students of AC circuits is the phrase *"Eli the Ice man"*, where the word "Eli" is supposed to remind you that voltage (E – representing *electromotive* force) comes before current (I) for inductive (L) circuits, and the word "Ice" is supposed to remind you that current (I) comes before voltage (E) for capacitive (C) circuits.

4.4 Realistic circuits

The previous few sections all began with the word "Purely" in their titles, which ought to tell you something about *real* components in AC circuits. In reality, there is no such thing as a purely resistive resistor, nor a purely inductive inductor, nor a purely capacitive capacitor. Real components may approach these ideals, but never perfectly embody them.

Resistors, for example, exhibit trace quantities of inductance and capacitance that are impossible to completely eliminate. Inductance is simply the storage of energy in a magnetic field, and since any current-carrying conductor forms a magnetic field around it (i.e. electromagnetism) this means *any* current-carrying component will harbor some inductance. Likewise for capacitance, being the storage of energy in an electric field formed between two or more conductors at differing potentials, which means *any* component experiencing a voltage between two or more terminals will form an electric field between those terminals and therefore will harbor some capacitance. Techniques[10] exist to minimize these "parasitic" effects, and so the phase shift between voltage and current for most resistors will be very nearly zero degrees, and therefore resistor power will be very nearly 100% real and 0% reactive.

Capacitors are never 100% reactive, mostly due to energy losses in their dielectric material. When an alternating electric field is applied across the plates of a capacitor from an applied AC voltage, the molecules of that material experience alternating physical stresses. These stresses cause vibration on a molecular level, which raises the temperature of the dielectric material. As that material warms, heat energy dissipates from it to the surrounding environment, and this constitutes real (not reactive) power. Modern capacitor dielectric materials are chemically engineered to minimize this energy dissipation, and so the phase angle between AC voltage and current for any capacitor will be very nearly -90^o, and power will be very nearly 100% reactive, but not perfectly so.

Inductors tend to be least-perfect form of reactive component, with phase shift between voltage and current approaching $+90^o$ but not quite achieving that ideal. They experience energy dissipation in multiple ways, most notably in the resistance of their wire coils and in various losses within their ferrous cores[11]. Wire resistance is a fairly obvious mechanism of energy loss in an inductor: any device requiring long lengths of wire to form electromagnetic coils will exhibit electrical resistance unless those wires are superconducting. Less obvious are energy losses in the magnetic core: like capacitor dielectrics, ferrous magnetic materials experience alternating physical stresses as their magnetic domains are forced to switch polarity with each cycle of the AC magnetic field. These alternating stresses cause some amount of temperature rise, which leads to energy being permanently drawn away from the circuit. Another loss mechanism within inductor cores is the inducing of AC electric currents within the ferrous core material, and since ferrous metals tend to be poor conductors it means these "eddy currents" may dissipate substantial amounts of energy passing through. These inductor imperfections can me minimized – copper losses minimized by using wires of larger diameter, and iron losses minimized by making the core's cross-section large to reduce magnetic field density as well as making the core *laminated* to reduce eddy currents – but both design strategies make the inductor bulkier and more massive.

[10] One such design technique applied to *wire-wound* resistors – which use a coil of fine wire as the resistive element – is to wrap that wire coil as many turns clockwise as counter-clockwise so that it will generate two self-canceling magnetic fields when current passes through, and therefore will not possess substantial inductance.

[11] These two loss mechanisms are generally referred to as *copper losses* and *iron losses*, respectively, by electrical power engineers.

So far we have referred to energy dissipation in inductive or capacitive components as being an imperfection, with the ideal inductor or capacitor being purely reactive and completely non-resistive. There do exist, however, a great many electrical components based on the actions of electric or magnetic fields that are actually designed to be dissipative in their behavior. These components utilize electric or magnetic fields to do useful work, and the performance of work *necessitates* energy leaving a circuit (i.e. true power).

Perhaps the most obvious example of such a component is the *electric motor*. An electric motor functions on the basis of magnetic fields produced by electric currents through coils of wire, those magnetic fields attracting and repelling each other to produce *torque* (i.e. twisting force) to spin a shaft. The whole purpose of an electric motor is to do mechanical work – *to move matter*. As matter is given potential and/or kinetic energy by means of a motor, the energy invested in that matter must come from the circuit powering the motor, and that energy generally does not return to the circuit which makes the motor a dissipative element even though it appears to be a collection of inductive coils. AC motors certainly exhibit inductive reactance, but here the reactance is the imperfection because the whole purpose of a motor is to translate all the given energy into mechanical work.

An audio loudspeaker is another example of an inductive component designed to dissipate energy, this time in the form of *sound*. Like an electric motor, a loudspeaker uses coils of wire to generate magnetic fields with the passage of AC electric current, those fields working to shake a paper "cone" to impart kinetic energy to air molecules which we perceive as sound. Here again, any inductance exhibited by a loudspeaker's coil is considered an imperfection, because it is the purpose of a loudspeaker to translate all of the electrical energy it's been given into sound, without returning any of the energy back to the rest of the circuit. In this next photograph, we see a small loudspeaker of the size one might find in a personal computer:

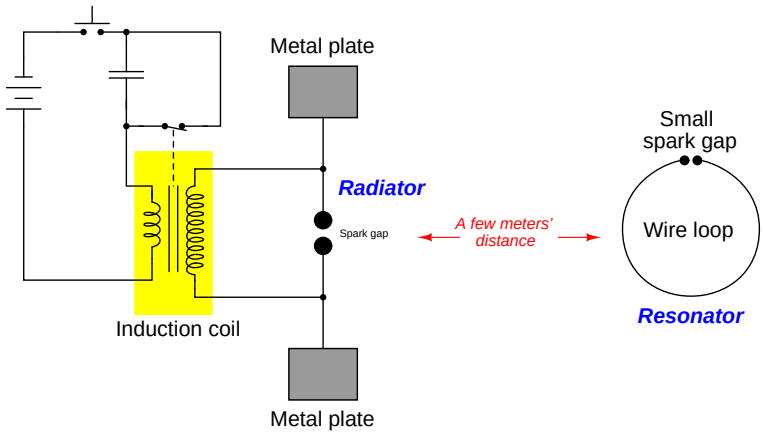
Here we see two photographs, one of an electric motor (colored black) powering a water pump (colored blue), and the other of a small speaker salvaged from an alarm-clock radio:

Another dissipative application of fields takes the form of *heating*, where we may use oscillating magnetic and/or electric fields to impart thermal energy to objects. *Induction heating* works by wrapping a coil of wire around some electrically conductive object and passing high-frequency alternating current through that coil. The oscillating magnetic field produced by the coil induces AC "eddy currents" in the conductive object, which then dissipate heat through that object's electrical resistance. *Dielectric heating* is the capacitive version of induction heating, where a non-conducting object is placed between two metal plates and subjected to an oscillating electric field by the application of a high-frequency AC voltage between those plates. In either case, the whole point of the assembly is to deliver thermal energy to some object from a circuit, with that thermal energy remaining in the object and never returning to the circuit. Thus, both inductive and dielectric heaters are dissipative devices, and any reactance they exhibit is undesired.

The following photographs show two examples of induction hearing. On the left is Dr. Harvey Rentschler of the Westinghouse Corporation in 1927 holding a glass bulb containing a sample of metal being inductively heated by the AC magnetic field of the wire coils surrounding the bulb. On the right is a section of large-diameter pipe being inductively heated by alternating current passed through loops of thick cable from an arc welding machine:

Special metal-melting furnaces called *induction furnaces* work on this same principle, heating metal to its melting temperature without the need for flame or any direct contact with any heat source. Some induction furnaces operate in a state of high vacuum so that the melted metal will not chemically react with oxygen or nitrogen in the air, the vacuum presenting no impediment to the transfer of energy from the induction coils to the melt.

Yet another example of a device using electric and magnetic fields in a dissipative fashion is a *radio transmitting antenna*. The Scottish physicist James Clerk Maxwell mathematically predicted in 1873 the existence of oscillating magnetic and electric fields that would support one another in empty space, akin to ripples in the surface of a pond, but propagating at the speed of light in all directions. Professor Heinrich Hertz demonstrated this fact in 1887 using an apparatus illustrated below. The "radiator" is the transmitting antenna and the "resonator" is the receiving antenna, a weak electric spark developing at the resonator's spark gap whenever the radiator's circuit is energized:



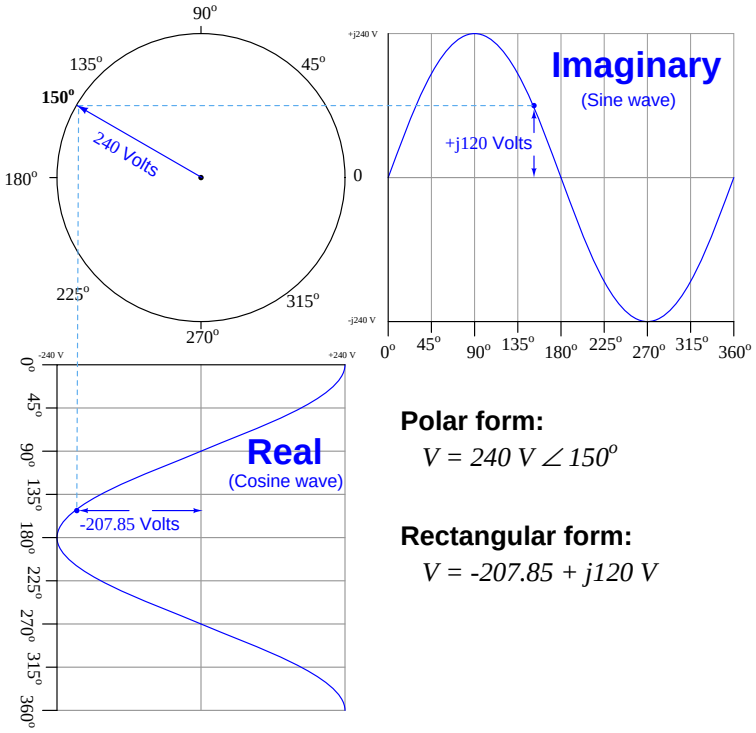
A modern microwave antenna appears in this photograph:

Close inspection of any antenna reveals nothing but short lengths of conductive metal having no moving parts. Those short lengths of metal tube comprising the antenna possess inductance (by virtue of their length) and capacitance (by virtue of their separation from each other), and these two properties allow the electromagnetic field to be generated when the antenna's elements are energized by an AC source. From this, and one might be tempted to conclude that an antenna ought to be

purely reactive, but the arrangement of those pieces facilitates radiation of electromagnetic waves in the desired direction(s) and this fact makes an antenna dissipative.

4.5 Applying phasors

After seeing how inductance and capacitance both react to the application of sinusoidal AC (Alternating Current), we may explore these concepts further by applying our knowledge of *phasors*. Sinusoidal waveforms are merely the projections of a circle's radius line as that line is rotated about the center. Thus, any sinusoidal wave may be described mathematically in terms of a radius line (called a *phasor*) rotating at a particular angular velocity. Complex numbers work well to express the dimensions of phasors: "polar form" representing the radius line length and angle, and "rectangular form" representing the respective positions on the corresponding cosine ("real") and sine ("imaginary") waves:



The equations $V = L\frac{dI}{dt}$ and $I = C\frac{dV}{dt}$ relate values of voltage and current at instantaneous moments in time, but not as complete waveforms. Phasors, however, do succinctly describe entire waves, and so if we use phasors to express voltage and current waveforms in these reactive circuits we should be able to apply these quantities in a manner similar to their application in Ohm's Law, and by doing so we may define a quantity similar to resistance but more universal in scope. If a phasor is given a fixed angle, that angle will represent the degree to which that AC quantity is *phase-shifted* from some other AC quantity in that same circuit (i.e. the amount that the waveform leads or lags another as both waveforms' phasors continually rotate).

First, let us consider the case of a purely inductive circuit. In this case, voltage leads current by 90^o. If we arbitrarily choose current as the "reference" waveform, we would say that I has a polar

angle of 0^o while V has a polar angle of 90^o. We could have just as easily chosen voltage as the reference waveform, in which case V would have a polar angle of 0^o and I would have a polar angle of -90^o (since voltage still leads current). Either way, the ratio of voltage to current, expressed in phasor form, is a quantity with a constant angle of 90^o. We will call this quantity *impedance* and use Z as its symbol.

Suppose the oscilloscope in our purely inductive circuit measured 21 Volts AC and 3 Amperes AC. The impedance of that inductance would be calculated as follows, setting current as the reference waveform for phasor angles:

$$Z_L = \frac{21 \text{ V}\angle 90^o}{3 \text{ A}\angle 0^o} = 7\,\Omega\angle 90^o$$

If we happened to set voltage as the phasor angle reference waveform, the impedance would be calculated just the same:

$$Z_L = \frac{21 \text{ V}\angle 0^o}{3 \text{ A}\angle -90^o} = 7\,\Omega\angle 90^o$$

In fact, the choice of phasor angle reference waveform is completely arbitrary because the phase shift between voltage and current for any pure inductance must be 90^o as a direct consequence of the $V = L\frac{dI}{dt}$ equation. Any quotient of two phasors 90 degrees apart (numerator leading denominator) must be 90^o. It is worthy of note that the impedance phase angle for a pure *resistance* will always be 0^o because there is no phase shift between voltage and current for a resistance.

The magnitude of this impedance (7 Ohms) is called *reactance*, and may be computed[12] as a function of frequency (f) and inductance (L):

$$X_L = 2\pi f L \qquad or \qquad X_L = \omega L$$

Where,

X_L = Reactance of the inductance (Ohms)
f = Frequency of the waveforms (Hertz, or cycles per second)
ω = Angular velocity or natural frequency of the waveforms (radians per second)
L = Inductance (Henrys)

[12] This involves some calculus, and is shown in section 5.1 on page 82.

Next, let us consider the case of a purely capacitive circuit. In this case, current leads voltage by 90^o. If we arbitrarily choose current as the "reference" waveform, we would say that I has a polar angle of 0^o while V has a polar angle of -90^o. We could have just as easily chosen voltage as the reference waveform, in which case V would have a polar angle of 0^o and I would have a polar angle of 90^o (since current still leads voltage). Either way, the ratio of voltage to current, expressed in phasor form, is a quantity with a constant angle of -90^o. We refer to this quantity as *impedance* just as we did in the inductive case, and continue to use Z as its symbol.

Suppose the oscilloscope in our purely capacitive circuit measured 18 Volts AC and 2 Amperes AC. The impedance of that capacitance would be calculated as follows, setting current as the reference waveform for phasor angles:

$$Z_C = \frac{18 \text{ V}\angle -90^o}{2 \text{ A}\angle 0^o} = 9\,\Omega\angle -90^o$$

If we happened to set voltage as the phasor angle reference waveform, the impedance would be calculated just the same:

$$Z_C = \frac{18 \text{ V}\angle 0^o}{2 \text{ A}\angle 90^o} = 9\,\Omega\angle -90^o$$

As before, the choice of phasor angle reference waveform is completely arbitrary because the phase shift between voltage and current for any pure capacitance must be -90^o as a direct consequence of the $I = C\frac{dV}{dt}$ equation. Any quotient of two phasors 90 degrees apart (numerator lagging denominator) must be -90^o.

The magnitude of this impedance (9 Ohms) is called *reactance*, and may be computed[13] as a function of frequency (f) and capacitance (C):

$$X_C = \frac{1}{2\pi f C} \qquad or \qquad X_C = \frac{1}{\omega C}$$

Where,

X_C = Reactance of the capacitance (Ohms)

f = Frequency of the waveforms (Hertz, or cycles per second)

ω = Angular velocity or natural frequency of the waveforms (radians per second)

C = Capacitance (Farads)

[13] This involves some calculus, and is shown in section 5.2 on page 85.

4.6 Converting reactance into impedance

As we have seen in the previous examples, the fundamental difference between *reactance* and *impedance* for either a pure capacitance or a pure inductance is that reactance is a *scalar* quantity (having magnitude but no direction) while impedance is a *phasor* quantity (i.e. expressed as a complex number). Thus, when you calculate reactance using either the $X_L = 2\pi fL$ formula for inductive reactance or the $X_C = \frac{1}{2\pi fC}$ formula for capacitive reactance, the result provided by your calculator is a scalar quantity and not a complex (phasor) quantity. We need to perform one more step in order to represent that reactance as an impedance, and that is to include the proper phase angle (either 90^o for an ideal inductive impedance or -90^o for an ideal capacitive impedance). How does one add a polar angle to a scalar number on an electronic calculator?

Let's work through a practical example to fully understand the problem and its solution. Suppose we are faced with computing the phasor impedance of a 0.22 μF capacitor at a frequency of 60 Hz. The first step, of course, is to calculate this capacitor's reactance value at that frequency:

$$X_C = \frac{1}{2\pi fC} = \frac{1}{(2\pi)(60 \text{ Hz})(0.22 \times 10^{-6} \text{ F})} = 12.057 \text{ k}\Omega$$

This value of 12.057 kΩ is just a reactance and not an impedance because it is not (yet) a complex number. We know that the proper complex expression of this as an impedance will be either 12.057 kΩ $\angle$ -90^o (polar form) or 0 - j12.057 kΩ (rectangular form), and so the obvious solution to the problem is to simply re-type `12.057E3` into our calculator with the phase angle or as a $-j$ quantity, and the calculator will now regard this as a complex number. However, re-typing computed values into a calculator is never a good idea: first, it invites keystroke errors; second, it unnecessarily truncates (rounds) the quantity because the calculator stores more significant digits in its memory than what it displays on the screen, which leads to unnecessary errors in later calculations using that quantity.

A simple solution for any calculator capable of performing complex-number arithmetic is to simply multiply the reactance value by a *unit phasor* having a magnitude of 1 (unity) and the desired phase angle. Thus, we could take our computed capacitive reactance value of 12.057 kΩ and multiply it either by 1 $\angle$ -90^o (polar) or multiply it by $-j$ (rectangular) and the result will be the correct impedance value as a phasor quantity, without suffering from unnecessary rounding or risking keystroke errors from having to re-enter the reactance value. If we were given an inductive reactance instead of a capacitive reactance, our unit phasor would be 1 $\angle$ 90^o (polar) or j (rectangular) to grant the computed reactance value a $+90^o$ impedance phase angle.

4.7 Natural frequency, reactance, and impedance

Most students of electronics learn the formulae $X_L = 2\pi fL$ and $X_C = \frac{1}{2\pi fC}$ when first encountering the concepts of inductive and capacitive reactance, and some of them wonder where the 2π coefficient comes from. This is actually a *conversion factor* for frequency, to convert the common unit of Hertz (i.e. cycles per second) into the more mathematically "natural" unit of *radians per second*. *Angular velocity* or *natural frequency* expressed in radians per second is always symbolized by the lower-case Greek letter omega (ω) rather than f which implies *frequency* in cycles per second (Hertz). Both, however, describe how rapidly a waveform such as an alternating voltage or an alternating current repeatedly cycles over time.

Re-writing both of these reactance formulae to use ω instead of $2\pi f$, we immediately see just how much simpler they appear:

$$X_L = \omega L \qquad\qquad X_C = \frac{1}{\omega C}$$

Extending these just one step further to represent *impedance* rather than merely reactance by including the imaginary operator $j = \sqrt{-1}$:

$$Z_L = j\omega L \qquad\qquad Z_C = \frac{1}{j\omega C} \quad \text{or} \quad Z_C = \frac{-j}{\omega C}$$

Refer to section 5.1 beginning on page 82 for inductance and section 5.2 beginning on page 85 for capacitance to see calculus-based derivations of these formulae. The simple justification for using ω as frequency is that the calculus operations of differentiation and integration are relatively easy to implement on imaginary-exponential functions (i.e. any function based on e^j), but more difficult to implement on trigonometric functions such as sine and cosine. Therefore, when starting with the "Ohm's Law" relations of voltage versus current for inductors ($V_L = L\frac{dI}{dt}$) and capacitors ($I_C = C\frac{dV}{dt}$) as a basis for deriving impedance, it makes practical sense to cast the waveform functions in imaginary-exponential terms because the ensuing calculus will be easier[14]. However, imaginary-exponential expressions of sinusoidal waveforms assume angles in radians rather than degrees (or complete cycles). Since there are exactly 2π radians in one complete cycle, one cycle per second (1 Hz) is equivalent to 2π radians per second.

Let's compare reactance calculations using f versus using ω. Applying 800 Hertz to a 47 milliHenry inductor yields 236.248 Ohms of reactance using the f-based formula:

$$X_L = 2\pi fL = 2\pi(800 \text{ Hz})(0.047 \text{ H}) = 236.248\,\Omega$$

If we convert 800 Hertz into radians per second ($\omega = 2\pi f = 5026.548$ radians per second) we arrive at the exact same 236.248 Ohms of inductive reactance using the ω-based formula:

$$X_L = \omega L = (5026.548 \text{ rad/s})(0.047 \text{ H}) = 236.248\,\Omega$$

[14] For example, the current through a capacitor energized by a voltage $e^{j\omega t}$ is C $\frac{d}{dt}\left(e^{j\omega t}\right)$ which is equal to $j\omega Ce^{j\omega t}$. Impedance is the ratio of voltage to current ($Z = \frac{V}{I}$), and so dividing that voltage ($e^{j\omega t}$) by the capacitor's current ($j\omega Ce^{j\omega t}$) simply results in $Z_C = \frac{1}{j\omega C}$.

Here we have a table comparing common mathematical expressions of electrical quantities, f-based versus ω-based:

Concept	**f-based**	**ω-based**
Inductive reactance	$X_L = 2\pi f L$	$X_L = \omega L$
Capacitive reactance	$X_C = \frac{1}{2\pi f C}$	$X_C = \frac{1}{\omega C}$
Inductive impedance	$Z_L = j2\pi f L$	$Z_L = j\omega L$
Capacitive impedance	$Z_C = \frac{1}{j2\pi f C}$	$Z_C = \frac{1}{j\omega C}$
Resonance	$f = \frac{1}{2\pi\sqrt{LC}}$	$\omega = \frac{1}{\sqrt{LC}}$

Clearly, formulae written in terms of ω instead of f are more compact and therefore preferred whenever the impedance of AC networks containing many inductors, capacitors, and/or resistors must be expressed in concise mathematical form.

One final note regarding capacitive impedance and the sign of j. While it is true that $Z_C = \frac{-j}{\omega C}$, it is equally true that $Z_C = \frac{1}{j\omega C}$. Showing the $-j$ term makes it more clear that the phase angle of a pure capacitance's impedance will be -90^o, but placing the j term in the denominator actually means the same thing. The following steps prove this mathematical equivalence:

$$Z_C = \frac{1}{j\omega C}$$

$$Z_C = \left(\frac{j}{j}\right)\left(\frac{1}{j\omega C}\right)$$

$$Z_C = \frac{j}{j^2\omega C}$$

$$Z_C = \frac{j}{(-1)\omega C}$$

$$Z_C = \frac{-j}{\omega C} \quad \text{or} \quad Z_C = -j\frac{1}{\omega C}$$

When writing complicated formulae describing networks containing capacitance it is customary to leave j as a positive quantity in the denominator, if for no other reason than to avoid inserting mathematically-unnecessary negative signs which make the resulting formulae easier to manipulate and solve. However, it is good to know that doing so still leaves the capacitive impedance with a negative phase angle even if that is not immediately apparent by inspection.

As we have seen, expressing the reactance and/or impedance of capacitors and inductors in terms of natural frequency (ω) is mathematically elegant, as the ω-based expressions have no need for the 2π conversion factor necessary to convert cycles per second (Hz) of conventional frequency into radians per second of natural frequency.

4.8 Complex frequency, reactance, and impedance

Natural frequency (ω) expressed in units of radians per second may seem unnecessarily complicated when ordinary frequency (f) expressed in units of cycles per second (Hz) seems most intuitive, despite reactance formulae containing fewer symbols using ω instead of f. A strong mathematical justification for ω, however, is its utility when used with imaginary-exponential functions. To understand why, we must review a very important and foundational equation in mathematics known as Euler's Relation:

$$e^{jx} = \cos x + j \sin x$$

Where,
e = Euler's number (approximately equal to 2.718281828)
j = The "imaginary operator" equal to $\sqrt{-1}$
x = Angle expressed in radians

This Relation shows the fundamental equivalence between two seemingly unrelated classes of functions – exponential functions and trigonometric functions. However, it is true only when the angle in question x is cast in units of radians. If we wish to describe the evolution of a sinusoidal wave in terms of frequency and time, we must use natural frequency in units of radians per second, multiplied ω by time (t, in seconds):

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

Where,
ω = Natural frequency in radians per second
t = Time in seconds

Exponential functions (e^x) are also useful for describing things other than cosine and sine waves. Notably, they are also used to describe natural growth and decay processes, using *real* numbers rather than imaginary numbers in the exponential term:

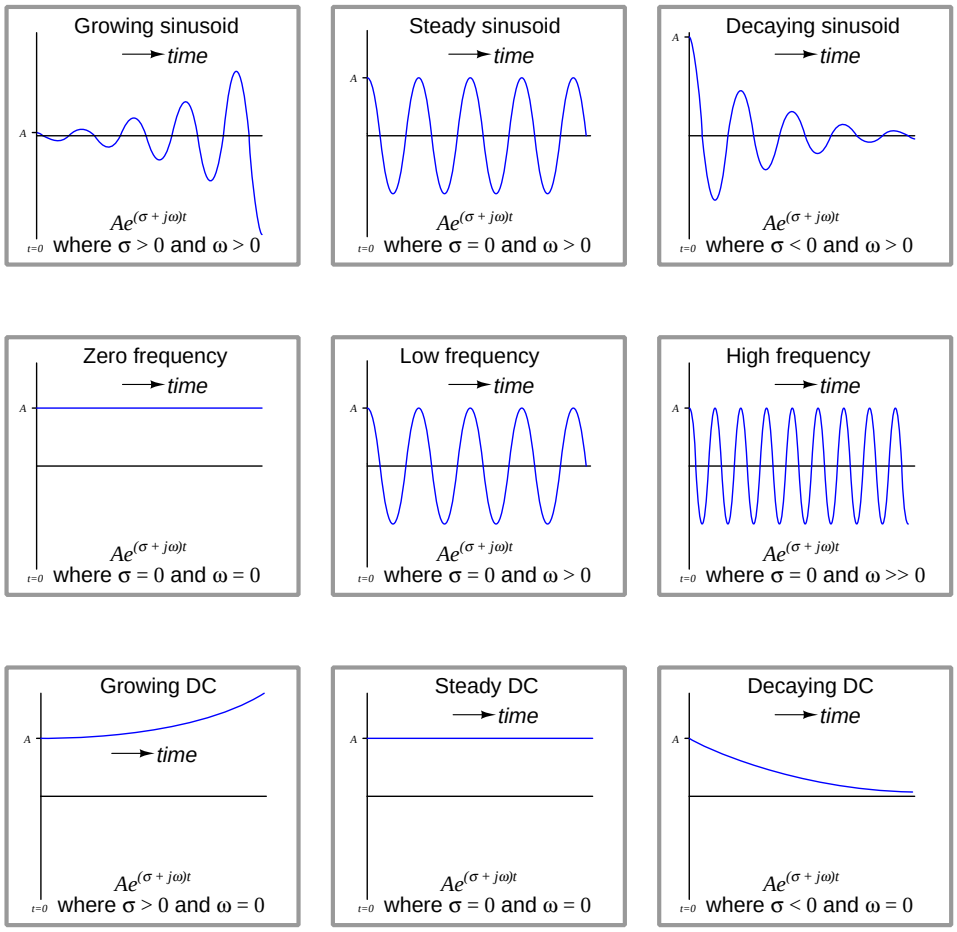
$$e^{\sigma t}$$

Where,
σ = Growth/decay rate in time constants per second ($\frac{1}{\text{sec}}$), also known as *Neper velocity*
t = Time in seconds

As every student of electronics learns, the decay of DC voltage and/or current for a de-energizing capacitor or inductor follows $e^{-\frac{t}{\tau}}$ where t is time and τ is the *time-constant* of the circuit, both measured in seconds. If we look at this familiar formula closely we will see that τ is really nothing more than the reciprocal of the growth/decay rate labeled by the lower-case Greek letter "sigma" (σ) shown above. In other words, $\tau = \frac{1}{\sigma}$ which makes $e^{\sigma t}$ equivalent to $e^{\frac{t}{\tau}}$. In fact, we may legitimately apply the unit of *time constants per second* to this rate. Thus, $e^{\sigma t}$ more generally describes anything growing exponentially in magnitude over time if σ is positive in value, or decaying toward zero over time if σ is negative. A zero value for σ means the quantity is stable over time, neither growing nor decaying.

If σ as a real number describes growth/decay rate in "time-constants per second" units (Neper velocity) and ω as an imaginary number ($j\omega$) describes the natural frequency (or angular velocity) of a sinusoid in "radians per second" units, we may combine these two values into a single complex-number sum called *complex frequency* which is able to describe a growing or decaying or stable sinusoid. By convention we use the variable s to represent this complex frequency ($s = \sigma + j\omega$), with $e^{(\sigma+j\omega)t}$ or simply e^{st} being the time-based function for that growing/decaying sinusoid.

Some time-domain graphs of sinusoids help illustrate the meaning of both σ and ω. Note the inclusion of the coefficient A describing the amplitude of the waveform at the starting time ($t = 0$):



Simple frequency, whether f or ω, merely describes the rate at which an AC signal oscillates. Complex frequency, by contrast, is additionally able to describe growth, decay, or stability. As such it is a more general and indeed more useful way to describe signals.

A practical example of a decaying sinusoid is a passive *tank circuit* gradually losing energy over time after being momentarily excited by an external energy source. A practical example of a growing sinusoid is an *oscillator* circuit starting up from an unpowered condition. The decaying DC graph

illustrates the behavior of a de-energizing inductor or capacitor network as mentioned previously, while a practical example of an exponentially growing DC signal is current through a bipolar junction transistor experiencing *thermal runaway*.

Calculations of impedance also benefit from the use of complex frequency. Instead of simply computing the number of Ohms and the phase angle of impedance for an inductor or capacitor, we may now additionally compute the number of Ohms offered by that component to the exponential growth or decay of that waveform.

$$Z_L = (\sigma + j\omega)L \qquad Z_C = \frac{1}{(\sigma + j\omega)C}$$

Again, the complex variable s is used as a convenient way to represent $\sigma + j\omega$, making the previous expressions of impedance extremely simple:

$$Z_L = sL \qquad Z_C = \frac{1}{sC}$$

ω-based expressions of impedance are now seen as *special cases* of s-based impedances where $\sigma = 0$, as proven by the following algebraic substitutions where we expand s into $\sigma + j\omega$ and then set σ equal to zero. These special cases assume AC signals of constant magnitude that neither grow nor decay over time:

$$Z_L = sL \qquad Z_C = \frac{1}{sC}$$

$$Z_L = (\sigma + j\omega)L \qquad Z_C = \frac{1}{(\sigma + j\omega)C}$$

$$Z_L = (0 + j\omega)L \qquad Z_C = \frac{1}{(0 + j\omega)C}$$

$$Z_L = j\omega L \qquad Z_C = \frac{1}{j\omega C}$$

Numerical examples serve well to clarify abstract concepts, so let's apply some complex frequency values to realistic inductor and capacitor values.

First, let's consider a 150 Hz signal applied to a 22 milliHenry inductor. This is a natural frequency of 942.48 radians per second[15]. If this signal is stable in magnitude, its σ value will be zero, and so the impedance offered by this inductor will be:

$$Z_L = sL = (\sigma + j\omega)L$$

$$Z_L = (0 + j942.48 \text{ s}^{-1})(0.022 \text{ H})$$

$$Z_L = 0 + j20.735\,\Omega = 20.735\,\Omega\angle 90^o$$

Next, let's make this 150 Hz signal *grow* exponentially at a rate of 1000 time constants per second. Re-calculating the inductor's impedance:

$$Z_L = (1000 + j942.48 \text{ s}^{-1})(0.022 \text{ H})$$

$$Z_L = 22 + j20.735\,\Omega = 30.231\,\Omega\angle 43.304^o$$

Our first calculation with a steady 150 Hz signal yielded an impedance of 20.735 Ohms at the customary phase angle of +90 degrees for an inductor. However, our second calculation using the exponentially growing 150 Hz signal gave us an impedance phase angle of only 43.304 degrees. Why is this the case? To answer this question we must recall the distinction between real and imaginary impedance values: real impedance values represent a net exchange of energy into or out of the circuit while imaginary impedance values represent energy alternately absorbed and released in equal measure. In a "steady-state AC" condition the inductor releases precisely as much energy as it earlier absorbed, which makes its impedance purely imaginary. However, when energized by a *growing* AC stimulus each successive cycle of that waveform is stronger than the last, which means the inductor is always absorbing more energy than it released at the close of the previous cycle. This means the inductor exhibits a net absorption of energy over time as the waveform grows stronger and stronger. Unlike a resistor which dissipates energy into heat where it never returns to the circuit, here the inductor's net intake of energy will eventually result in a release of that accumulated energy if and when the waveform decays in value.

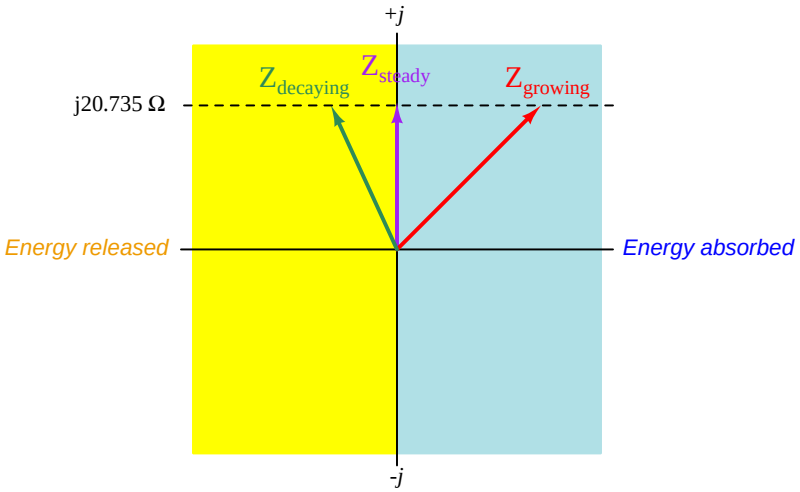
If we were to take this inductor and energize it instead with a 150 Hz signal *decaying* at a rate of −440 time constants per second, we would see its impedance phase angle surpass 90 degrees:

$$Z_L = (-440 + j942.48 \text{ s}^{-1})(0.022 \text{ H})$$

$$Z_L = -9.680 + j20.735\,\Omega = 22.883\,\Omega\angle 115.03^o$$

[15] A common way of expressing "per second" as a unit of measurement is to abbreviate "seconds" using the letter "s" and then apply a −1 power to that unit abbreviation. Therefore, 942.48 radians per second is typically written as 942.48 s^{-1}. Please note that the letter "s" here is a symbol for the unit *seconds*, not to be confused with the variable s which is complex frequency. It is customary to use different fonts to distinguish variables from unit abbreviations. Here, the variable is "s" while the unit is "s".

Plotting these three inductive impedances on a phasor diagram divided into "Energy absorbed" and "Energy released" halves, we see the effect of the signal's growth and decay rate on the inductor's impedance:



In all three cases the imaginary portion of the impedance is the same ($j20.735$ Ohms), but the growth or decay of the signal over time contributes a net absorption or net release, respectively, of energy by the inductor over multiple cycles.

Continuing our quantitative exploration of complex frequency, let us now consider a capacitor subjected to a similar treatment of stable, growing, and decaying signals. For the sake of illustration, let's assume a frequency of 2.3 kHz (14451.3 radians per second) and a capacitance of 0.01 microFarads.

First, the stable signal neither growing nor decaying over time:

$$Z_C = \frac{1}{sC} = \frac{1}{(\sigma + j\omega)C}$$

$$Z_C = \frac{1}{(0 + j14451.3 \text{ s}^{-1})(0.01 \times 10^{-6} \text{ F})}$$

$$Z_C = 0 - j6919.8\,\Omega = 6919.8\,\Omega \angle -90^o$$

Next, calculating capacitive impedance for a signal growing at a rate of 13000 time constants per second at the same frequency of 2.3 kHz:

$$Z_C = \frac{1}{(13000 + j14451.3 \text{ s}^{-1})(0.01 \times 10^{-6} \text{ F})}$$

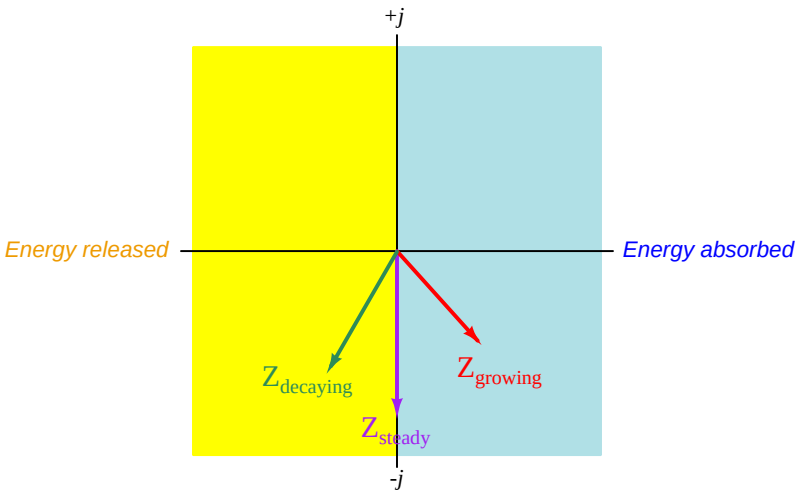
$$Z_C = 3440.6 - j3824.7\,\Omega = 5144.5\,\Omega \angle -48.026^o$$

Lastly, calculating capacitive impedance for a signal decaying at a rate of -9000 time constants per second at the same frequency of 2.3 kHz:

$$Z_C = \frac{1}{(-9000 + j14451.3 \text{ s}^{-1})(0.01 \times 10^{-6} \text{ F})}$$

$$Z_C = -3105.2 - j4986.0\,\Omega = 5873.8\,\Omega \angle -121.91^o$$

Plotting these three capacitive impedances on a phasor diagram divided into "Energy absorbed" and "Energy released" halves:



Once again we see how an AC signal of constant magnitude over time results in an impedance phase angle precisely one-quarter cycle (i.e. -90^o), which means the component absorbs and releases energy in equal measure with each cycle. However, when the AC signal's oscillations grow in strength over time the reactive component absorbs more energy than it releases over multiple cycles; when the oscillations decay the component releases more energy than it absorbs over multiple cycles.

4.9 Concept review

So far this tutorial has covered a lot of ground, and it is a good idea to stop here and review what we've learned so far:

- Resistance (R) always functions as a load in a circuit, converting the potential energy of charge carriers passing through into heat. This energy leaves the circuit never to return, and its rate of dissipation over time is called *true power*.
- Both inductance (L) and capacitance (C) *react* to changing quantities of current and voltage, respectively. When energized by AC, they *exchange* energy (alternately absorbing and releasing energy), while dissipating none. The rate over time at which energy exchanges in this manner is called *reactive power*.
- Voltage drops or rises across an inductance in proportion to the rate that current through it varies over time: $V = L\frac{dI}{dt}$. This mathematical relationship results in a pure inductance's voltage leading its current by 90^o when energized by sinusoidal AC.
- Current flows through a capacitance in proportion to the rate that voltage across it varies over time: $I = C\frac{dV}{dt}$. This mathematical relationship results in a pure capacitance's current leading its voltage by 90^o when energized by sinusoidal AC.
- No inductor or capacitor is purely reactive, but capacitors tend to be purer reactances than inductors.
- Some components such as electric motors and radio transmitting antennae exploit magnetic and/or electric fields, but are nevertheless dissipative rather than reactive because the energy invested in those fields leaves the circuit to do useful work.
- Impedance (Z) is the phasor ratio of voltage to current ($Z = \frac{V}{I}$), measured in Ohms (Ω). Inductive impedance (Z_L) always has a polar angle of 90^o while capacitive impedance (Z_C) always has a polar angle of -90^o. Resistive impedance (Z_R) always has a polar angle of 0^o because voltage and current are in-phase for a resistance.
- Reactance (X) is the magnitude of a pure inductance's or capacitance's opposition to current. It is measured in Ohms just like resistance, the difference being reactance only exchanges energy back and forth while resistance dissipates energy.
- Inductive reactance is a function of inductance and frequency: $X_L = 2\pi fL$
- Capacitive reactance is a function of capacitance and frequency: $X_C = \frac{1}{2\pi fC}$
- Any reactance value may be converted into a phasor impedance value by multiplying by a *unit phasor*: $1 \angle 90^o$ for inductive impedance and $1 \angle -90^o$ for a capacitive impedance. This operation will attach the proper phase angle to the reactance value to make it a complex-number *phasor* impedance.

It is also a good idea at this juncture to pose the question: "Why does any of this matter to us?" The answer to this very valid question is quite simple: *if we express all resistances and reactances as phasor impedance (Z) values, and all voltages and currents in phasor form as well, we may analyze any sinusoidal AC circuit as readily as a similar DC circuit.* With the availability of electronic calculators[16] capable of performing complex-number arithmetic as easily as real-number arithmetic, AC circuit analysis becomes very straightforward.

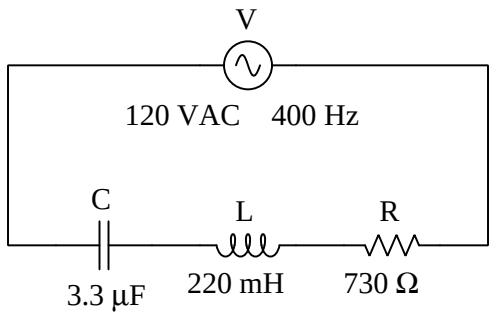
Throughout the rest of this tutorial we will gain a more thorough understanding of these principles through application to example circuits. Over and over again we will follow these rules: *express all passive component values as impedances*, *regard all impedances in the same manner we would regard resistances in a DC circuit*, *apply Ohm's Law and Kirchhoff's Laws as we would in a DC circuit.* The only difference between DC circuit analysis and AC circuit analysis for us is the use of complex numbers rather than real numbers: all impedances must be expressed in complex (phasor) form, as well as all voltages and all currents. All computations using Ohm's and Kirchhoff's Laws must be done with complex numbers.

As with all textbook examples, a good reading strategy is to *follow along with the example*, performing each and every computation as you read the text and then returning to the text to check that your numerical results agree. This ensures you can properly use your calculator, and that you correctly understand the steps being explained. The same reading strategy applies to non-mathematical explanations as well: when the text describes a thought experiment, try imagining that thought experiment on your own, then return to the text to see if your result agrees. Technical reading is very different from recreational reading: it is "hands-on" in that the text describes some real thing (or at least a simulated thing) that you can experience for yourself. Avail yourself of this capability, and be an *active* reader of the text!

[16] As an aid to performing these computations it is strongly advised that you obtain a scientific calculator capable of performing complex-number arithmetic. The alternative is ugly. Complex numbers are best added and subtracted in *rectangular* form, and best multiplied and divided in *polar* form. Since circuit analysis calculations involve all these arithmetic operations, performing complex-number arithmetic without the help of a calculator designed to do that task involves many trigonometric conversions back and forth between rectangular and polar forms, a tedious and error-prone task to say the least. I speak from years of experience when I say that such analysis of even simple series-parallel AC circuits can necessitate pages of written notes, and a far greater number of calculations than any comparable DC circuit would deserve. If you truly desire to perfect this skill, and/or your instructor demands you perform AC circuit calculations using nothing more than a trigonometric calculator, I still recommend obtaining a calculator capable of managing this tedium if only to relieve your mind of these computational details so that you may quickly gain proficiency in the conceptual analysis of AC circuits. Then, when necessary, abandon the helpful calculator and perform the arithmetic "the hard way" *after* you have familiarized yourself with the application of fundamental laws and principles to AC circuit analysis. At least this way, you will divide the problem of learning AC circuits into two smaller sub-problems: (1) mastering the principles, and (2) mastering the arithmetic. I have encountered far too many students of electricity and electronics who know #2 (the math) but never learned #1 (how and why to consistently apply it).

4.10 Example circuit #1

Our first example circuit will be a circuit consisting of a capacitor, inductor, and resistor connected in series. The source is 120 VAC[17] at a frequency of 400 Hertz:



The very first thing we should do is calculate the impedance of the three passive components. The resistor's impedance is the simplest to determine: $Z_R = 730\Omega \angle 0^o$, being the given resistance value in Ohms with a phase angle of zero[18]. In order to calculate impedances for the other two components, we must first calculate their reactances from their given values and the circuit's frequency:

$$X_C = \frac{1}{2\pi fC} = \frac{1}{(2\pi)(400 \text{ Hz})(3.3\,\mu\text{F})} = 120.57\,\Omega$$

$$X_L = 2\pi fL = (2\pi)(400 \text{ Hz})(220\,\text{mH}) = 552.92\,\Omega$$

Please note that these two reactance values merely express the degrees to which these components oppose the passage of charge carriers by means of storing and releasing energy. In order to be fully useful to us in our analysis, we need to express them as *complex* phasor values, which means attaching phase angles to them. Since we know purely inductive impedance always has a phase angle of 90^o and purely capacitive impedance always has a phase angle of -90^o, our next step is as simple as attaching[19] these angle values to their respective reactance values:

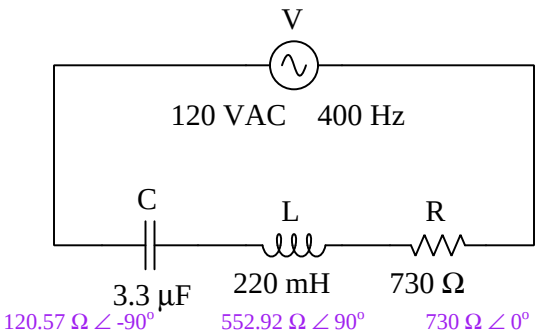
$$Z_C = 120.57\,\Omega \angle -90^o \qquad Z_L = 552.92\,\Omega \angle 90^o$$

[17] Presumably, this is an RMS value. Whatever qualifier it may be (e.g. peak, peak-to-peak, or RMS), all subsequent results will bear the same qualifier. It is customary in engineering to use RMS values unless otherwise specified, so we will assume this is 120 Volts RMS, which means all calculated values of voltage and current in this circuit will also be Volts RMS.

[18] When entering this phasor value into an electronic calculator capable of handling complex numbers, there is actually no need to enter the zero degree angle, since a polar complex number with an angle of zero degrees is entirely real with no "imaginary" component.

[19] Performing this "attachment" in a calculator capable of handling complex-number arithmetic is as simple as *multiplying* each of the computed reactance values by a *unit phasor* having a polar magnitude of 1 and the required angle. For example, $120.57 \times (1 \angle -90^o) = 120.57 \angle -90^o$. Once this complex value is shown on the calculator's display, *store it in one of the calculator's memory locations for future retrieval.* I recommend against *typing* $120.57 \angle -90^o$ because this introduces an unnecessary rounding error, as well as invites keystroke errors. Immediately manipulating the reactance value into a complex impedance value and then storing it in memory is a great labor-saving and error-avoiding strategy!

Just to track our progress and clarify the meaning of our work thus far, I will document this new information in the circuit diagram:



If this were a DC circuit, our next steps would be obvious: add all the resistance values together (because series resistances add to make the total resistance), then divide that total resistance into the total source voltage to solve for current (applying Ohm's Law), then multiply that shared current value (since series components experience equal current) by each of the resistances individually to compute voltage drops (applying Ohm's Law). Phasor representation of impedances, voltages, and currents makes this AC analysis just that simple. The only difference is, we perform *all* calculations using complex numbers instead of using regular ("real") numbers.

Finding total impedance by applying the principle that series impedances add:

$$Z_T = Z_C + Z_L + Z_R = (120.57\,\Omega \angle -90^o) + (552.92\,\Omega \angle 90^o) + (730\,\Omega \angle 0^o)$$

$$Z_T = 848.43\,\Omega \angle 30.64^o$$

Note the total impedance's phase angle: being neither 0^o nor 90^o nor -90^o means this circuit's power will be partially dissipative and partially exchanged (i.e. part resistive and part reactive). This stands to reason, as only the resistor dissipates energy, while the capacitor and inductor alternately absorb and release energy.

Next, finding the circuit's current by applying Ohm's Law using total voltage and total impedance. For this we must arbitrarily choose a phase angle for the source, since it is the only source in the circuit. By convention we will set the source's phase angle at 0^o (i.e. we will use the circuit's source voltage as the *phase reference* for our calculations):

$$I = \frac{V}{Z} = \frac{120\text{ VAC}\angle 0^o}{848.43\,\Omega \angle 30.64^o} = 141.44\text{ mA} \angle -30.64^o$$

Note the -30.64^o angle for this circuit's current phasor. This means current is phase-shifted 30.64^o *lagging* behind the source voltage at 0^o.

Lastly, finding individual component voltages by applying Ohm's Law using total current and individual impedances:

$$V_C = IZ_C = (141.44 \text{ mA} \angle -30.64^o)(120.57\,\Omega \angle -90^o) = 17.05 \text{ V} \angle -120.64^o$$

$$V_L = IZ_L = (141.44 \text{ mA} \angle -30.64^o)(552.92\,\Omega \angle 90^o) = 78.20 \text{ V} \angle 59.36^o$$

$$V_R = IZ_R = (141.44 \text{ mA} \angle -30.64^o)(730\,\Omega \angle 0^o) = 103.25 \text{ V} \angle -30.64^o$$

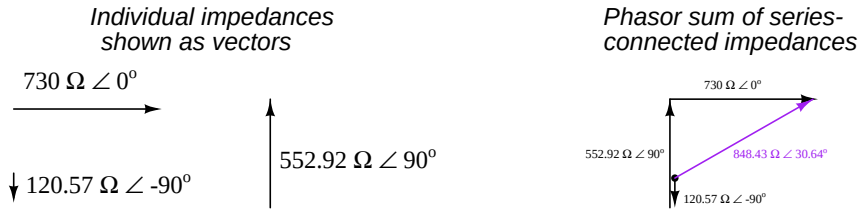
As a final check of our work, we may add all three component voltages together to see if the sum indeed is equal to the source voltage (applying Kirchhoff's Voltage Law):

$$V_T = V_C + V_L + V_R$$

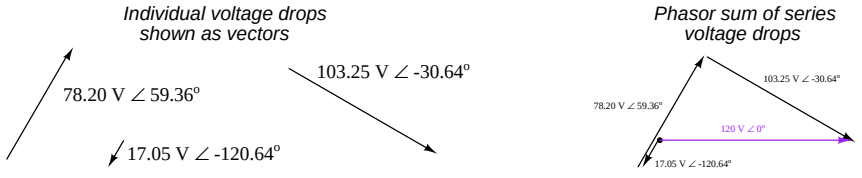
$$120 \text{ V} \angle 0^o = (17.05 \text{ V} \angle -120.64^o) + (78.20 \text{ V} \angle 59.36^o) + (103.25 \text{ V} \angle -30.64^o)$$

Please note how the results are all fairly non-intuitive. It may not seem like 120.57 Ohms and 552.92 Ohms and 730 Ohms should sum to a total of 848.43 Ohms, but they do because these are phasors with their own *directions* as well as magnitudes. Likewise, it may not seem correct that 17.05 Volts and 78.20 Volts and 103.25 Volts add up to 120 Volts, but they do because these three voltages are *out of phase* with each other. We simply aren't accustomed to mentally summing *complex* numbers, and so these sums appear quite strange. However, they are all mathematically correct. Intuition is better served when we represent these complex-number values in the form of *phasor diagrams*, where addition is equivalent to graphically stacking phasors end-to-end:

Phasor diagram of series impedances

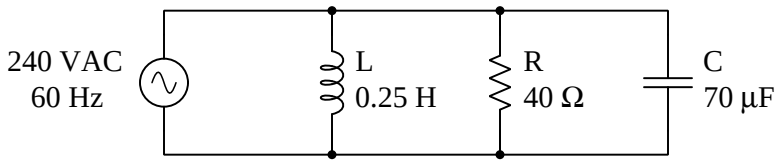


Phasor diagram of series voltage drops



4.11 Example circuit #2

Our second example circuit consists of a capacitor, inductor, and resistor connected in parallel. The source is 240 VAC at a frequency of 60 Hertz:



We will follow the same general procedure as last time: convert all passive component values into phasor impedances, then apply the same principles of parallel circuits, Ohm's Law, and Kirchhoff's Laws we would use if this were a DC circuit.

First, computing impedance values for the resistor, inductor, and capacitor. For the resistor this is as simple as attaching a 0 degree angle to the given value in Ohms. For the inductor and capacitor we must use the respective reactance formula to calculate magnitude and then attach the correct phase angle:

$$Z_R = 40\,\Omega\,\angle 0^o$$

$$Z_L = 2\pi f L\,\angle 90^o = (2\pi)(60\text{ Hz})(0.25\text{ H})\,\angle 90^o = 94.248\,\Omega\,\angle 90^o$$

$$Z_C = \frac{1}{2\pi f C}\,\angle -90^o = \frac{1}{(2\pi)(60\text{ Hz})(70\,\mu\text{F})}\angle -90^o = 37.894\,\Omega\,\angle -90^o$$

Next, finding total impedance by applying the principle that parallel impedances diminish:

$$Z_T = \frac{1}{\frac{1}{Z_C} + \frac{1}{Z_L} + \frac{1}{Z_R}}$$

$$Z_T = \frac{1}{\frac{1}{37.894\,\Omega\,\angle -90^o} + \frac{1}{94.248\,\Omega\,\angle 90^o} + \frac{1}{40\,\Omega\,\angle 0^o}}$$

$$Z_T = 33.826\,\Omega\,\angle -32.259^o$$

Total current is simply the quotient of source voltage over total impedance, as per Ohm's Law. We will assume a source voltage phase angle of 0 degrees unless otherwise stated, using that voltage as the circuit's "phase reference" for our calculations:

$$I_T = \frac{V}{Z_T} = \frac{240\text{ VAC}\,\angle 0^o}{33.826\,\Omega\,\angle -32.259^o} = 7.0952\text{ A}\,\angle 32.259^o$$

Lastly, finding individual component (branch) currents using Ohm's Law, based on the knowledge that these are connected in parallel with the source and that parallel-connected components share the same voltage:

$$I_L = \frac{V}{Z_L} = \frac{240 \text{ VAC} \angle 0^o}{94.248\,\Omega \angle 90^o} = 2.5465 \text{ A} \angle -90^o$$

$$I_C = \frac{V}{Z_C} = \frac{240 \text{ VAC} \angle 0^o}{37.894\,\Omega \angle -90^o} = 6.3335 \text{ A} \angle 90^o$$

$$I_R = \frac{V}{Z_R} = \frac{240 \text{ VAC} \angle 0^o}{40\,\Omega \angle 0^o} = 6.000 \text{ A} \angle 0^o$$

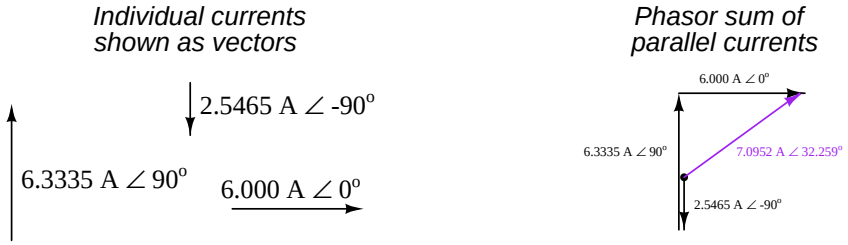
A good way to check our work is to verify the phasor sum of all parallel branch currents equals the total current, based on the principle that parallel branch currents add to equal total current:

$$I_T = I_L + I_C + I_R$$

$$7.0952 \text{ A} \angle 32.259^o = (2.5465 \text{ A} \angle -90^o) + (6.3335 \text{ A} \angle 90^o) + (6.000 \text{ A} \angle 0^o)$$

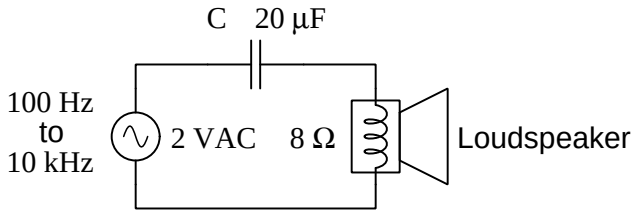
As with example circuit #1, the results are fairly non-intuitive. It may not seem like 2.5465 Amperes and 6.3335 Amperes and 6.000 Amperes should sum to a total of 7.0952 Amperes, but they do because these are phasors with their own *directions* as well as magnitudes.

Phasor diagram of parallel currents



4.12 Example circuit #3

In this third example we will apply the concepts of resistance, reactance, and impedance to a practical AC circuit called an *audio filter*. The purpose of a "filter" circuit is to attenuate (diminish) signals of certain frequency. For example, a *high pass filter* circuit allows signals having high frequency values to "pass through" with minimum attenuation, while significantly attenuating signals having low frequency values. High pass filters are used as the basis for "treble" adjustments in audio systems, allowing high-pitched sounds to be heard more than low-pitched sounds. A very simple form of high pass filter used to screen out low frequency sounds from reaching a loudspeaker is seen in the following schematic diagram:



The loudspeaker consists of a moving wire coil, a stationary permanent magnet, and a paper cone attached to the coil. Alternating current passing through the coil causes it to thrust to and fro, driving the paper cone in such a way as to create sound waves in the air. Despite the fact that the electrical element of a loudspeaker is a *coil*, a loudspeaker behaves more like a resistor than an inductor. The reason for this is the fact that work is required to shake air molecules, this work constitutes energy leaving the electrical circuit in the form of sound just as a resistor causes energy to leave a circuit in the form of heat. A well-designed loudspeaker translates nearly all of its energy into sound, and for this reason we will model the loudspeaker as an 8 Ω *resistor* and not an 8 Ω *inductor*.

Before we apply any arithmetic to this problem, let us *qualitatively* analyze this circuit's operation. The impedance offered by the loudspeaker (as a dissipative component) should remain constant at 8 Ω (at an angle of 0 degrees). The capacitor's impedance magnitude, however, will vary with frequency according to the capacitive reactance formula $X_C = \frac{1}{2\pi f C}$. As frequency increases, capacitive reactance decreases, and vice-versa. If we consider two extremes[20] of frequency (zero and infinite), we may readily perceive why this is circuit functions as a high pass filter. At zero frequency (DC), the capacitor's impedance becomes infinite which means it *blocks* all signal from reaching the loudspeaker in the same manner that an *open* fault would block the signal. At infinite frequency, the capacitor's impedance diminishes to zero which means it *passes* the signal in the same manner that a direct *short* would pass the signal. With zero output at zero Hertz and full output at infinite Hertz, this circuit clearly acts to block low frequencies and pass high frequencies.

Now that we have confirmed the frequency-filtering behavior of this simple circuit, we shall proceed with quantitative analysis. The diagram shows us a source whose signal frequency varies

[20] I like to refer to this analytical strategy as *limiting cases*, which means we consider cases that are as extreme as possible. For frequency, nothing can be less than zero and nothing can be greater than infinity. Considering these two extremes on the frequency spectrum often causes the circuit being analyzed to simplify. In this case we see the capacitor being reduced to either an open fault or a shorted fault.

from 100 Hertz to 10,000 Hertz (a factor of 100 to 1, or 2 orders of magnitude[21]). A sensible analysis over this large of a frequency span would be to calculate output (loudspeaker) voltage values at order-of-magnitude intervals: at 100 Hertz, at 1,000 Hertz, and at 10,000 Hertz.

Capacitive impedance is the only component value that will vary as a function of frequency, so we will calculate Z_C at these three frequencies:

$$Z_C \text{ at } 100 \text{ Hz } = \frac{1}{(2\pi)(100 \text{ Hz})(20\,\mu\text{F})} \angle -90^o = 79.577\,\Omega \angle -90^o$$

$$Z_C \text{ at } 1,000 \text{ Hz } = \frac{1}{(2\pi)(1000 \text{ Hz})(20\,\mu\text{F})} \angle -90^o = 7.9577\,\Omega \angle -90^o$$

$$Z_C \text{ at } 10,000 \text{ Hz } = \frac{1}{(2\pi)(10000 \text{ Hz})(20\,\mu\text{F})} \angle -90^o = 0.79577\,\Omega \angle -90^o$$

A simple way for us to calculate loudspeaker voltage is to use an AC version of our familiar (DC) voltage divider formula, where output voltage is equal to input voltage multiplied by the ratio of load impedance to total series impedance. In our case the load is the 8 Ω (resistive) loudspeaker, and the total impedance will be the phasor sum of that loudspeaker resistance plus the capacitor's impedance. The source voltage is a constant 2 Volts:

$$V_{load} = V_{source} \left(\frac{Z_{load}}{Z_{load} + Z_C} \right)$$

$$V_{load} \text{ at } 100 \text{ Hz} = 2 \text{ V} \left(\frac{8\,\Omega \angle 0^o}{(8\,\Omega \angle 0^o) + (79.577\,\Omega \angle -90^o)} \right) = 0.20005 \text{ V} \angle 84.259^o$$

$$V_{load} \text{ at } 1,000 \text{ Hz} = 2 \text{ V} \left(\frac{8\,\Omega \angle 0^o}{(8\,\Omega \angle 0^o) + (7.9577\,\Omega \angle -90^o)} \right) = 1.41795 \text{ V} \angle 44.848^o$$

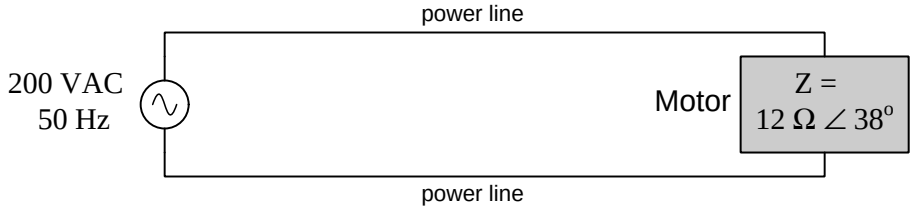
$$V_{load} \text{ at } 10,000 \text{ Hz} = 2 \text{ V} \left(\frac{8\,\Omega \angle 0^o}{(8\,\Omega \angle 0^o) + (0.79577\,\Omega \angle -90^o)} \right) = 1.99018 \text{ V} \angle 5.6806^o$$

From these results we see two trends evident as signal frequency increases: the loudspeaker's driving voltage increases toward the source voltage value (2 Volts), and the phase shift decreases toward zero. If we imagine signal frequency increasing toward infinity, we would expect the load voltage to equal the source voltage (2 Volts) while the phase angle equals zero degrees: as if we had replaced the capacitor by a short length of wire. Phase shift is not particularly important for a simple audio speaker system such as this, but the increased driving voltage tells us the loudspeaker will become louder as pitch (frequency) increases, which is precisely what a high pass (treble) filter should do.

[21] An "order of magnitude" refers to a power of ten. For example, a ten-fold ratio is a single order of magnitude because $10 = 10^1$. Likewise, a hundred-fold ratio represents two orders of magnitude because $100 = 10^2$. Similarly, a thousand-fold ratio represents three orders of magnitude ($1000 = 10^3$), and so on.

4.13 Example circuit #4

In this fourth example we will apply the concepts of resistance, reactance, and impedance to another practical AC circuit, the purpose of this one being to minimize the amount of current passing through power lines. Consider the following circuit consisting of an AC voltage source, long length of power lines, and an electric motor with its impedance expressed in phasor form:



Calculating the amount of current in these two power lines is a simple matter of applying Ohm's Law. Once again we will assume a zero degree phase angle for the voltage source because there is no other waveform given as a phase angle reference:

$$I = \frac{V}{Z} = \frac{200 \text{ V } \angle 0^o}{12\,\Omega \angle 38^o} = 16.667 \text{ A } \angle -38^o$$

The fact that the motor's impedance phase angle is not zero degrees, and consequently neither is the line current's phase angle equal to the source's angle, tells us that *not all* of the current in this circuit goes into doing useful work. An impedance phase angle of zero degrees would mean the motor was entirely dissipative in nature, meaning all of the electrical energy delivered to the motor through the power lines would leave the circuit in the form of mechanical work. An impedance phase angle of either $+90$ or -90 degrees would mean *none* of the energy at that motor was dissipated, but rather it was all alternately absorbed and returned. An impedance phase angle between 0 and ± 90 exclusive means *some* energy is dissipated at the motor but the rest is stored and released. The fact that this particular motor's phase angle has a positive rather than a negative value suggests its mechanism of energy storage is *inductive*[22] rather than *capacitive*.

In electric power distribution systems, reactive power is generally undesirable because it constitutes a greater amount of current in the power lines than what is actually necessary to do useful work at the loads. Power lines are thermally limited in how much current they may safely carry, and so too much reactive power in a power system leads to unnecessarily overloaded power lines.

A solution to this problem is to add another component to the circuit, situated as close to the motor as possible. This other component must have the *opposite* type of reactance (i.e. capacitive versus inductive) in order to cancel out the reactive power of the motor. In a case like this where the motor has a slightly inductive nature, we will need to connect a *capacitor* in parallel to satisfy its reactive needs. Energy alternately exchanged between the motor's inductance and the capacitor's capacitance will result in high current only between those two components, while the power lines connecting them both to the source need only carry as much current as necessary to satisfy the motor's dissipation.

[22] Since electric motors are constructed of wire coils wrapped around iron poles, it makes perfect sense that an electric motor would exhibit at least some inductive characteristics.

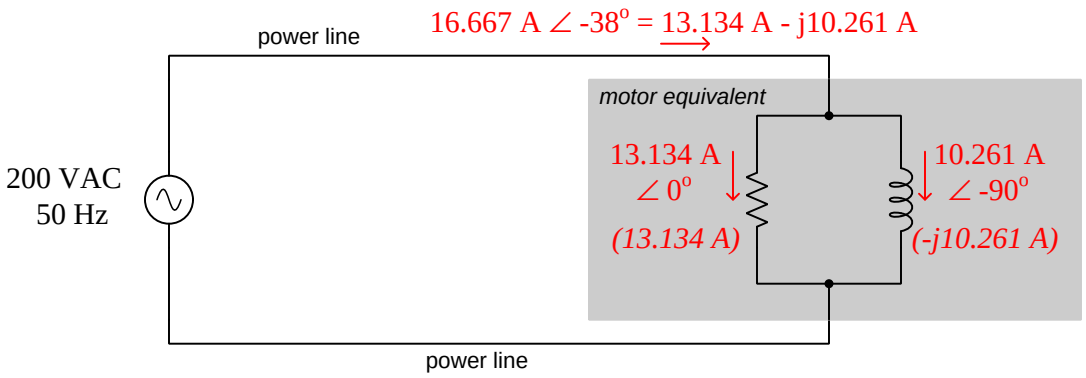
The challenge, of course, is calculating the correct size of capacitor to connect to this motor. A capacitor that is too small won't satisfy enough of the motor's reactive power demand, while a capacitor that is too large will create its own excessive reactive power problem. In order to do this, we must analyze the total current value to see just how much of that current is reactive and how much is resistive.

If we use a calculator to convert the polar-form current value of 16.667 A $\angle -38^o$ into *rectangular* form, we shall get a clearer view of reactive versus resistive current at the motor:

$$16.667 \text{ A} \angle -38^o = 13.134 \text{ A} - j10.261 \text{ A}$$

While polar form shows us the total magnitude of the current (16.667 Amperes) and the phase-shift between its waveform and the source's voltage waveform (-38^o lagging behind the source voltage), rectangular form shows us the *real* current (13.134 Amperes) doing useful work at the motor versus the *imaginary* current ($-j10.261$ Amperes) merely exchanged back and forth between the motor and the source. It is as if *two* different AC currents exist on the same power lines: one current which is 13.134 Amperes at an angle of 0 degrees and another AC current which is 10.261 Amperes at an angle of -90 degrees.

One way to illustrate this is to model the motor as a network of *two* parallel-connected components, one resistive[23] with a current of 13.134 A $\angle$ 0^o and one inductive with a current of 10.261 A $\angle$ -90^o. The power line current is simply the sum of these parallel branch currents:



Capacitors are almost purely reactive devices by nature. As such, their impedance values are practically equal to their reactance values, with an impedance phase angle almost exactly equal to -90^o. Thus, current passing through a capacitor connected to the AC source will have a phase angle of nearly $+90^o$. A capacitor connected in parallel with the motor and drawing 10.261 A $\angle$ 90^o of current will satisfy the motor's reactive current requirement of 10.261 A $\angle$ -90^o, canceling that out in the power lines to leave only the 13.134 A $\angle$ 0^o of current doing useful work.

[23] We are using a resistor symbol here only to represent the dissipative nature of the motor's mechanical function. An *actual* resistor would dissipate all its energy in the form of heat, whereas a motor converts this electrical energy into mechanical work. All we mean to show here is that the 13.134 Amperes of current are involved with energy leaving the circuit, without specifying exactly what form that dissipated energy takes as it leaves.

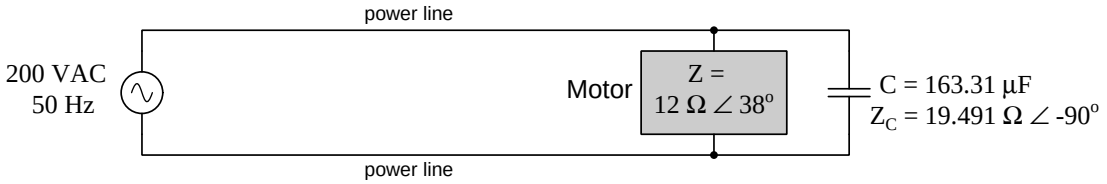
We may use Ohm's Law to calculate the necessary capacitive impedance (Z_C) based on source voltage (V) and our necessary capacitor current (I_C), then use the magnitude of that impedance as the capacitor's reactance value (X_C), from which we may calculate the necessary capacitance (C):

$$Z_C = \frac{V}{I_C} = \frac{200 \text{ V } \angle 0^o}{10.261 \angle 90^o} = 19.491 \, \Omega \angle -90^o$$

$$\text{Therefore, } X_C = 19.491 \, \Omega$$

$$C = \frac{1}{2\pi f X_C} = \frac{1}{(2\pi)(50 \text{ Hz})(19.491 \, \Omega)} = 163.31 \, \mu\text{F}$$

As a final check of our work, we may compute total (parallel) impedance for the capacitor and the motor, to ensure that the Z_T phase angle is equal to 0 degrees:



$$Z_T = \frac{1}{\frac{1}{Z_{motor}} + \frac{1}{Z_C}} = \frac{1}{\frac{1}{12 \, \Omega \angle 38^o} + \frac{1}{19.491 \, \Omega \angle -90^o}} = 15.228 \, \Omega \angle 0^o$$

Calculating power line current with this new total impedance value:

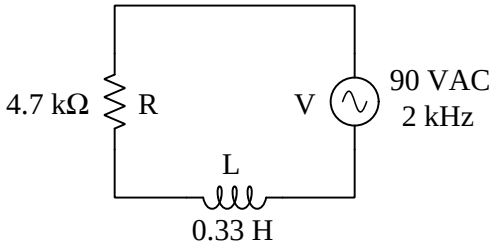
$$I = \frac{V}{Z_T} = \frac{200 \text{ V } \angle 0^o}{15.228 \, \Omega \angle 0^o} = 13.134 \text{ A } \angle 0^o$$

Sure enough, the inclusion of the capacitor in parallel with the motor brought the power line current down from 16.667 Amperes to 13.134 Amperes, which is the value shown earlier passing through the dissipative "component" within the motor (modeled as a parallel-connected resistance and inductance). The motor does precisely the same amount of useful mechanical work as it did before, but with less current through the power lines because the motor's reactive current needs are being supplied by the capacitor rather than being supplied by the source.

4.14 Example circuit #5

In the previous examples we have assumed the use of an electronic calculator capable of performing arithmetic operations on complex numbers. This is a great labor-saving tool for beginning students, as the use of a such a tool frees their minds to focus on the circuit concepts (e.g. laws and principles) without getting bogged down by the sheer work of the arithmetic. In this example, however, we will demonstrate how to perform all the complex-number arithmetic without this tool.

The following schematic diagram shows the circuit for consideration. Our task is to calculate all component voltages and currents:



If this were a DC resistor circuit, we would begin by calculating total (series) resistance, then use Ohm's Law to determine circuit current ($I = \frac{V}{R}$), then use Ohm's Law again to compute each resistor's voltage drop ($V_{R1} = IR_1$; $V_{R2} = IR_2$). Our approach for this circuit will be much the same, except we will need to first determine each component's impedance (in complex form, as a phasor) before applying any properties of series circuits or Ohm's Law.

Beginning with impedances, the resistor's impedance is as follows. We will write this phasor impedance value in both polar and rectangular forms, because each will be useful to us later:

$$Z_R = 4700\,\Omega\,\angle 0^o \qquad \text{or} \qquad 4700\,\Omega + j0\,\Omega$$

The inductor's impedance magnitude must be determined from its reactance value (X_L):

$$X_L = 2\pi fL = (2\pi)(2000\text{ Hz})(0.33\text{ H}) = 4146.9\,\Omega$$

Expressing this reactance as a phasor impedance, in both polar and rectangular forms just like the resistor's:

$$Z_L = 4146.9\,\Omega\,\angle 90^o \qquad \text{or} \qquad 0\,\Omega + j4146.9\,\Omega$$

Now we are ready to begin our actual analysis of the circuit, following the same steps we would if it were DC rather than AC.

Our first analytical step is to compute total (series) impedance, by adding Z_R with Z_L. Since rectangular notation lends itself best to addition, we will use $4700\Omega+j0\Omega$ for the resistor's impedance and $0\ \Omega + j4146.9\ \Omega$ for the inductor's impedance. Adding two complex numbers together in rectangular form is as simple as adding the real portions together, then adding the imaginary portions together, then combining real and imaginary sums into one rectangular-form complex number:

$$Z_T = Z_R + Z_L$$

$$Z_T = (4700\ \Omega + j0\ \Omega) + (0\ \Omega + j4146.9\ \Omega)$$

$$Z_T = 4700\ \Omega + j4146.9\ \Omega$$

Our next step is to divide source voltage by total impedance. Division is best performed with complex numbers expressed in *polar* notation, which means we must convert $4700\ \Omega + j4146.9\ \Omega$ into polar form before we proceed any further. Polar magnitude is calculated using the Pythagorean Theorem, treating the polar magnitude as the hypotenuse of a right triangle, with the real portion of the rectangular form representing the adjacent side length and the imaginary portion of the rectangular form representing the opposite side length:

$$Z(\text{mag}) = \sqrt{(4700\ \Omega)^2 + (4146.9\ \Omega)^2} = 6267.9\ \Omega$$

We're not done with the conversion yet! The next sub-step is to calculate the polar phase angle. This is done using a trigonometric function, either the arc-sine of the opposite over hypotenuse, the arc-cosine of the adjacent over the hypotenuse, or the arc-tangent of the opposite over the adjacent. I choose the latter approach:

$$Z(\text{ang}) = \tan^{-1}\left(\frac{4146.9\ \Omega}{4700\ \Omega}\right) = 41.423^o$$

Now we have the total impedance converted from rectangular form into polar form, ready to use in the next step:

$$Z_T = 4700\ \Omega + j4146.9\ \Omega = 6267.9\ \Omega\ \angle 41.423^o$$

Our next step, of course, is dividing total source voltage by total impedance to calculate current. To perform this division, we must express the source voltage (90 Volts AC) as a complex number as well, and will assign to it an angle of 0 degrees since no other phase reference has been given to us. Dividing two complex numbers in polar form is as simple as dividing the magnitudes, then subtracting the angles, then combining the magnitude quotient and angle difference into one polar-form complex number:

$$I = \frac{V}{Z_T} = \frac{90\ \text{V}\ \angle 0^o}{6267.9\ \Omega\ \angle 41.423^o} = 14.359\ \text{mA}\ \angle -41.423^o$$

Finally, calculating voltages for the resistor and inductor, we take the common current value we just calculated and multiply it by each of the component impedance values, respectively. Multiplication with complex numbers, like division, is best performed in polar notation. This is as simple as multiplying the magnitudes, then adding the angles, then combining the magnitude product and angle sum into one polar-form number:

$$V_R = IZ_R = (14.359 \text{ mA} \angle -41.423^o)(4700\,\Omega \angle 0^o) = 67.487 \text{ V} \angle -41.423^o$$

$$V_L = IZ_L = (14.359 \text{ mA} \angle -41.423^o)(4146.9\,\Omega \angle 90^o) = 59.545 \text{ V} \angle 48.577^o$$

At this point we have computed all voltage and current values, and stand ready to check our work. A good way to verify our calculations are correct is to add the two component voltages together to ensure their sum equals the total (source) voltage, in accordance with the properties of series circuits and Kirchhoff's Voltage Law. Unfortunately, though, since addition of complex numbers is best performed in rectangular form, this means converting both of the polar-form voltage values into rectangular form before we may proceed with our check.

To convert a polar-form complex number into rectangular form, multiply the polar magnitude by the cosine of the angle to compute the real portion and multiply the polar magnitude by the sine of the angle to compute the imaginary portion. Beginning with the resistor's voltage:

$$(67.487 \text{ V}) \cos(-41.423^o) = 50.605 \text{ V}$$

$$(67.487 \text{ V}) \sin(-41.423^o) = -j44.650 \text{ V}$$

$$\text{Therefore, } V_R = 50.605 \text{ V} - j44.650 \text{ V}$$

Next, converting the inductor's voltage value from polar form into rectangular form:

$$(59.545 \text{ V}) \cos(48.577^o) = 39.395 \text{ V}$$

$$(59.545 \text{ V}) \sin(48.577^o) = j44.650 \text{ V}$$

$$\text{Therefore, } V_L = 39.395 \text{ V} + j44.650 \text{ V}$$

Now, *finally*, we may check our work by verifying that the source voltage equals the sum of our calculated resistor and inductor voltages:

$$V_{source} = V_R + V_L = (50.605 \text{ V} - j44.650 \text{ V}) + (39.395 \text{ V} + j44.650 \text{ V})$$

$$V_{source} = 90 \text{ V} + j0 \text{ V}$$

Indeed this proves true: 90 Volts + j0 Volts is the same as 90 Volts $\angle$ 0^o.

As you can see, performing all the necessary complex-number calculations for even a very simple AC circuit such as this is rather laborious, and error-prone[24]. This is why I strongly recommend the use of an electronic calculator to manage the complex-number arithmetic, especially when first learning how to analyze AC circuits. The sheer amount of work involved with converting between polar and rectangular forms, and then performing multiple real calculations for each complex-number calculation, invites error and frustration. This is especially true when one considers the fact that students and professionals alike are expected to use an electronic calculator to compute trigonometric functions, squares, and square-roots anyway[25]. Why not just use a more capable calculator and avoid a lot of tedium?

Just for fun, let's see this same circuit analysis done in polar form, using a calculator:

$$Z_R = 4700\,\Omega\ \angle 0^o$$

$$Z_L = (2\pi fL)\ \angle 90^o = (2\pi)(2000\text{ Hz})(0.33\text{ H})\ \angle 90^o = 4146.9\,\Omega\ \angle 90^o$$

$$Z_T = Z_R + Z_L = (4700\,\Omega\ \angle 0^o) + (4146.9\,\Omega\ \angle 90^o) = 6267.9\,\Omega\ \angle 41.423^o$$

$$I = \frac{V}{Z_T} = \frac{90\text{ V}\ \angle 0^o}{6267.9\,\Omega\ \angle 41.423^o} = 14.359\text{ mA}\ \angle -41.423^o$$

$$V_R = IZ_R = (14.359\text{ mA}\ \angle -41.423^o)(4700\,\Omega\ \angle 0^o) = 67.487\text{ V}\ \angle -41.423^o$$

$$V_L = IZ_L = (14.359\text{ mA}\ \angle -41.423^o)(4146.9\,\Omega\ \angle 90^o) = 59.545\text{ V}\ \angle 48.577^o$$

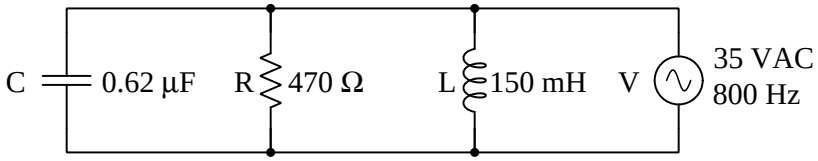
$$V_{source} = V_R + V_L = (67.487\text{ V}\ \angle -41.423^o) + (59.545\text{ V}\ \angle 48.577^o) = 90\text{ V}\ \angle 0^o$$

[24] Working through this example, I ended up using all seven memory locations on my TI-36X Pro calculator just so I would avoid any rounding errors and/or keystroke errors otherwise incurred by re-entering intermediate results. This is a noteworthy point as well: rounding errors tend to accumulate in a really bad way when performing arithmetic operations such as squares, square roots, and trigonometric functions. I have learned the hard way that neglecting to store and then recall those values using the calculator's memory is a recipe for significant error!

[25] I am just old enough to remember learning simple trigonometric functions in grade school through the use of *trig tables*, since electronic hand calculators were not yet affordable for elementary-level students. Some time prior to high school I taught myself how to calculate square roots by hand, from an old textbook. As soon as I owned my first electronic calculator, all that was pushed aside and I never went back. One would literally be considered foolish, or perhaps just masochistic, to compute square roots and trig functions without a calculator of some sort. I will say, however, that I think there is still merit in teaching the use of a *slide rule* (a type of analog mechanical calculator), because that particular obsolete tool very powerfully demonstrates how logarithms transform multiplication and division problems into simpler addition and subtraction problems, and it also teaches the user to manage decimal points on their own.

4.15 Example circuit #6

Once again we will approach the analysis of an AC reactive circuit without the use of an electronic calculator capable of performing arithmetic operations on complex numbers, this time for a *parallel* circuit rather than series. Here is our sample circuit, with our task being to calculate all component voltages and currents within:



As usual, a good first step is to express each component value as a complex impedance. For the resistor, this is elementary:

$$Z_R = 470\,\Omega\,\angle 0^o \qquad \text{or} \qquad 470\,\Omega + j0\,\Omega$$

For the inductor, we first calculate inductive reactance, and then express that as an impedance:

$$X_L = 2\pi f L = (2\pi)(800\text{ Hz})(0.15\text{ H}) = 753.98\,\Omega$$

$$Z_L = 753.98\,\Omega\,\angle 90^o \qquad \text{or} \qquad 0\,\Omega + j753.98\,\Omega$$

We also do the same for the capacitor, first calculating capacitive reactance and then expressing that as a complex impedance:

$$X_C = \frac{1}{2\pi f C} = \frac{1}{(2\pi)(800\text{ Hz})(0.62\,\mu\text{F})} = 320.88\,\Omega$$

$$Z_C = 320.88\,\Omega\,\angle -90^o \qquad \text{or} \qquad 0\,\Omega - j320.88\,\Omega$$

Now that we know how each component will react at the source frequency of 800 Hz, we may begin to apply foundational principles of parallel networks. Perhaps the most directly-applicable principle of parallel networks to utilize here is the fact that all parallel-connected components share the same voltage, in this case 35 Volts AC. Then, with each component having a known impedance and a known voltage, we may apply Ohm's Law (dividing voltage by impedance) to calculate current through each.

As we saw previously, dividing two complex numbers in polar form is as simple as dividing the magnitudes, then subtracting the angles, then combining the magnitude quotient and angle difference into one polar-form complex number. Calculating each component's current in this manner:

$$I_R = \frac{V}{Z_R} = \frac{35 \text{ V } \angle 0^o}{470\,\Omega \angle 0^o} = 74.468 \text{ mA } \angle 0^o$$

$$I_L = \frac{V}{Z_L} = \frac{35 \text{ V } \angle 0^o}{753.98\,\Omega \angle 90^o} = 46.420 \text{ mA } \angle -90^o$$

$$I_C = \frac{V}{Z_C} = \frac{35 \text{ V } \angle 0^o}{320.88\,\Omega \angle -90^o} = 109.08 \text{ mA } \angle 90^o$$

The next parallel-network principle to apply is the fact that currents add in parallel to make total (source) current. In order to add these complex-number current values, we must translate each one into rectangular form, then simply add all their real portions together and add all their imaginary portions together:

$$I_R = 74.468 \text{ mA } \angle 0^o = 74.468 + j0 \text{ mA}$$

$$I_L = 46.420 \text{ mA } \angle -90^o = 0 - j46.420 \text{ mA}$$

$$I_C = 109.08 \text{ mA } \angle 90^o = 0 + j109.08 \text{ mA}$$

$$I_{total} = (74.468 \text{ mA }) + j(109.08 \text{ mA } - 46.420 \text{ mA}) = 74.468 \text{ mA } + j62.656 \text{ mA}$$

In order to determine how much current an AC ammeter would register if connected in series with the source, we must convert this rectangular-form expression of I_{total} into polar form using the Pythagorean Theorem to find the polar magnitude and the arc-tangent function to find the polar angle:

$$I_{total}(\text{mag}) = \sqrt{(74.468 \text{ mA})^2 + (62.656 \text{ mA})^2} = 97.320 \text{ mA}$$

$$I_{total}(\text{ang}) = \tan^{-1}\left(\frac{62.656 \text{ mA}}{74.468 \text{ mA}}\right) = 40.077^o$$

Therefore, total (source) current in this circuit is 97.320 mA $\angle$ 40.077^o, which means an ammeter connected in series with one of the source terminals should register 97.320 milliAmperes.

Something else we may do with this total current value is to calculate the network's total impedance using Ohm's Law ($Z_{total} = \frac{V_{total}}{I_{total}}$). As usual, polar form works well for division in that we simply divide magnitudes and subtract angles:

$$Z_{total} = \frac{V_{total}}{I_{total}} = \frac{35 \text{ V } \angle 0^o}{97.320 \text{ mA } \angle 40.077^o} = 359.64\Omega \angle -40.077^o$$

Since this is a parallel network, one might be inclined to try calculating total impedance as the first analytical step following the determination of all component impedances, rather than calculate component currents as previously shown. While a calculation of total impedance is certainly possible, it ends up being fairly laborious because of the sheer number of arithmetic operations necessary to evaluate the $\frac{1}{\frac{1}{Z_R}+\frac{1}{Z_L}+\frac{1}{Z_C}}$ formula. These operations may be enumerated as follows:

1. Reciprocate all impedances in polar form: reciprocate each polar magnitude, and reverse the sign of each angle. The resulting quantity in each case is called an *admittance* (Y), and is analogous to *conductance* being the reciprocal of resistance ($G = \frac{1}{R}$). Like conductance, admittance is measured in the unit of *Siemens*.

2. Convert each of these admittances into rectangular form in preparation for addition. Each real component is the magnitude times the cosine of the angle; each imaginary component is the magnitude times the sine of the angle.

3. Add the three rectangular-form admittances together to compute total (parallel) admittance. This involves adding together all real components, and then adding together all imaginary components. At this point in the process you now know the value of the large fraction's denominator, in rectangular form.

4. Convert this total admittance from rectangular form into polar form in preparation for final reciprocation. Polar magnitude is found using the Pythagorean Theorem, and the polar angle using the arc-tangent function.

5. Finally, reciprocate this total (parallel) admittance value to arrive at total impedance. This means reciprocating the polar magnitude and reversing the sign of the angle.

As one might guess, this lengthy set of calculations is fraught with potential for error. A single dropped sign, rounded result, or keystroke error is sufficient to corrupt the final result.

Chapter 5

Derivations and Technical References

This chapter is where you will find mathematical derivations too detailed to include in the tutorial, and/or tables and other technical reference material.

5.1 Derivation of inductive impedance

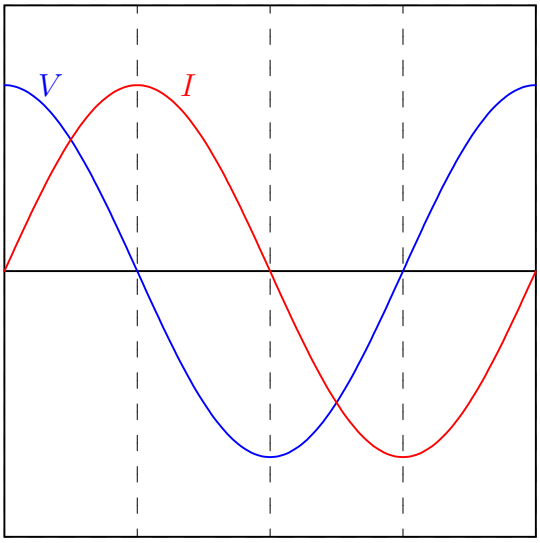
The relationship between voltage (V) and current (I) for a pure[1] inductance (L) is based on rates of change, specifically the rate-of-change of current over time ($\frac{dI}{dt}$). One might think of the following equation as being the inductive equivalent of Ohm's Law, because it fundamentally relates voltage and current for any inductance:

$$V = L\frac{dI}{dt}$$

Where,

V = Voltage across the inductance's terminals (Volts)
L = Inductance (Henrys)
$\frac{dI}{dt}$ = Rate-of-change of current over time (Amperes per second)

If we pass a sinusoidal AC current through an inductance, the voltage at any given point in time will be proportional to the current's instantaneous rate-of-change. This means the voltage will be zero at those points in time when the current waveform is *level* (i.e. zero slope, at every peak) and maximum when the current waveform crosses the zero line (i.e. at maximum steepness). These notable points in time are marked on the following plot by dashed lines:



Therefore, when we apply a sinusoidal current through a pure inductor, the voltage waveform is also a sinusoid but phase-shifted by 90 degrees leading. In other words, *voltage leads current by 90 degrees for a pure inductance.*

[1] By "pure" it is meant that the inductance in question dissipates no energy whatsoever, but losslessly absorbs and releases energy. This is nearly impossible to achieve in practice, but superconducting inductors come very close to this ideal.

This same conclusion may be reached symbolically through the application of simple calculus, if we substitute a time-based sine function for I:

$$V = L\frac{dI}{dt} \qquad \text{and} \qquad I = \sin t$$

$$V = L\frac{d}{dt}(\sin t)$$

$$V = L\cos t$$

Since the derivative ($\frac{d}{dt}$) of a sine function is a cosine function, the voltage waveform must be a cosine wave, and we know cosine waves lead sine waves by 90 degrees.

We may use a more sophisticated approach of defining the AC current as a complex exponential function[2] of time rather than merely as a trigonometric function of time:

$$V = L\frac{dI}{dt} \qquad \text{and} \qquad I = e^{j\omega t}$$

$$V = L\frac{d}{dt}(e^{j\omega t})$$

$$V = j\omega L e^{j\omega t}$$

Just as electrical resistance is mathematically defined as the ratio of voltage to current ($R = \frac{V}{I}$), impedance is also defined as the ratio of voltage to current. Taking the voltage and current exponential functions together as a ratio, we see that inductive impedance reduces to a simple expression lacking any exponential term:

$$Z = \frac{V}{I}$$

$$Z = \frac{j\omega L e^{j\omega t}}{e^{j\omega t}}$$

$$Z = j\omega L$$

Unlike V and I which are both rotating phasors having both real and imaginary components alternating over time, Z is a *constant* imaginary quantity. The magnitude of Z depends on both frequency (ω) and inductance (L): the higher the frequency, the greater the impedance; the greater the inductance, the greater the impedance. The positive imaginary "direction" of Z (i.e. a phasor angle of 90^o) reflects the constant 90 degree phase shift between voltage (leading) and current (lagging) for an inductance. Expressed in non-phasor form, ωL is the *reactance* (X) of the inductor. When expressing frequency in Hertz rather than radians per second, it is more common to see reactance computed as $X_L = 2\pi f L$.

[2] Recall Euler's Relation, where $e^{ix} = \cos x + i \sin x$. Using j rather than i to signify the imaginary unit quantity, and using ω to signify the angular velocity (frequency) of the waveform, the time-domain wave function will be $e^{j\omega t}$ which is equal to $\cos \omega t + j \sin \omega t$. This is the *phasor* expression of the current waveform, having both a real and an imaginary component each of which varies over time. The advantage of this approach is that exponential functions are extremely simple to differentiate, as $\frac{d}{dx}e^{kx} = ke^{kx}$.

An even more sophisticated and versatile expression of an AC voltage uses *complex frequency* represented by the variable s, where s is the complex sum of a real growth/decay rate (σ) and an imaginary natural frequency ($j\omega$):

$$s = \sigma + j\omega$$

Where,
s = Complex frequency in $\frac{1}{\text{seconds}}$
σ = Growth/decay rate in time constants per second
ω = Natural frequency in radians per second

Applying this complex frequency as a current through the inductor and differentiating the inductive "Ohm's Law" formula:

$$V = L\frac{dI}{dt} \qquad \text{and} \qquad I = e^{st}$$

$$V = L\frac{d}{dt}(e^{st})$$

$$V = sLe^{st}$$

Defining inductive impedance as the ratio of inductor voltage to inductor current:

$$Z = \frac{V}{I}$$

$$Z = \frac{sLe^{st}}{e^{st}}$$

$$Z = sL$$

The impedance (in Ohms) is now a function of complex frequency, varying with growth/decay rate as well as with plain frequency. The real growth/decay rate dictates whether the component will experience a net absorption or a net release of energy over time (i.e. behaving as a load versus as a source) while the imaginary frequency dictates how often energy gets absorbed and released.

5.2 Derivation of capacitive impedance

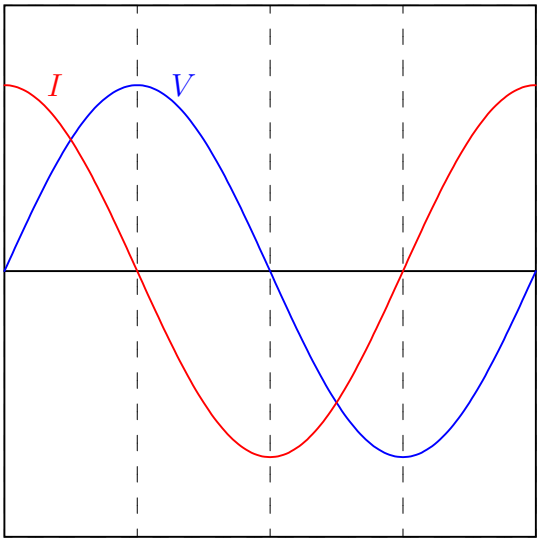
The relationship between voltage (V) and current (I) for a pure[3] capacitance (C) is based on rates of change, specifically the rate-of-change of voltage over time ($\frac{dV}{dt}$). One might think of the following equation as being the capacitive equivalent of Ohm's Law, because it fundamentally relates voltage and current for any capacitance:

$$I = C\frac{dV}{dt}$$

Where,

I = Current through the capacitance's terminals (Amperes)
C = Capacitance (Farads)
$\frac{dV}{dt}$ = Rate-of-change of voltage over time (Volts per second)

If we impress a sinusoidal AC voltage across a capacitance, the current at any given point in time will be proportional to the voltage's instantaneous rate-of-change. This means the current will be zero at those points in time when the voltage waveform is *level* (i.e. zero slope, at every peak) and maximum when the voltage waveform crosses the zero line (i.e. at maximum steepness). These notable points in time are marked on the following plot by dashed lines:



Therefore, when we apply a sinusoidal voltage across a pure capacitance, the current waveform is also a sinusoid but phase-shifted by 90 degrees leading. In other words, *current leads voltage by 90 degrees for a pure capacitance.*

[3] By "pure" it is meant that the capacitance in question dissipates no energy whatsoever, but losslessly absorbs and releases energy. Real capacitors come remarkably close to this ideal.

This same conclusion may be reached symbolically through the application of simple calculus, if we substitute a time-based sine function for V:

$$I = C\frac{dV}{dt} \qquad \text{and} \qquad V = \sin t$$

$$I = C\frac{d}{dt}(\sin t)$$

$$I = C\cos t$$

Since the derivative ($\frac{d}{dt}$) of a sine function is a cosine function, the current waveform must be a cosine wave, and we know cosine waves lead sine waves by 90 degrees.

We may use a more sophisticated approach of defining the AC voltage as a complex exponential function[4] of time rather than merely as a trigonometric function of time:

$$I = C\frac{dV}{dt} \qquad \text{and} \qquad V = e^{j\omega t}$$

$$I = C\frac{d}{dt}(e^{j\omega t})$$

$$I = j\omega C e^{j\omega t}$$

Just as electrical resistance is mathematically defined as the ratio of voltage to current ($R = \frac{V}{I}$), impedance is also defined as the ratio of voltage to current. Taking the voltage and current exponential functions together as a ratio, we see that capacitive impedance reduces to a simple expression lacking any exponential term:

$$Z = \frac{V}{I}$$

$$Z = \frac{e^{j\omega t}}{j\omega C e^{j\omega t}}$$

$$Z = \frac{1}{j\omega C} \qquad \text{or} \qquad Z = -j\frac{1}{\omega C}$$

Unlike V and I which are both rotating phasors having both real and imaginary components alternating over time, Z is a *constant* imaginary quantity. The magnitude of Z depends on both frequency (ω) and capacitance (C): the higher the frequency, the less the impedance; the greater the capacitance, the less the impedance. The negative imaginary "direction" of Z (i.e. a phasor angle of -90^o) reflects the constant 90 degree phase shift between voltage (lagging) and current (leading) for a capacitance. Expressed in non-phasor form, $\frac{1}{\omega C}$ is the *reactance* (X) of the capacitance. When

[4] Recall Euler's Relation, where $e^{ix} = \cos x + i \sin x$. Using j rather than i to signify the imaginary unit quantity, and using ω to signify the angular velocity (frequency) of the waveform, the time-domain wave function will be $e^{j\omega t}$ which is equal to $\cos \omega t + j \sin \omega t$. This is the *phasor* expression of the current waveform, having both a real and an imaginary component each of which varies over time. The advantage of this approach is that exponential functions are extremely simple to differentiate, as $\frac{d}{dx}e^{kx} = ke^{kx}$.

expressing frequency in Hertz rather than radians per second, it is more common to see capacitive reactance computed as $X_C = \frac{1}{2\pi f C}$.

An even more sophisticated and versatile expression of an AC voltage uses *complex frequency* represented by the variable s, where s is the complex sum of a real growth/decay rate (σ) and an imaginary natural frequency ($j\omega$):

$$s = \sigma + j\omega$$

Where,
s = Complex frequency in $\frac{1}{\text{seconds}}$
σ = Growth/decay rate in time constants per second
ω = Natural frequency in radians per second

Applying this complex frequency as a voltage across the capacitor and differentiating the capacitive "Ohm's Law" formula:

$$I = C\frac{dV}{dt} \qquad \text{and} \qquad V = e^{st}$$

$$I = C\frac{d}{dt}(e^{st})$$

$$I = sCe^{st}$$

Defining capacitive impedance as the ratio of capacitor voltage to capacitor current:

$$Z = \frac{V}{I}$$

$$Z = \frac{e^{st}}{sCe^{st}}$$

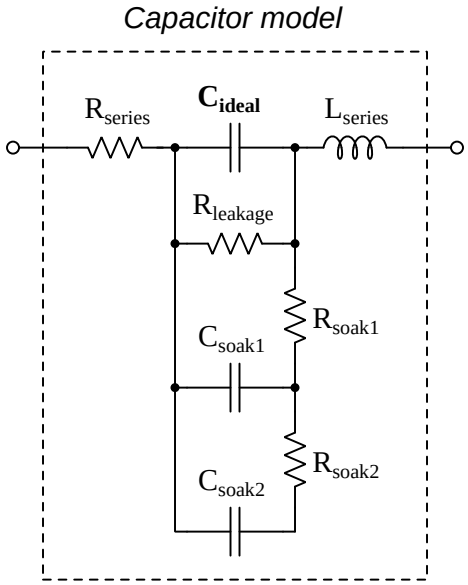
$$Z = \frac{1}{sC}$$

The impedance (in Ohms) is now a function of complex frequency, varying with growth/decay rate as well as with plain frequency. The real growth/decay rate dictates whether the component will experience a net absorption or a net release of energy over time (i.e. behaving as a load versus as a source) while the imaginary frequency dictates how often energy gets absorbed and released.

5.3 Capacitor parasitics

5.3.1 Model of a real capacitor

An ideal capacitor exhibits only capacitance, with no inductance, resistance, or other characteristics to interfere. Real capacitors exhibit all these phenomena to varying degrees, and we collectively refer to these undesirable traits as *parasitic effects*. The following diagram models some of the parasitic effects observed in real capacitors:



In addition to the capacitance the capacitor is supposed to exhibit (C_{ideal}), the capacitor also has parasitic resistance (R_{series}, also known as *Equivalent Series Resistance*, or *ESR*), parasitic inductance (L_{series}), and additional energy storage in the form of *soakage* (also known as *dielectric absorption*) whereby the dielectric substance itself absorbs and releases energy after relatively long periods of time compared to the main (ideal) capacitance.

Some of these parasitic effects – such as leakage resistance and soakage – affect the capacitor's performance in DC applications. Most of the other parasitic effects cause problems in AC and pulsed applications. For example, the effective series capacitor-inductor combination formed by C_{ideal} and L_{series} will cause *resonance* to occur at a particular AC frequency, resulting in much less reactance at that frequency than what would be predicted by the capacitive reactance formula $X_C = \frac{1}{2\pi fC}$.

Next we will explore common mechanisms for each of these effects.

5.3.2 Parasitic resistance in capacitors

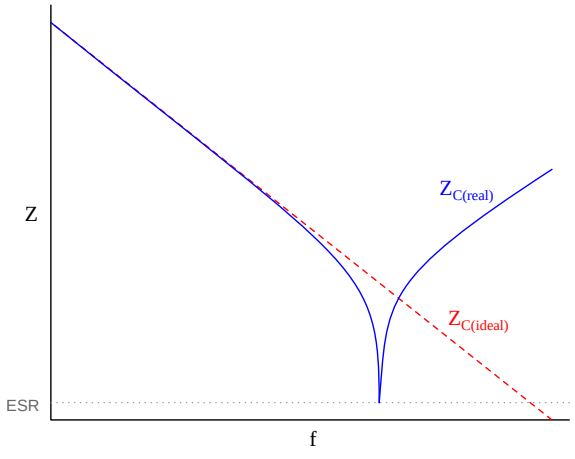
Wire resistance, of course, plays a part in this parasitic effect, but this is generally very small due to the short conductor lengths we typically see with capacitors. More significant is *dielectric losses* – energy dissipation caused by the stressing and relaxation of dipoles within the dielectric material – which act like resistance because energy ends up leaving the component (in the form of heat) and not returning to the circuit. Electrolytic capacitors have an additional source of parasitic resistance, in the form of the electrolytic gel substance used to make electrical contact from the metal-foil "plate" to the surface of the dielectric layer.

Another form of parasitic resistance within a capacitor behaves like a resistor connected in parallel with the ideal capacitance ($R_{leakage}$), resulting from the dielectric not being a perfectly insulating medium. This parasitic characteristic results in a small current passing through the capacitor even when the voltage across the capacitor is steady (i.e. $\frac{dV}{dt}$ is zero).

5.3.3 Parasitic inductance in capacitors

Any time a magnetic field forms around a current-carrying conductor, energy is stored in that magnetic field. We call this magnetic-based energy-storing capability *inductance*, and of course all capacitors must have some inductance due to the wire leads serving as connection points to the capacitor's metal plates. Much of a capacitor's parasitic inductance may be minimized by maintaining short lead lengths as it attaches to a printed-circuit board.

Parasitic inductance is a problem for capacitors in AC applications because inductive reactance (X_L) tends to cancel out capacitive reactance (X_C). If we plot the impedance of a capacitor as a function of frequency, we would expect an ideal capacitor to manifest a straight-line descent on a logarithmic plot. However, what we see is that at a certain frequency the series parasitic inductance resonates with the capacitance leaving only parasitic resistance (ESR), and then past that frequency the inductive effects overshadow the capacitance:



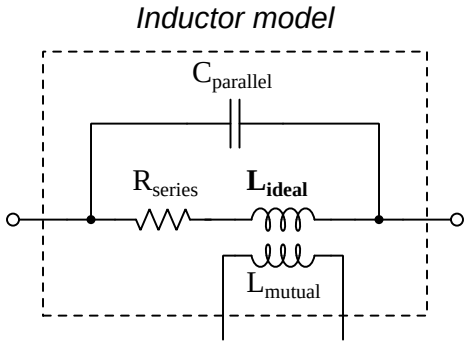
5.3.4 Other parasitic effects in capacitors

Soakage is an interesting effect resulting from dipole relaxation within the dielectric material itself, and may be modeled (as shown) by a series of resistor-capacitor subnetworks. This effect is especially prominent in aluminum electrolytic capacitors, and may be easily demonstrated by discharging a capacitor (by briefly connecting a shorting wire across the capacitor's terminals) and then monitoring the capacitor's DC voltage slowly "recover" with no connection to an external source.

5.4 Inductor parasitics

5.4.1 Model of a real inductor

An ideal inductor exhibits only inductance, with no capacitance, resistance, or other characteristics to interfere. Real inductors exhibit all these phenomena to varying degrees, and we collectively refer to these undesirable traits as *parasitic effects*. The following diagram models some of the parasitic effects observed in real inductors:



In addition to the inductance the inductor is supposed to exhibit (L_{ideal}), the inductor also has parasitic resistance (R_{series}, also known as *Equivalent Series Resistance*, or *ESR*), parasitic capacitance ($C_{parallel}$), and mutual inductance (L_{mutual}) with nearby wires and components.

Some of these parasitic effects – such as equivalent series resistance – affect the inductor's performance in DC applications. Most of the other parasitic effects cause problems in AC and pulsed applications. For example, the effective inductor-capacitor "tank circuit" formed by L_{ideal} and $C_{parallel}$ will cause *resonance* to occur at a particular AC frequency, resulting in much more reactance at that frequency than what would be predicted by the inductive reactance formula $X_L = 2\pi fL$.

Next we will explore common mechanisms for each of these effects.

5.4.2 Parasitic resistance in inductors

Wire resistance plays a dominant role in this parasitic effect due to the typically long lengths of wire necessary to wind the coil that forms most inductors. Wire resistance is not the only dissipative mechanism at work, though. Other losses include magnetic hysteresis of the iron core material as well as *eddy currents* induced in the iron core. An "eddy" current is a circulating electric current induced within the iron core of an inductor, made possible by the fact that iron is an electrically-conductive material as well as being ferromagnetic. These circulating currents do no useful work, and dissipate energy in the form of heating the iron. They may be minimized by forming the iron core from pieces of iron that are electrically insulated from one another, e.g. forming the iron core from *laminated sheets* or *powdered particles* of iron where each sheet or particle is electrically insulated from the next by a layer of non-conductive material on its outer surface.

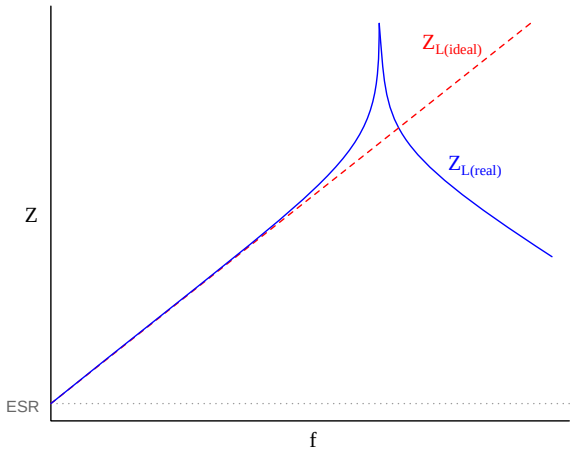
The series resistance of an inductor is always frequency-dependent. In DC conditions (i.e. frequency of zero Hertz) there will be the basic wire resistance of the coil at play. As frequency increases from zero, however, both the magnetic core losses from hysteresis and eddy currents also increase which add to the DC resistance to form a larger ESR. At extremely high frequencies the *skin effect*[5] further adds to the inductor's ESR.

[5] At high frequencies, electric current travels more toward the outer surface of a conductor rather than through the conductor's entire cross-section, effectively decreasing the conductor's cross-sectional area (gauge) as frequency rises.

5.4.3 Parasitic capacitance in inductors

Any time an electric field forms between two conductors, energy is stored in that electric field. We call this electric-based energy-storing capability *capacitance*, and of course all inductors must have some capacitance due to the insulating media between wire leads as well as between adjacent turns of wire within the coil (and between the wire turns and the iron core).

Parasitic capacitance is a problem for inductors in AC applications because capacitive reactance (X_C) tends to cancel out inductive reactance (X_L). If we plot the impedance of an inductor as a function of frequency, we would expect an ideal inductor to manifest a straight-line ascent on a logarithmic plot. However, what we see is that at a certain frequency the parallel parasitic capacitance resonates with the inductance to create a nearly-infinite impedance, and then past that frequency the capacitive effects overshadow the inductance:



Precious little may be done to eliminate parasitic capacitance within any inductor, whereas parasitic inductance is fairly easy to minimize within a capacitor. This explains why when faced with an equivalent choice between a circuit design using capacitors and a circuit design using inductors, capacitors nearly always win. Simply put, it is easier to make a nearly-ideal capacitor than it is to make a nearly-ideal inductor.

This also explains why the self-resonant frequency of most inductors is much lower than the self-resonant frequency of most capacitors: all other factors being equal, an inductor will have more parasitic capacitance in it than an equivalent capacitor will have parasitic inductance within it, making the LC product greater for the inductor than for the capacitor.

5.4.4 Other parasitic effects in inductors

Mutual inductance occurs whenever adjacent conductors' magnetic fields link with one another, which is difficult to avoid especially in physically dense circuit layouts. This parasitic effect may be minimized by proper placement of inductive components (e.g. keeping them spaced as far apart from each other as possible, orienting their axes perpendicular to each other rather than parallel) as well as by core designs with strong magnetic field containment (e.g. toroidal cores contain their magnetic fields better than rectangular cores).

5.4.5 Inductor testing

The following list shows typical tests for inductors at the time of manufacture:

- **DC winding resistance** – the simplest of these to measure, using a multimeter
- **Winding inductance/impedance** – performed by exciting the winding with an AC source and comparing voltage/current, typically using either an LCR meter or an impedance analyzer
- **Equivalent winding resistance** – based on the ratio of AC voltage to AC current (only the portions of each in-phase with each other) representing winding resistance as well as energy losses in the magnetic core
- **Core saturation** – performed by exciting the winding with a sinusoidal AC current and measuring self-induced winding voltage until a non-sinusoidal waveshape is detected, or more crudely by determining when the self-induced voltage no longer increases proportionately to increased excitation current
- **Insulation resistance** – performed with a "Hi-Pot" (high-potential) insulation tester, generally connecting one terminal to either or both ends of the winding and the other tester terminal to the conductive core material, this infers insulation resistance by applying (typically) several thousand Volts DC while measuring the small current passing through the stressed insulation

A more thorough battery of tests used during transformer design development might include the following:

- **BH loop** – performed by exciting the winding with an AC voltage and measuring the self-induced voltage across it, the phasor portion of that induced voltage shifted by 90 degrees from the current waveform representing the rate-of-change of magnetic flux which may be mathematically integrated to infer the amount of magnetic flux, those two variables being plotted using an oscilloscope's X-Y mode much like "curve tracing" for a semiconductor component ; often performed at different temperatures to fully characterize the magnetic properties of the core material
- **High-voltage impulse** – similar to an insulation resistance test, this uses a high-voltage DC pulse applied across the winding insulation to stress it
- **Volt-second** – performed by applying a DC voltage pulse to the winding while measuring current through it, the ideal response being a linear-ramping current whose magnitude is proportional to the Volt-second product of the applied pulse
- **Overload** – subjecting a winding to dramatic overcurrent to thermally and mechanically stress the inductor as it might be in an overloaded power system scenario
- **External magnetic field rejection** – subjecting the entire inductor to an external (alternating) magnetic field and measuring any voltage induced across any winding, usually tested with the magnetic field applied across different physical axes
- **Vibration** – not an electrical test per se, but a useful stressor when followed by repeated electrical tests to ensure electrical/magnetic properties have not substantially changed following the vibration

Chapter 6

Programming References

A powerful tool for mathematical modeling is text-based *computer programming*. This is where you type coded commands in text form which the computer is able to interpret. Many different text-based languages exist for this purpose, but we will focus here on just two of them, *C++* and *Python*.

6.1 Programming in C++

One of the more popular text-based computer programming languages is called *C++*. This is a *compiled* language, which means you must create a plain-text file containing C++ code using a program called a *text editor*, then execute a software application called a *compiler* to translate your "source code" into instructions directly understandable to the computer. Here is an example of "source code" for a very simple C++ program intended to perform some basic arithmetic operations and print the results to the computer's console:

```
#include <iostream>
using namespace std;

int main (void)
{
  float x, y;

  x = 200;
  y = -560.5;

  cout << "This simple program performs basic arithmetic on" << endl;
  cout << "the two numbers " << x << " and " << y << " and then" << endl;
  cout << "displays the results on the computer's console." << endl;

  cout << endl;

  cout << "Sum = " << x + y << endl;
  cout << "Difference = " << x - y << endl;
  cout << "Product = " << x * y << endl;
  cout << "Quotient of " << x / y << endl;

  return 0;
}
```

Computer languages such as C++ are designed to make sense when read by human programmers. The general order of execution is left-to-right, top-to-bottom just the same as reading any text document written in English. Blank lines, indentation, and other "whitespace" is largely irrelevant in C++ code, and is included only to make the code more pleasing[1] to view.

[1] Although not included in this example, *comments* preceded by double-forward slash characters (`//`) may be added to source code as well to provide explanations of what the code is supposed to do, for the benefit of anyone reading it. The compiler application will ignore all comments.

Let's examine the C++ source code to explain what it means:

- `#include <iostream>` and `using namespace std;` are set-up instructions to the compiler giving it some context in which to interpret your code. The code specific to your task is located between the brace symbols ({ and }, often referred to as "curly-braces").

- `int main (void)` labels the "Main" function for the computer: the instructions within this function (lying between the { and } symbols) it will be commanded to execute. Every complete C++ program contains a `main` function at minimum, and often additional functions as well, but the `main` function is where execution always begins. The `int` declares this function will return an *integer* number value when complete, which helps to explain the purpose of the `return 0;` statement at the end of the `main` function: providing a numerical value of zero at the program's completion as promised by `int`. This returned value is rather incidental to our purpose here, but it is fairly standard practice in C++ programming.

- Grouping symbols such as (parentheses) and {braces} abound in C, C++, and other languages (e.g. Java). Parentheses typically group data to be processed by a function, called *arguments* to that function. Braces surround lines of executable code belonging to a particular function.

- The `float` declaration reserves places in the computer's memory for two *floating-point* variables, in this case the variables' names being `x` and `y`. In most text-based programming languages, variables may be named by single letters or by combinations of letters (e.g. `xyz` would be a single variable).

- The next two lines assign numerical values to the two variables. Note how each line terminates with a semicolon character (`;`) and how this pattern holds true for most of the lines in this program. In C++ semicolons are analogous to periods at the ends of English sentences. This demarcation of each line's end is necessary because C++ ignores whitespace on the page and doesn't "know" otherwise where one line ends and another begins.

- All the other instructions take the form of a `cout` command which prints characters to the "standard output" stream of the computer, which in this case will be text displayed on the console. The double-less-than symbols (`<<`) show data being sent *toward* the `cout` command. Note how verbatim text is enclosed in quotation marks, while variables such as `x` or mathematical expressions such as `x - y` are not enclosed in quotations because we want the computer to display the numerical values represented, not the literal text.

- Standard arithmetic operations (add, subtract, multiply, divide) are represented as `+`, `-`, `*`, and `/`, respectively.

- The `endl` found at the end of every `cout` statement marks the end of a line of text printed to the computer's console display. If not for these `endl` inclusions, the displayed text would resemble a run-on sentence rather than a paragraph. Note the `cout << endl;` line, which does nothing but create a blank line on the screen, for no reason other than esthetics.

After saving this *source code* text to a file with its own name (e.g. `myprogram.cpp`), you would then *compile* the source code into an *executable* file which the computer may then run. If you are using a console-based compiler such as *GCC* (very popular within variants of the Unix operating system[2], such as Linux and Apple's OS X), you would type the following command and press the Enter key:

```
g++ -o myprogram.exe myprogram.cpp
```

This command instructs the `GCC` compiler to take your source code (`myprogram.cpp`) and create with it an executable file named `myprogram.exe`. Simply typing `./myprogram.exe` at the command-line will then execute your program:

```
./myprogram.exe
```

If you are using a graphic-based C++ development system such as Microsoft Visual Studio[3], you may simply create a new console application "project" using this software, then paste or type your code into the example template appearing in the editor window, and finally run your application to test its output.

As this program runs, it displays the following text to the console:

```
This simple program performs basic arithmetic on
the two numbers 200 and -560.5 and then
displays the results on the computer's console.

Sum = -360.5
Difference = 760.5
Product = -112100
Quotient of -0.356824
```

As crude as this example program is, it serves the purpose of showing how easy it is to write and execute simple programs in a computer using the C++ language. As you encounter C++ example programs (shown as source code) in any of these modules, feel free to directly copy-and-paste the source code text into a text editor's screen, then follow the rest of the instructions given here (i.e. save to a file, compile, and finally run your program). You will find that it is generally easier to

[2] A very functional option for users of Microsoft Windows is called *Cygwin*, which provides a Unix-like console environment complete with all the customary utility applications such as GCC!

[3] Using Microsoft Visual Studio community version 2017 at the time of this writing to test this example, here are the steps I needed to follow in order to successfully compile and run a simple program such as this: (1) Start up Visual Studio and select the option to create a New Project; (2) Select the Windows Console Application template, as this will perform necessary set-up steps to generate a console-based program which will save you time and effort as well as avoid simple errors of omission; (3) When the editing screen appears, type or paste the C++ code within the `main()` function provided in the template, deleting the "Hello World" `cout` line that came with the template; (4) Type or paste any preprocessor directives (e.g. `#include` statements, `namespace` statements) necessary for your code that did not come with the template; (5) Lastly, under the Debug drop-down menu choose either Start Debugging (F5 hot-key) or Start Without Debugging (Ctrl-F5 hotkeys) to compile ("Build") and run your new program. Upon execution a console window will appear showing the output of your program.

learn computer programming by closely examining others' example programs and modifying them than it is to write your own programs starting from a blank screen.

6.2 Programming in Python

Another text-based computer programming language called *Python* allows you to type instructions at a terminal prompt and receive immediate results without having to compile that code. This is because Python is an *interpreted* language: a software application called an *interpreter* reads your source code, translates it into computer-understandable instructions, and then executes those instructions in one step.

The following shows what happens on my personal computer when I start up the Python interpreter on my personal computer, by typing `python3`[4] and pressing the Enter key:

```
Python 3.7.2 (default, Feb 19 2019, 18:15:18)
[GCC 4.1.2] on linux
Type "help", "copyright", "credits" or "license" for more information.
>>>
```

The `>>>` symbols represent the prompt within the Python interpreter "shell", signifying readiness to accept Python commands entered by the user.

Shown here is an example of the same arithmetic operations performed on the same quantities, using a Python interpreter. All lines shown preceded by the `>>>` prompt are entries typed by the human programmer, and all lines shown without the `>>>` prompt are responses from the Python interpreter software:

```
>>> x = 200
>>> y = -560.5
>>> x + y
-360.5
>>> x - y
760.5
>>> x * y
-112100.0
>>> x / y
-0.35682426404995538
>>> quit()
```

[4] Using version 3 of Python, which is the latest at the time of this writing.

More advanced mathematical functions are accessible in Python by first entering the line `from math import *` which "imports" these functions from Python's math *library* (with functions identical to those available for the C programming language, and included on any computer with Python installed). Some examples show some of these functions in use, demonstrating how the Python interpreter may be used as a scientific calculator:

```
>>> from math import *
>>> sin(30.0)
-0.98803162409286183
>>> sin(radians(30.0))
0.49999999999999994
>>> pow(2.0, 5.0)
32.0
>>> log10(10000.0)
4.0
>>> e
2.7182818284590451
>>> pi
3.1415926535897931
>>> log(pow(e,6.0))
6.0
>>> asin(0.7071068)
0.78539819000368838
>>> degrees(asin(0.7071068))
45.000001524425265
>>> quit()
```

Note how trigonometric functions assume angles expressed in *radians* rather than *degrees*, and how Python provides convenient functions for translating between the two. Logarithms assume a base of e unless otherwise stated (e.g. the `log10` function for common logarithms).

The interpreted (versus compiled) nature of Python, as well as its relatively simple syntax, makes it a good choice as a person's first programming language. For complex applications, interpreted languages such as Python execute slower than compiled languages such as C++, but for the very simple examples used in these learning modules speed is not a concern.

Another Python math library is `cmath`, giving Python the ability to perform arithmetic on complex numbers. This is very useful for AC circuit analysis using *phasors*[5] as shown in the following example. Here we see Python's interpreter used as a scientific calculator to show series and parallel impedances of a resistor, capacitor, and inductor in a 60 Hz AC circuit:

```
>>> from math import *
>>> from cmath import *
>>> r = complex(400,0)
>>> f = 60.0
>>> xc = 1/(2 * pi * f * 4.7e-6)
>>> zc = complex(0,-xc)
>>> xl = 2 * pi * f * 1.0
>>> zl = complex(0,xl)
>>> r + zc + zl
(400-187.38811239154882j)
>>> 1/(1/r + 1/zc + 1/zl)
(355.837695813625+125.35793777619385j)
>>> polar(r + zc + zl)
(441.717448903332, -0.4381072059213295)
>>> abs(r + zc + zl)
441.717448903332
>>> phase(r + zc + zl)
-0.4381072059213295
>>> degrees(phase(r + zc + zl))
-25.10169387356105
```

When entering a value in rectangular form, we use the `complex()` function where the arguments are the real and imaginary quantities, respectively. If we had opted to enter the impedance values in polar form, we would have used the `rect()` function where the first argument is the magnitude and the second argument is the angle in radians. For example, we could have set the capacitor's impedance (`zc`) as $X_C \angle -90^o$ with the command `zc = rect(xc,radians(-90))` rather than with the command `zc = complex(0,-xc)` and it would have worked the same.

Note how Python defaults to rectangular form for complex quantities. Here we defined a 400 Ohm resistance as a complex value in rectangular form (400 +j0 Ω), then computed capacitive and inductive reactances at 60 Hz and defined each of those as complex (phasor) values ($0 - jX_c$ Ω and $0 + jX_l$ Ω, respectively). After that we computed total impedance in series, then total impedance in parallel. Polar-form representation was then shown for the series impedance (441.717 Ω $\angle$ -25.102^o). Note the use of different functions to show the polar-form series impedance value: `polar()` takes the complex quantity and returns its polar magnitude and phase angle in *radians*; `abs()` returns just the polar magnitude; `phase()` returns just the polar angle, once again in radians. To find the polar phase angle in degrees, we nest the `degrees()` and `phase()` functions together.

The utility of Python's interpreter environment as a scientific calculator should be clear from these examples. Not only does it offer a powerful array of mathematical functions, but also unlimited

[5] A "phasor" is a voltage, current, or impedance represented as a complex number, either in rectangular or polar form.

assignment of variables as well as a convenient text record[6] of all calculations performed which may be easily copied and pasted into a text document for archival.

It is also possible to save a set of Python commands to a text file using a text editor application, and then instruct the Python interpreter to execute it at once rather than having to type it line-by-line in the interpreter's shell. For example, consider the following Python program, saved under the filename `myprogram.py`:

```
x = 200
y = -560.5

print("Sum")
print(x + y)

print("Difference")
print(x - y)

print("Product")
print(x * y)

print("Quotient")
print(x / y)
```

As with C++, the interpreter will read this source code from left-to-right, top-to-bottom, just the same as you or I would read a document written in English. Interestingly, whitespace *is* significant in the Python language (unlike C++), but this simple example program makes no use of that.

To execute this Python program, I would need to type `python myprogram.py` and then press the Enter key at my computer console's prompt, at which point it would display the following result:

```
Sum
-360.5
Difference
760.5
Product
-112100.0
Quotient
-0.35682426405
```

As you can see, syntax within the Python programming language is simpler than C++, which is one reason why it is often a preferred language for beginning programmers.

[6] Like many command-line computing environments, Python's interpreter supports "up-arrow" recall of previous entries. This allows quick recall of previously typed commands for editing and re-evaluation.

If you are interested in learning more about computer programming in *any* language, you will find a wide variety of books and free tutorials available on those subjects. Otherwise, feel free to learn by the examples presented in these modules.

Chapter 7

Questions

This learning module, along with all others in the ModEL collection, is designed to be used in an inverted instructional environment where students independently read[1] the tutorials and attempt to answer questions on their own *prior* to the instructor's interaction with them. In place of lecture[2], the instructor engages with students in Socratic-style dialogue, probing and challenging their understanding of the subject matter through inquiry.

Answers are not provided for questions within this chapter, and this is by design. Solved problems may be found in the Tutorial and Derivation chapters, instead. The goal here is *independence*, and this requires students to be challenged in ways where others cannot think for them. Remember that you always have the tools of *experimentation* and *computer simulation* (e.g. SPICE) to explore concepts!

The following lists contain ideas for Socratic-style questions and challenges. Upon inspection, one will notice a strong theme of *metacognition* within these statements: they are designed to foster a regular habit of examining one's own thoughts as a means toward clearer thinking. As such these sample questions are useful both for instructor-led discussions as well as for self-study.

[1] Technical reading is an essential academic skill for any technical practitioner to possess for the simple reason that the most comprehensive, accurate, and useful information to be found for developing technical competence is in textual form. Technical careers in general are characterized by the need for continuous learning to remain current with standards and technology, and therefore any technical practitioner who cannot read well is handicapped in their professional development. An excellent resource for educators on improving students' reading prowess through intentional effort and strategy is the book textitReading For Understanding – How Reading Apprenticeship Improves Disciplinary Learning in Secondary and College Classrooms by Ruth Schoenbach, Cynthia Greenleaf, and Lynn Murphy.

[2] Lecture is popular as a teaching method because it is easy to implement: any reasonably articulate subject matter expert can talk to students, even with little preparation. However, it is also quite problematic. A good lecture always makes complicated concepts seem easier than they are, which is bad for students because it instills a false sense of confidence in their own understanding; reading and re-articulation requires more cognitive effort and serves to verify comprehension. A culture of teaching-by-lecture fosters a debilitating dependence upon direct personal instruction, whereas the challenges of modern life demand independent and critical thought made possible only by gathering information and perspectives from afar. Information presented in a lecture is ephemeral, easily lost to failures of memory and dictation; text is forever, and may be referenced at any time.

General challenges following tutorial reading

- Summarize as much of the text as you can in one paragraph of your own words. A helpful strategy is to explain ideas as you would for an intelligent child: as simple as you can without compromising too much accuracy.
- Simplify a particular section of the text, for example a paragraph or even a single sentence, so as to capture the same fundamental idea in fewer words.
- Where did the text make the most sense to you? What was it about the text's presentation that made it clear?
- Identify where it might be easy for someone to misunderstand the text, and explain why you think it could be confusing.
- Identify any new concept(s) presented in the text, and explain in your own words.
- Identify any familiar concept(s) such as physical laws or principles applied or referenced in the text.
- Devise a proof of concept experiment demonstrating an important principle, physical law, or technical innovation represented in the text.
- Devise an experiment to disprove a plausible misconception.
- Did the text reveal any misconceptions you might have harbored? If so, describe the misconception(s) and the reason(s) why you now know them to be incorrect.
- Describe any useful problem-solving strategies applied in the text.
- Devise a question of your own to challenge a reader's comprehension of the text.

General follow-up challenges for assigned problems

- Identify where any fundamental laws or principles apply to the solution of this problem, especially before applying any mathematical techniques.
- Devise a thought experiment to explore the characteristics of the problem scenario, applying known laws and principles to mentally model its behavior.
- Describe in detail your own strategy for solving this problem. How did you identify and organized the given information? Did you sketch any diagrams to help frame the problem?
- Is there more than one way to solve this problem? Which method seems best to you?
- Show the work you did in solving this problem, even if the solution is incomplete or incorrect.
- What would you say was the most challenging part of this problem, and why was it so?
- Was any important information missing from the problem which you had to research or recall?
- Was there any extraneous information presented within this problem? If so, what was it and why did it not matter?
- Examine someone else's solution to identify where they applied fundamental laws or principles.
- Simplify the problem from its given form and show how to solve this simpler version of it. Examples include eliminating certain variables or conditions, altering values to simpler (usually whole) numbers, applying a limiting case (i.e. altering a variable to some extreme or ultimate value).
- For quantitative problems, identify the real-world meaning of all intermediate calculations: their units of measurement, where they fit into the scenario at hand. Annotate any diagrams or illustrations with these calculated values.
- For quantitative problems, try approaching it qualitatively instead, thinking in terms of "increase" and "decrease" rather than definite values.
- For qualitative problems, try approaching it quantitatively instead, proposing simple numerical values for the variables.
- Were there any assumptions you made while solving this problem? Would your solution change if one of those assumptions were altered?
- Identify where it would be easy for someone to go astray in attempting to solve this problem.
- Formulate your own problem based on what you learned solving this one.

General follow-up challenges for experiments or projects

- In what way(s) was this experiment or project easy to complete?
- Identify some of the challenges you faced in completing this experiment or project.

- Show how thorough documentation assisted in the completion of this experiment or project.
- Which fundamental laws or principles are key to this system's function?
- Identify any way(s) in which one might obtain false or otherwise misleading measurements from test equipment in this system.
- What will happen if (component X) fails (open/shorted/etc.)?
- What would have to occur to make this system unsafe?

7.1 Conceptual reasoning

These questions are designed to stimulate your analytic and synthetic thinking[3]. In a Socratic discussion with your instructor, the goal is for these questions to prompt an extended dialogue where assumptions are revealed, conclusions are tested, and understanding is sharpened. Your instructor may also pose additional questions based on those assigned, in order to further probe and refine your conceptual understanding.

Questions that follow are presented to challenge and probe your understanding of various concepts presented in the tutorial. These questions are intended to serve as a guide for the Socratic dialogue between yourself and the instructor. Your instructor's task is to ensure you have a sound grasp of these concepts, and the questions contained in this document are merely a means to this end. Your instructor may, at his or her discretion, alter or substitute questions for the benefit of tailoring the discussion to each student's needs. The only absolute requirement is that each student is challenged and assessed at a level equal to or greater than that represented by the documented questions.

It is far more important that you convey your *reasoning* than it is to simply convey a correct answer. For this reason, you should refrain from researching other information sources to answer questions. What matters here is that *you* are doing the thinking. If the answer is incorrect, your instructor will work with you to correct it through proper reasoning. A correct answer without an adequate explanation of how you derived that answer is unacceptable, as it does not aid the learning or assessment process.

You will note a conspicuous lack of answers given for these conceptual questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. The best way by far is to debate the answers with fellow students and also with the instructor during the Socratic dialogue sessions intended to be used with these learning modules. Reasoning through challenging questions with other people is an excellent tool for developing strong reasoning skills.

Another means of checking your conceptual answers, where applicable, is to use circuit simulation software to explore the effects of changes made to circuits. For example, if one of these conceptual questions challenges you to predict the effects of altering some component parameter in a circuit, you may check the validity of your work by simulating that same parameter change within software and seeing if the results agree.

[3] *Analytical* thinking involves the "disassembly" of an idea into its constituent parts, analogous to dissection. *Synthetic* thinking involves the "assembly" of a new idea comprised of multiple concepts, analogous to construction. Both activities are high-level cognitive skills, extremely important for effective problem-solving, necessitating frequent challenge and regular practice to fully develop.

7.1.1 Reading outline and reflections

"Reading maketh a full man; conference a ready man; and writing an exact man" – Francis Bacon

Francis Bacon's advice is a blueprint for effective education: reading provides the learner with knowledge, writing focuses the learner's thoughts, and critical dialogue equips the learner to confidently communicate and apply their learning. Independent acquisition and application of knowledge is a powerful skill, well worth the effort to cultivate. To this end, students should read these educational resources closely, journal their own reflections on the reading, and discuss in detail their findings with classmates and instructor(s). You should be able to do all of the following after reading any instructional text:

☑ Briefly SUMMARIZE THE TEXT in the form of a journal entry documenting your learning as you progress through the course of study. Share this summary in dialogue with your classmates and instructor. Journaling is an excellent self-test of thorough reading because you cannot clearly express what you have not read or did not comprehend.

☑ Demonstrate ACTIVE READING STRATEGIES, including verbalizing your impressions as you read, simplifying long passages to convey the same ideas using fewer words, annotating text and illustrations with your own interpretations, working through mathematical examples shown in the text, cross-referencing passages with relevant illustrations and/or other passages, identifying problem-solving strategies applied by the author, etc. Technical reading is a special case of problem-solving, and so these strategies work precisely because they help solve any problem: paying attention to your own thoughts (metacognition), eliminating unnecessary complexities, identifying what makes sense, paying close attention to details, drawing connections between separated facts, and noting the successful strategies of others.

☑ Identify IMPORTANT THEMES, especially GENERAL LAWS and PRINCIPLES, expounded in the text and express them in the simplest of terms as though you were teaching an intelligent child. This emphasizes connections between related topics and develops your ability to communicate complex ideas to anyone.

☑ Form YOUR OWN QUESTIONS based on the reading, and then pose them to your instructor and classmates for their consideration. Anticipate both correct and incorrect answers, the incorrect answer(s) assuming one or more plausible misconceptions. This helps you view the subject from different perspectives to grasp it more fully.

☑ Devise EXPERIMENTS to test claims presented in the reading, or to disprove misconceptions. Predict possible outcomes of these experiments, and evaluate their meanings: what result(s) would confirm, and what would constitute disproof? Running mental simulations and evaluating results is essential to scientific and diagnostic reasoning.

☑ Specifically identify any points you found CONFUSING. The reason for doing this is to help diagnose misconceptions and overcome barriers to learning.

7.1.2 Foundational concepts

Correct analysis and diagnosis of electric circuits begins with a proper understanding of some basic concepts. The following is a list of some important concepts referenced in this module's tutorial. Define each of them in your own words, and be prepared to illustrate each of these concepts with a description of a practical example and/or a live demonstration.

Energy

Conservation of Energy

Limiting cases as a problem-solving strategy

Power

Electrical source

Electrical load

Direct Current (DC)

Alternating Current (AC)

Resistance

Ohm's Law

Joule's Law

Inductance

Capacitance

True power

Reactive power

Reactance

Impedance

Phasor

Phase angle

Properties of series AC circuits

Properties of parallel AC circuits

Kirchhoff's Voltage Law for AC circuits

Kirchhoff's Current Law for AC circuits

7.1.3 Parasitic inductance and capacitance

Not only do reactive components unavoidably possess some parasitic ("stray") resistance, but they also exhibit parasitic reactance of the *opposite* kind. For instance, inductors are bound to have a small amount of capacitance built-in, and capacitors are bound to have a small amount of inductance built-in. These effects are not intentional, but they exist anyway. Such unintentional effects are called *parasitic*.

Describe how a small amount of capacitance comes to exist within an inductor, and how a small amount of inductance comes to exist within a capacitor. Explain what it is about the construction of these two reactive components that allows the existence of "opposite" characteristics.

Additionally, small amounts of inductance and capacitance exist based on how components are mounted to circuit boards when constructing real circuits. Give examples of both parasitic inductance and parasitic capacitance existing as a function of component mounting.

Challenges

- In some circuits these parasitic effects are undesirable. Identify methods to minimize them.

7.1.4 Parasitic reactance between parallel wires

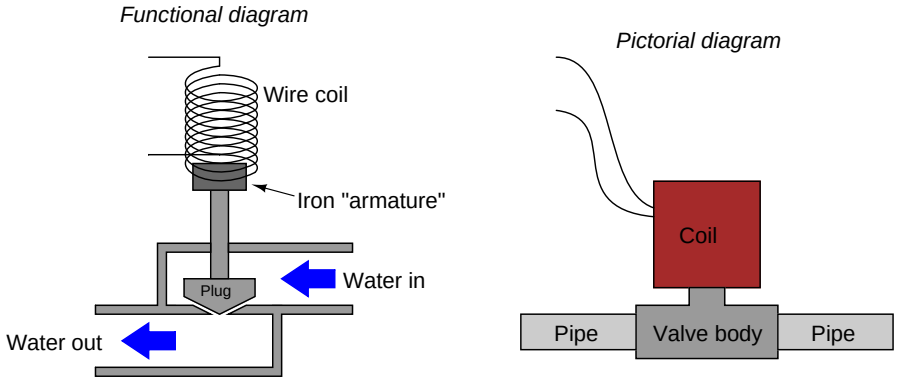
Parasitic inductance and capacitance is present any time wires are other over significant distances. Suppose two wires are run parallel to each other, carrying current in opposite directions. Explain what happens to their parasitic inductance and parasitic capacitance when moved closer to each other.

Challenges

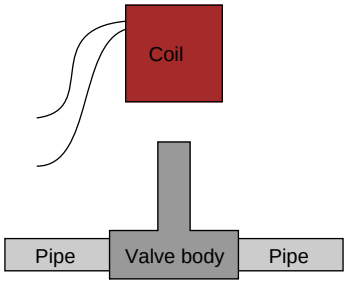
- When students first learn about electricity and how to build electric circuits, they ignore parasitic effects. Explain how it is possible for a beginning student to ignore these effects, yet construct and test working circuits that perform as expected without consideration to these effects.

7.1.5 Solenoid valve maintenance

A *solenoid valve* is a mechanical shutoff device actuated by electricity. An electromagnet coil produces an attractive force on an iron "armature" which then either opens or closes a valve mechanism to control the flow of some fluid. Shown here are two different types of illustrations, both showing a solenoid valve:

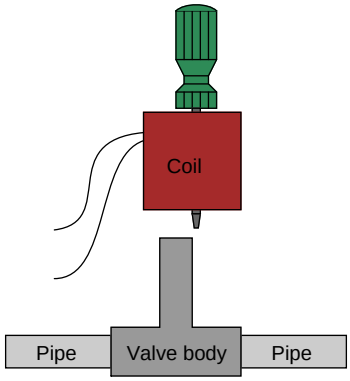


Some solenoid valves are constructed in such a way that the coil assembly may be removed from the valve body, separating these two pieces so that maintenance work may be done on one without interfering with the other. Of course, this means the valve mechanism will no longer be actuated by the magnetic field, but at least one piece may be worked upon without having to remove the other piece from whatever it may be connected to:



This is commonly done when replacement of the valve mechanism is needed. First, the coil is lifted off the valve mechanism, then the maintenance technician is free to remove the valve body from the pipes and replace it with a new valve body. Lastly, the coil is re-installed on the new valve body and the solenoid is once more ready for service, all without having to electrically disconnect the coil from its power source.

However, if this is done while the coil is energized, it will overheat and burn up in just a few minutes. To prevent this from happening, the maintenance technicians have learned to insert a steel screwdriver through the center hole of the coil while it is removed from the valve body, like this:



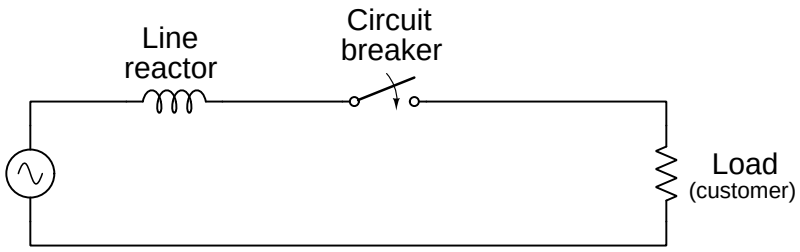
With the steel screwdriver shank taking the place of the iron armature inside the valve body, the coil will not overheat and burn up even if continually powered. Explain the nature of the problem (why the coil tends to burn up when separated from the valve body) and also why a screwdriver put in place of the iron armature works to prevent this from happening.

Challenges

- What do you suppose the screwdriver would *feel* like when inserted into the energized solenoid coil?

7.1.6 Substation power line reactors

A common fixture of electrical power substations is the so-called *line reactor*, which is just an air-core inductor connected in series with the power line. A simplified schematic diagram shows where this "reactor" is located in the circuit, in series with the circuit breaker[4]:



An actual photograph showing three substation line reactors appears here:

The purpose of line reactors in a power system is to make the circuit breaker's job easier. In other words, the circuit breaker will be able to more reliably "trip" and isolate power from a fault with line reactors in place. Based on what you know about inductance, explain why this is.

It is technically possible to replace line reactors with series *resistors* and achieve a similar effect. Explain why reactors are preferable to resistors in this application.

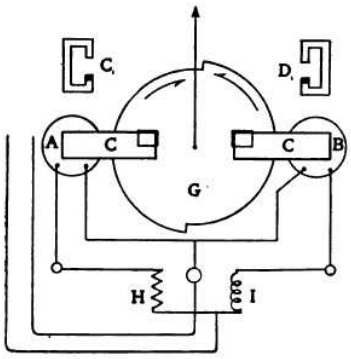
Challenges

- Would a *capacitor* fulfill the same practical function as a line reactor, if substituted? Explain why or why not.

[4] A *circuit breaker* is a special switch designed to open very quickly in the event of a *fault* where line current suddenly becomes very large. An example of such a fault would be a tree branch falling down and shorting across two conductors of a power line: sensors would detect the abnormally high current in this circuit, and the circuit breaker would quickly open to shut off the flow of current. Fault currents tend to be extremely large (*tens of thousands of Amperes* are typical for faults, in a power conductor that might normally carry less than a hundred Amperes of current!), and this causes wear and tear within the circuit breaker as the contacts open and a powerful arc develops between them.

7.1.7 Electromagnetic frequency meter

The following illustration and text came from page 52 of *Cassier's Magazine*, Volume 42, issued in the year 1912. In this article (*"New Types of Frequency Meters"* by William Smith), multiple designs are presented for meters measuring the frequency of AC in early power systems. This particular design uses a pair of electromagnets (labeled A and B) to produce opposing torques[5] on a disk (G) made of thin aluminum metal, which is connected to a pointer (the vertical line and arrowhead) indicating frequency:



DETAIL OF FREQUENCY METER

The torque of each magnet is proportional to the square of the current in the coil and to the frequency. At normal frequency the torques balance when the disk is in the middle position. If the frequency increases, the current in coil *A*, which is in series with resistance only, remains unchanged and its torque increases because of the higher frequency; the current in coil *B*, which is in series with reactance, dimin-

52

Coil A imparts a clockwise torque to the disk, while coil B imparts a counter-clockwise torque. Component "H" is a resistor, and component "I" is an inductor. Based on this knowledge, as well as the brief description given in the magazine's text, explain how this mechanism serves to measure frequency of the applied AC.

Which direction of the pointer's movement indicates a greater frequency, clockwise or counter-clockwise?

How would this mechanism behave if the inductor (I) were replaced by a resistor equal in value

[5] "Torque" simply refers to a twisting force. For those who are interested, the means by which an AC-excited electromagnet is able to create a torque on an aluminum disk is called a *shaded-pole* structure. The concept of shaded poles goes well beyond the scope of this tutorial, but for the time being it is sufficient to know that it produces a lateral force on any electrically conductive material placed between the poles of the electromagnet.

to H?

How would this mechanism behave if the inductor (I) were replaced by a capacitor?

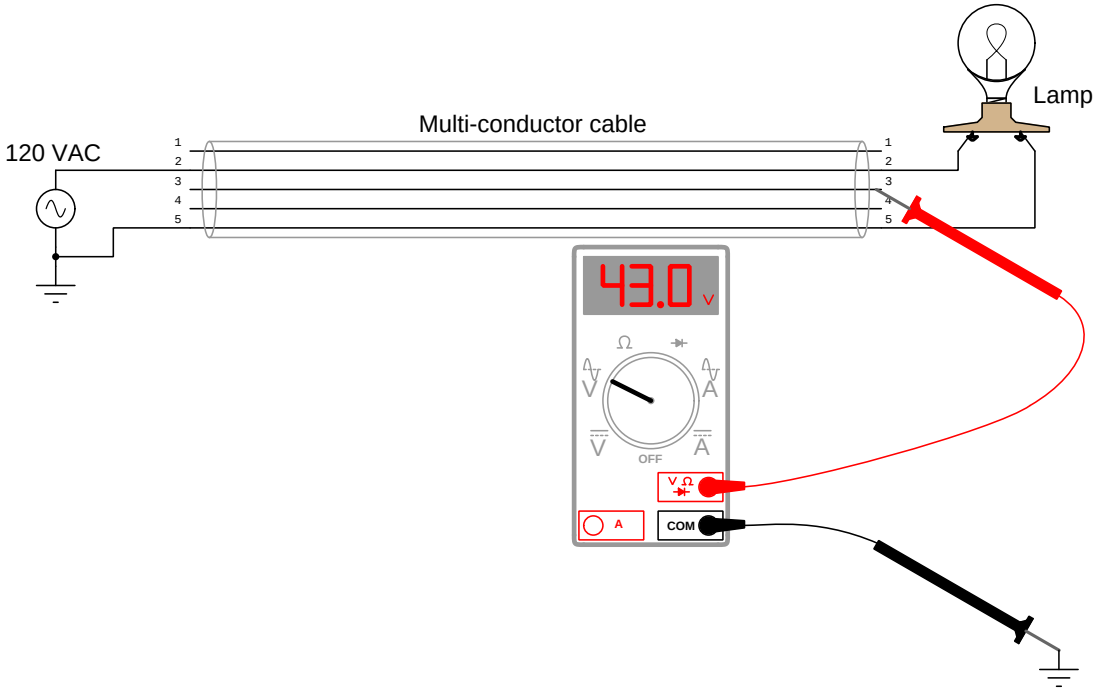
The two electromagnet coils themselves behave more like resistors than inductors. Explain why this is.

Challenges

- What factor(s) influence frequency in an AC power system?
- Explain the impact of inserting a large(r) iron core inside of inductor I.

7.1.8 Phantom voltage measurements

A technician uses a digital multimeter to check for the presence of dangerous voltage between conductor #3 and Earth ground prior to touching that conductor. This safety check is done as a matter of precaution, even though the technician has reason to believe that conductor is electrically distinct (i.e. not electrically common to any other conductors) and should not be "live". To this technician's surprise, he measures 43 Volts AC between conductor #3 and ground!



Puzzled, this technician consults another technician to ask how it is possible to measure a possibly dangerous voltage level between a conductor and ground when the conductor in question is not connected to anything else. "Oh," says the other technician, "That's probably just a *phantom voltage*. Don't worry about it!"

Explain this phenomenon. Exactly how is the AC voltmeter registering a voltage on what should be a "dead" wire?

Should you be concerned about the safety hazard of so-called "phantom voltages"? Why or why not?

Some voltmeters made for general electrical use (rather than precision electronic use) are specially built to have less input impedance than typical: tens of kiloOhms instead of megaOhms. This design helps minimize "phantom" voltage measurements. Explain how a voltmeter with less input impedance (also known as *insertion resistance*) than usual is less liable to be "fooled" when taking measurements on an unattached conductor such as this.

Challenges

- Are "phantom" voltages strictly an AC phenomenon, or may they manifest in DC circuits as well?
- Identify some parameters of the multi-conductor cable which could be modified (e.g. length, wire gauge, insulation thickness, etc.) to exacerbate the "phantom" voltages effect.

7.2 Quantitative reasoning

These questions are designed to stimulate your computational thinking. In a Socratic discussion with your instructor, the goal is for these questions to reveal your mathematical approach(es) to problem-solving so that good technique and sound reasoning may be reinforced. Your instructor may also pose additional questions based on those assigned, in order to observe your problem-solving firsthand.

Mental arithmetic and estimations are strongly encouraged for all calculations, because without these abilities you will be unable to readily detect errors caused by calculator misuse (e.g. keystroke errors).

You will note a conspicuous lack of answers given for these quantitative questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. My advice is to use circuit simulation software such as SPICE to check the correctness of quantitative answers. Refer to those learning modules within this collection focusing on SPICE to see worked examples which you may use directly as practice problems for your own study, and/or as templates you may modify to run your own analyses and generate your own practice problems.

Completely worked example problems found in the Tutorial may also serve as "test cases[6]" for gaining proficiency in the use of circuit simulation software, and then once that proficiency is gained you will never need to rely[7] on an answer key!

[6] In other words, set up the circuit simulation software to analyze the same circuit examples found in the Tutorial. If the simulated results match the answers shown in the Tutorial, it confirms the simulation has properly run. If the simulated results disagree with the Tutorial's answers, something has been set up incorrectly in the simulation software. Using every Tutorial as practice in this way will quickly develop proficiency in the use of circuit simulation software.

[7] This approach is perfectly in keeping with the instructional philosophy of these learning modules: *teaching students to be self-sufficient thinkers*. Answer keys can be useful, but it is even more useful to your long-term success to have a set of tools on hand for checking your own work, because once you have left school and are on your own, there will no longer be "answer keys" available for the problems you will have to solve.

7.2.1 Miscellaneous physical constants

Note: constants shown in **bold** type are *exact*, not approximations. Values inside of parentheses show one standard deviation (σ) of uncertainty in the final digits: for example, the magnetic permeability of free space value given as $1.25663706212(19) \times 10^{-6}$ H/m represents a center value (i.e. the location parameter) of $1.25663706212 \times 10^{-6}$ Henrys per meter with one standard deviation of uncertainty equal to $0.0000000000019 \times 10^{-6}$ Henrys per meter.

Avogadro's number (N_A) = **6.02214076** $\times 10^{23}$ **per mole** (mol^{-1})

Boltzmann's constant (k) = **1.380649** $\times 10^{-23}$ **Joules per Kelvin** (J/K)

Electronic charge (e) = **1.602176634** $\times 10^{-19}$ **Coulomb** (C)

Faraday constant (F) = **96,485.33212...** $\times 10^{4}$ **Coulombs per mole** (C/mol)

Magnetic permeability of free space (μ_0) = $1.25663706212(19) \times 10^{-6}$ Henrys per meter (H/m)

Electric permittivity of free space (ϵ_0) = $8.8541878128(13) \times 10^{-12}$ Farads per meter (F/m)

Characteristic impedance of free space (Z_0) = 376.730313668(57) Ohms (Ω)

Gravitational constant (G) = $6.67430(15) \times 10^{-11}$ cubic meters per kilogram-seconds squared ($m^3/kg\text{-}s^2$)

Molar gas constant (R) = **8.314462618... Joules per mole-Kelvin** (J/mol-K) = 0.08205746(14) liters-atmospheres per mole-Kelvin

Planck constant (h) = **6.62607015** $\times 10^{-34}$ **joule-seconds** (J-s)

Stefan-Boltzmann constant (σ) = **5.670374419...** $\times 10^{-8}$ **Watts per square meter-Kelvin**^{4} ($W/m^2 \cdot K^4$)

Speed of light in a vacuum (c) = **299,792,458 meters per second** (m/s) = 186282.4 miles per second (mi/s)

Note: All constants taken from NIST data "Fundamental Physical Constants – Complete Listing", from `http://physics.nist.gov/constants`, National Institute of Standards and Technology (NIST), 2018 CODATA Adjustment.

7.2.2 Resistances and reactances expressed as impedances

Express the impedance (Z) in both polar and rectangular mathematical forms for each of the following components, and also sketch a phasor diagram representing each impedance value:

- A resistor with 500 Ω of resistance
- An inductor with 1.2 kΩ of reactance
- A capacitor with 950 Ω of reactance
- A resistor with 22 kΩ of resistance
- A capacitor with 50 kΩ of reactance
- An inductor with 133 Ω of reactance

Challenges

- Explain the conceptual distinction between resistance and reactance.

7.2.3 Imperfect inductor

Real inductors and capacitors are never purely reactive. There will inevitably be some resistance intrinsic to these devices as well.

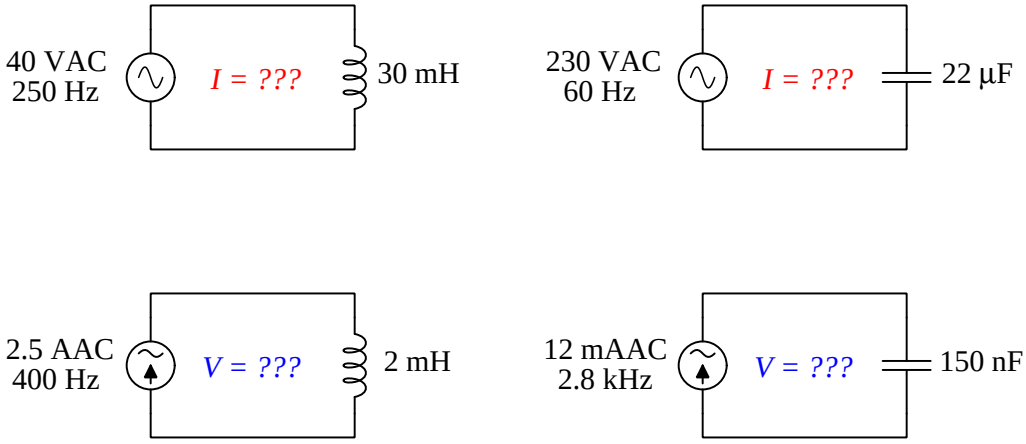
Suppose an inductor has 57 Ω of winding resistance, and 1500 Ω of reactance at a particular frequency. How would this combination be expressed as a single impedance? State your answer in both polar and rectangular forms.

Challenges

- Wire resistance is the dominant form of resistance within real inductors, but not the only form. Identify at least one other form of energy dissipation contributing to the resistance of a real inductor.

7.2.4 Simple AC circuits

Calculate the unknown quantities in the following circuits, expressing your answers in both rectangular and polar form, assuming the source in each case is the phase reference (i.e. its phase angle is zero):



- Upper-left:
- Upper-right:
- Lower-left:
- Lower-right:

Challenges

- Sketch phasor diagrams showing each circuit's voltage, current, and impedance values in graphical form.
- Predict the effect in each of these circuits of the frequency increasing.

7.2.5 Series RC circuit with AC voltage source

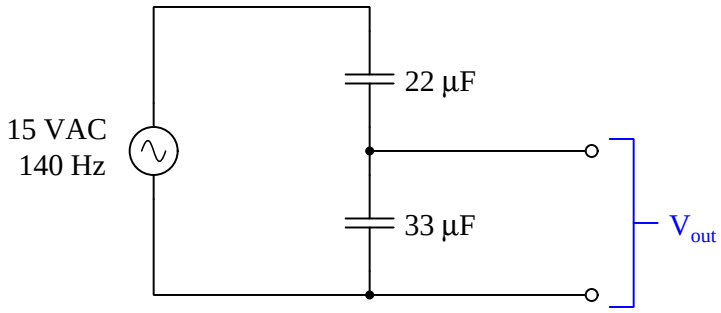
Suppose an AC voltage source outputs 48 Volts RMS at a frequency of 60 Hertz, and powers a simple series circuit consisting of a 230 Ohm resistor and a 10 microFarad capacitor. Express the circuit current and all component voltages as complex numbers in polar form.

Challenges

- How would an increase in frequency affect these parameters?

7.2.6 Capacitive voltage divider

Calculate the voltage output by this capacitive voltage divider circuit:



Challenges

- Explain why altering the frequency of the voltage source has no effect on the voltage division ratio.

7.2.7 Protective relay burdens

A type of instrument used in electric power systems called a *protective relay* receives a current input signal from the power system. That AC current passes through a set of electromagnetic coils, called *restraint coils*, which helps the relay perform its intended function. Those restraint coils within the relay impose a certain amount of impedance (called "burden") upon the current signal, the value of which depending on the system frequency and the relay's configuration. The following is a screenshot from the instruction manual for a General Electric model CFD22A high-speed differential relay:

BURDENS

The burden of the restraint coils at rated amps is given below.

RATED FREQUENCY	CONTINUOUS RATING--AMPS	ONE-SECOND RATING--AMPS	R	X	Z	VA	PF
60	5.0	220	0.040	0.057	0.070	1.75	0.57
50	5.0	220	0.034	0.048	0.059	1.47	0.58
50	1.0	42	1.14	1.64	2.00	2.00	0.57

Express the impedance of the restraint coils for this model of protective relay assuming a system frequency of 60 Hertz and the relay being configured for a continuous current rating of 5.0 Amperes. Write your answer in two different forms: *polar* and *rectangular*.

Express the impedance of the restraint coils for this model of protective relay assuming a system frequency of 50 Hertz and the relay being configured for a continuous current rating of 1.0 Amperes. Write your answer in two different forms: *polar* and *rectangular*.

Calculate the voltage dropped by the 5.0 Ampere-rated restraint coils at a signal current value of 3.0 Amperes $\angle$ 24^o and a frequency of 60 Hz.

Calculate the inductance (L) of the 5.0 Ampere-rated restraint coil based on the information you see here in the table.

Challenges

- Since this protective relay is designed to measure a current signal, one might think of it as being a special form of ammeter. What should the ideal "burden" of an ammeter be?
- Why would the *resistance* (R) value of a restraint coil vary with frequency, as it appears based on the 0.040 Ω value at 60 Hz and the 0.034 Ω value at 50 Hz?

7.2.8 Parallel RC circuit with AC voltage source

Suppose an AC voltage source outputs 30 Volts RMS at a frequency of 60 Hertz, and powers a simple parallel circuit consisting of a 410 Ohm resistor and a 22 microFarad capacitor. Express all current values as complex numbers in polar form.

Challenges

- How would an increase in frequency affect these parameters?

7.3 Diagnostic reasoning

These questions are designed to stimulate your deductive and inductive thinking, where you must apply general principles to specific scenarios (deductive) and also derive conclusions about the failed circuit from specific details (inductive). In a Socratic discussion with your instructor, the goal is for these questions to reinforce your recall and use of general circuit principles and also challenge your ability to integrate multiple symptoms into a sensible explanation of what's wrong in a circuit. Your instructor may also pose additional questions based on those assigned, in order to further challenge and sharpen your diagnostic abilities.

As always, your goal is to fully *explain* your analysis of each problem. Simply obtaining a correct answer is not good enough – you must also demonstrate sound reasoning in order to successfully complete the assignment. Your instructor's responsibility is to probe and challenge your understanding of the relevant principles and analytical processes in order to ensure you have a strong foundation upon which to build further understanding.

You will note a conspicuous lack of answers given for these diagnostic questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. The best way by far is to debate the answers with fellow students and also with the instructor during the Socratic dialogue sessions intended to be used with these learning modules. Reasoning through challenging questions with other people is an excellent tool for developing strong reasoning skills.

Another means of checking your diagnostic answers, where applicable, is to use circuit simulation software to explore the effects of faults placed in circuits. For example, if one of these diagnostic questions requires that you predict the effect of an open or a short in a circuit, you may check the validity of your work by simulating that same fault (substituting a very high resistance in place of that component for an open, and substituting a very low resistance for a short) within software and seeing if the results agree.

7.3.1 Identifying an unmarked component

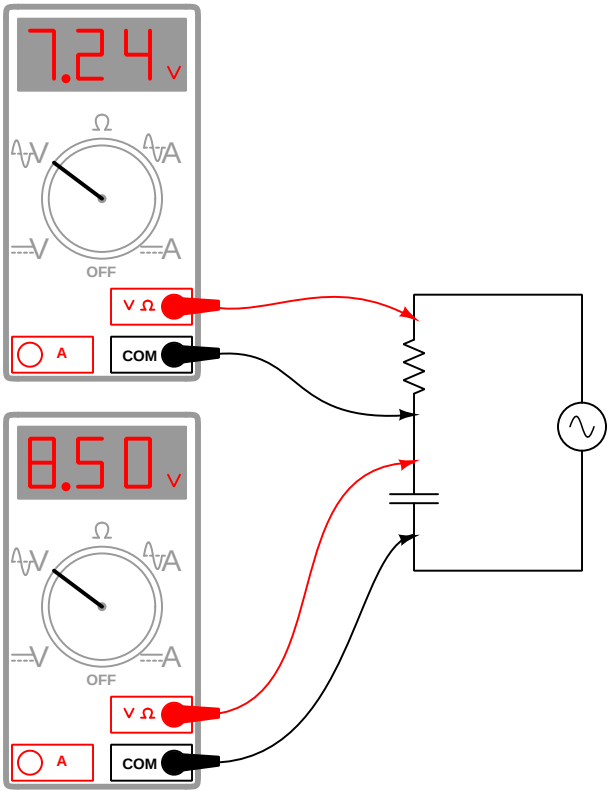
Suppose you were given a component and told it was either a resistor, an inductor, or a capacitor. The component is unmarked, and impossible to visually identify. Explain what steps you would take to electrically identify what type of component it was, and what its value was, without the use of any test equipment except a signal generator, a multimeter (capable of measuring nothing but voltage, current, and resistance), and some miscellaneous passive components (resistors, capacitors, inductors, switches, etc.). Demonstrate your technique if possible.

Challenges

- Explain how you could use similar (or identical?) techniques to test a component suspected as being damaged.

7.3.2 Incorrect voltage calculation

A student measures voltage drops in an AC circuit using two AC voltmeters and arrives at the following measurements:



Knowing that voltages add in series circuits, the student sums 7.24 Volts and 8.50 Volts to arrive at 15.74 Volts. However, this result is incorrect. Explain what the student did wrong, and then calculate the proper series-total voltage in this circuit.

Challenges

- Calculate a set of possible values for the capacitor and resistor that would generate these same voltage drops in a real circuit. Hint: you must also decide on a value of frequency for the power source.

Appendix A

Problem-Solving Strategies

The ability to solve complex problems is arguably one of the most valuable skills one can possess, and this skill is particularly important in any science-based discipline.

- Study principles, not procedures. Don't be satisfied with merely knowing how to compute solutions – learn *why* those solutions work.
- Identify what it is you need to solve, identify all relevant data, identify all units of measurement, identify any general principles or formulae linking the given information to the solution, and then identify any "missing pieces" to a solution. Annotate all diagrams with this data.
- Sketch a diagram to help visualize the problem. When building a real system, always devise a plan for that system and analyze its function *before* constructing it.
- Follow the units of measurement and meaning of every calculation. If you are ever performing mathematical calculations as part of a problem-solving procedure, and you find yourself unable to apply each and every intermediate result to some aspect of the problem, it means you don't understand what you are doing. Properly done, every mathematical result should have practical meaning for the problem, and not just be an abstract number. You should be able to identify the proper units of measurement for each and every calculated result, and show where that result fits into the problem.
- Perform "thought experiments" to explore the effects of different conditions for theoretical problems. When troubleshooting real systems, perform *diagnostic tests* rather than visually inspecting for faults, the best diagnostic test being the one giving you the most information about the nature and/or location of the fault with the fewest steps.
- Simplify the problem until the solution becomes obvious, and then use that obvious case as a model to follow in solving the more complex version of the problem.
- Check for exceptions to see if your solution is incorrect or incomplete. A good solution will work for *all* known conditions and criteria. A good example of this is the process of testing scientific hypotheses: the task of a scientist is not to find support for a new idea, but rather to *challenge* that new idea to see if it holds up under a battery of tests. The philosophical

principle of *reductio ad absurdum* (i.e. disproving a general idea by finding a specific case where it fails) is useful here.

- Work "backward" from a hypothetical solution to a new set of given conditions.
- Add quantities to problems that are qualitative in nature, because sometimes a little math helps illuminate the scenario.
- Sketch graphs illustrating how variables relate to each other. These may be quantitative (i.e. with realistic number values) or qualitative (i.e. simply showing increases and decreases).
- Treat quantitative problems as qualitative in order to discern the relative magnitudes and/or directions of change of the relevant variables. For example, try determining what happens if a certain variable were to increase or decrease before attempting to precisely calculate quantities: how will each of the dependent variables respond, by increasing, decreasing, or remaining the same as before?
- Consider limiting cases. This works especially well for qualitative problems where you need to determine which direction a variable will change. Take the given condition and magnify that condition to an extreme degree as a way of simplifying the direction of the system's response.
- Check your work. This means regularly testing your conclusions to see if they make sense. This does *not* mean repeating the same steps originally used to obtain the conclusion(s), but rather to use some other means to check validity. Simply repeating procedures often leads to *repeating the same errors* if any were made, which is why alternative paths are better.

Appendix B

Instructional philosophy

B.1 First principles of learning

- **Anyone can learn anything** given appropriate time, effort, resources, challenges, encouragement, and expectations. Dedicating time and investing effort are the student's responsibility; providing resources, challenges, and encouragement are the teacher's responsibility; high expectations are a responsibility shared by both student and teacher.

- **Transfer is not automatic.** The human mind has a natural tendency to compartmentalize information, which means the process of taking knowledge learned in one context and applying it to another usually does not come easy and therefore should never be taken for granted.

- **Learning is iterative.** The human mind rarely learns anything perfectly on the first attempt. Anticipate mistakes and plan for multiple tries to achieve full understanding, using the lessons of those mistakes as feedback to guide future attempts.

- **Information is absorbed, but understanding is created.** Facts and procedures may be memorized easily enough by repeated exposure, but the ability to reliably apply principles to novel scenarios only comes through intense personal effort. This effort is fundamentally creative in nature: casting new concepts in one's own words, running mental experiments to perceive patterns and trends, etc. These acts of making knowledge "one's own" need not be perfect in order to be effective, as the value lies in the activity and not necessarily the finished product. Depth of understanding and permanence of memory are both directly related to mental effort during the learning process – i.e. *easy learning is shallow and fleeting.*

- **Education trumps training.** There is no such thing as an entirely isolated subject, as all fields of knowledge are connected. Training is narrowly-focused and task-oriented. Education is broad-based and principle-oriented. When preparing for a life-long technical career, education beats training every time.

- **Character matters.** Poor habits are more destructive than deficits of knowledge or skill. This is especially true in collective endeavors, where a team's ability to function depends on trust between its members. Simply put, no one wants an untrustworthy person on their team. An essential component of education then, is character development.

- **People learn to be responsible by bearing responsibility.** An irresponsible person is someone who has never *had* to be responsible for anything that mattered enough to them. Just as anyone can learn anything, anyone can become responsible if the personal cost of irresponsibility becomes high enough.

- **What gets measured, gets done.** Accurate and relevant assessment of learning is key to ensuring all students learn. Therefore, it is imperative to measure what matters.

- **Failure is nothing to fear.** Every human being fails, and fails in multiple ways at multiple times. Eventual success only happens when we don't stop trying.

B.2 Proven strategies for instructors

- Assume every student is capable of learning anything they desire given the proper conditions. Treat them as capable adults by granting real responsibility and avoiding artificial incentives such as merit or demerit points.
- Create a consistent culture of high expectations across the entire program of study. Demonstrate and encourage patience, persistence, and a healthy sense of self-skepticism. Anticipate and de-stigmatize error. Teach respect for the capabilities of others as well as respect for one's own fallibility.
- Replace lecture with "inverted" instruction, where students first encounter new concepts through reading and then spend class time in Socratic dialogue with the instructor exploring those concepts and solving problems individually. There is a world of difference between observing someone solve a problem versus actually solving a problem yourself, and so the point of this form of instruction is to place students in a position where they *cannot* passively observe.
- Require students to read extensively, write about what they learn, and dialogue with you and their peers to sharpen their understanding. Apply Francis Bacon's advice that "reading maketh a full man; conference a ready man; and writing an exact man". These are complementary activities helping students expand their confidence and abilities.
- Use artificial intelligence (AI) to challenge student understanding rather than merely provide information. Find productive ways for AI to critique students' clarity of thought and of expression, for example by employing AI as a Socratic-style interlocutor or as a reviewer of students' journals. Properly applied, AI has the ability to expand student access to critical review well outside the bounds of their instructor's reach.
- Build frequent and rapid feedback into the learning process so that students know at all times how well they are learning, to identify problems early and fix them before they grow. Model the intellectual habit of self-assessing and self-correcting your own understanding (i.e. a cognitive *feedback loop*), encouraging students to do the same.
- Use "mastery" as the standard for every assessment, which means the exam or experiment or project must be done with 100% competence in order to pass. Provide students with multiple opportunity for re-tries (different versions of the assessment every time).
- Require students to devise their own hypotheses and procedures on all experiments, so that the process is truly a scientific one. Have students assess their proposed experimental procedures for risk and devise mitigations for those risks. Let nothing be pre-designed about students' experiments other than a stated task (i.e. what principle the experiment shall test) at the start and a set of demonstrable knowledge and skill objectives at the end.
- Have students build as much of their lab equipment as possible: building power sources, building test assemblies[1], and building complete working systems (no kits!). In order to provide

[1] In the program I teach, every student builds their own "Development Board" consisting of a metal chassis with DIN rail, terminal blocks, and an AC-DC power supply of their own making which functions as a portable lab environment they can use at school as well as take home.

this same "ground-up" experience for every new student, this means either previous students take their creations with them, or the systems get disassembled in preparation for the new students, or the systems grow and evolve with each new student group.

- Incorporate external accountability for you and for your students, continuously improving the curriculum and your instructional methods based on proven results. Have students regularly network with active professionals through participation in advisory committee meetings, service projects, tours, jobshadows, internships, etc. Practical suggestions include requiring students to design and build projects for external clients (e.g. community groups, businesses, different departments within the institution), and also requiring students attend all technical advisory committee meetings and to dialogue with the industry representatives at those meetings.
- Repeatedly explore difficult-to-learn concepts across multiple courses, so that students have multiple opportunities to build their understanding.
- Relate all new concepts, whenever possible, to previous concepts and to relevant physical laws. Challenge each and every student, every day, to *reason* from concept to concept and to explain the logical connections between. Challenge students to verify their conclusions by multiple approaches (e.g. double-checking their work using different methods). Ask *"Why?"* often.
- Maintain detailed records on each student's performance and share these records privately with them. These records should include academic performance as well as professionally relevant behavioral tendencies.
- Hold mandatory "check-in" meetings between all program faculty and each new student during their first term. Offer these to all other students as an option, except for any students continuing to manifest unprofessional behaviors, poor academic performance, or who have some other need for a face-to-face meeting with faculty.
- Address problems while they are small, before they grow larger. This is equally true for tutoring technical concepts as it is for helping students build professional habits.
- Build rigorous quality control into the curriculum to ensure every student masters every important concept, and that the mastery is retained over time. This includes (1) review questions added to every exam to re-assess knowledge taught in previous terms, (2) cumulative exams at the end of every term to re-assess all important concepts back to the very beginning of the program, and (3) review assessments in practical (hands-on) coursework to ensure critically-important skills were indeed taught and are still retained. What you will find by doing this is that it actually boosts retention of students by ensuring that important knowledge gets taught and is retained over long spans of time. In the absence of such quality control, student learning and retention tends to be spotty and this contributes to drop-out and failure rates later in their education.
- Finally, *never rush learning.* Education is not a race. Give your students ample time to digest complex ideas, as you continually remind yourself of just how long it took you to achieve mastery! Long-term retention and the consistently correct application of concepts are always the result of *focused effort over long periods of time* which means there are no shortcuts to learning.

B.3 Proven strategies for students

The single most important piece of advice I have for any student of any subject is to take responsibility for your own development in all areas of life including mental development. Expecting others in your life to entirely guide your own development is a recipe for disappointment. This is just as true for students enrolled in formal learning institutions as it is for auto-didacts pursuing learning entirely on their own. Learning to think in new ways is key to being able to gainfully use information, to make informed decisions about your life, and to best serve those you care about. With this in mind, I offer the following advice to students:

- **Approach all learning as valuable**. No matter what course you take, no matter who you learn from, no matter the subject, there is something useful in every learning experience. If you don't see the value of every new experience, you are not looking closely enough!
- **Continually challenge yourself**. Let other people take shortcuts and find easy answers to easy problems. The purpose of education is to stretch your mind, in order to shape it into a more powerful tool. This doesn't come by taking the path of least resistance. An excellent analogy for an empowering education is productive physical exercise: becoming stronger, more flexible, and more persistent only comes through intense personal effort.
- **Master the use of language**. This includes reading extensively, writing every day, listening closely, and speaking articulately. To a great extent language channels and empowers thought, so the better you are at wielding language the better you will be at grasping abstract concepts and articulating them not only for your benefit but for others as well.
- **Do not limit yourself to the resources given to you**. Read books that are not on the reading list. Run experiments that aren't assigned to you. Form study groups outside of class. Take an entrepreneurial approach to your own education, as though it were a business you were building for your future benefit.
- **Express and share what you learn**. Take every opportunity to teach what you have learned to others, as this will not only help them but will also strengthen your own understanding[2].
- Realize that **no one can give you understanding**, just as no one can give you physical fitness. These both must be *built*.
- **Above all, recognize that learning is hard work, and that a certain level of frustration is unavoidable**. There are times when you will struggle to grasp some of these concepts, and that struggle is a natural thing. Take heart that it will yield with persistent and varied[3] effort, and never give up! That concepts don't immediately come to you is not a sign of something wrong, but rather of something right: that you have found a worthy challenge!

[2] On a personal note, I was surprised to learn just how much my own understanding of electronics and related subjects was strengthened by becoming a teacher. When you are tasked every day with helping other people grasp complex topics, it catalyzes your own learning by giving you powerful incentives to study, to articulate your thoughts, and to reflect deeply on the process of learning.

[3] As the old saying goes, "Insanity is trying the same thing over and over again, expecting different results." If you find yourself stumped by something in the text, you should attempt a different approach. Alter the thought experiment, change the mathematical parameters, do whatever you can to see the problem in a slightly different light, and then the solution will often present itself more readily.

B.4 Design of these learning modules

"The unexamined circuit is not worth energizing" – Socrates (if he had taught electricity)

These learning modules, although useful for self-study, were designed to be used in a formal learning environment where a subject-matter expert challenges students to digest the content and exercise their critical thinking abilities in the answering of questions and in the construction and testing of working circuits. Every effort has been made to embed the following instructional and assessment philosophies within:

- The first goal of education is to enhance clear and independent thought, in order that every student reach their fullest potential in a highly complex and inter-dependent world. Robust reasoning is *always* more important than particulars of any subject matter, because its application is universal.
- Literacy is fundamental to independent learning and thought because text continues to be the most efficient way to communicate complex ideas over space and time. Those who cannot read with ease are limited in their ability to acquire knowledge and perspective.
- Articulate communication is fundamental to work that is complex and interdisciplinary.
- Faulty assumptions and poor reasoning are best corrected through challenge, not presentation. The rhetorical technique of *reductio ad absurdum* (disproving an assertion by exposing an absurdity) works well to discipline student's minds, not only to correct the problem at hand but also to learn how to detect and correct future errors.
- Important principles should be repeatedly explored and widely applied throughout a course of study, not only to reinforce their importance and help ensure their mastery, but also to showcase the interconnectedness and utility of knowledge.

These learning modules were expressly designed to be used in an "inverted" teaching environment[4] where students first read the introductory and tutorial chapters on their own, then individually attempt to answer the questions and construct working circuits according to the experiment and project guidelines. The instructor never lectures, but instead meets regularly with each individual student to review their progress, answer questions, identify misconceptions, and challenge the student to new depths of understanding through further questioning. Regular meetings between instructor and student should resemble a Socratic[5] dialogue, where questions serve as scalpels to dissect topics and expose assumptions. The student passes each module only after consistently demonstrating their ability to logically analyze and correctly apply all major concepts in each question or project/experiment. The instructor must be vigilant in probing each student's understanding to ensure they are truly *reasoning* and not just *memorizing*. This is why "Challenge" points appear throughout, as prompts for students to think deeper about topics and as starting points for instructor queries. Sometimes these challenge points require additional knowledge that hasn't been covered in the series to answer in full. This is okay, as the major purpose of the Challenges is to stimulate analysis and synthesis on the part of each student.

The instructor must possess enough mastery of the subject matter and awareness of students' reasoning to generate their own follow-up questions to practically any student response. Even completely correct answers given by the student should be challenged by the instructor for the purpose of having students practice articulating their thoughts and defending their reasoning. Conceptual errors committed by the student should be exposed and corrected not by direct instruction, but rather by reducing the errors to an absurdity[6] through well-chosen questions and thought experiments posed by the instructor. Becoming proficient at this style of instruction requires time and dedication, but the positive effects on critical thinking for both student and instructor are spectacular.

An inspection of these learning modules reveals certain unique characteristics. One of these is a bias toward thorough explanations in the tutorial chapters. Without a live instructor to explain concepts and applications to students, the text itself must fulfill this role. This philosophy results in lengthier explanations than what you might typically find in a textbook, each step of the reasoning process fully explained, including footnotes addressing common questions and concerns students raise while learning these concepts. Each tutorial seeks to not only explain each major concept in sufficient detail, but also to explain the logic of each concept and how each may be developed

[4] In a traditional teaching environment, students first encounter new information via *lecture* from an expert, and then independently apply that information via *homework*. In an "inverted" course of study, students first encounter new information via *homework*, and then independently apply that information under the scrutiny of an expert. The expert's role in lecture is to simply *explain*, but the expert's role in an inverted session is to *challenge*, *critique*, and if necessary *explain* where gaps in understanding still exist.

[5] Socrates is a figure in ancient Greek philosophy famous for his unflinching style of questioning. Although he authored no texts, he appears as a character in Plato's many writings. The essence of Socratic philosophy is to leave no question unexamined and no point of view unchallenged. While purists may argue a topic such as electric circuits is too narrow for a true Socratic-style dialogue, I would argue that the essential thought processes involved with scientific reasoning on *any* topic are not far removed from the Socratic ideal, and that students of electricity and electronics would do very well to challenge assumptions, pose thought experiments, identify fallacies, and otherwise employ the arsenal of critical thinking skills modeled by Socrates.

[6] This rhetorical technique is known by the Latin phrase *reductio ad absurdum*. The concept is to expose errors by counter-example, since only one solid counter-example is necessary to disprove a universal claim. As an example of this, consider the common misconception among beginning students of electricity that voltage cannot exist without current. One way to apply *reductio ad absurdum* to this statement is to ask how much current passes through a fully-charged battery connected to nothing (i.e. a clear example of voltage existing without current).

from "first principles". Again, this reflects the goal of developing clear and independent thought in students' minds, by showing how clear and logical thought was used to forge each concept. Students benefit from witnessing a model of clear thinking in action, and these tutorials strive to be just that.

Another feature of these learning modules is that they do not treat topics in isolation. Rather, important concepts are introduced early in the series, and appear repeatedly as stepping-stones toward other concepts in subsequent modules. This helps to avoid the "compartmentalization" of knowledge, demonstrating the inter-connectedness of concepts and simultaneously reinforcing them. Each module is fairly complete in itself, reserving the beginning of its tutorial to a review of foundational concepts.

To high standards of education,

Tony R. Kuphaldt

Appendix C

Tools used

I am indebted to the developers of many open-source software applications in the creation of these learning modules. The following is a list of these applications with some commentary on each.

You will notice a theme common to many of these applications: a bias toward *code*. Although I am by no means an expert programmer in any computer language, I understand and appreciate the flexibility offered by code-based applications where the user (you) enters commands into a plain ASCII text file, which the software then reads and processes to create the final output. Code-based computer applications are by their very nature *extensible*, while WYSIWYG (What You See Is What You Get) applications are generally limited to whatever user interface the developer makes for you.

The `GNU/Linux` computer operating system

> There is so much to be said about Linus Torvalds' `Linux` and Richard Stallman's `GNU` project. First, to credit just these two individuals is to fail to do justice to the *mob* of passionate volunteers who contributed to make this amazing software a reality. I first learned of `Linux` back in 1996, and have been using this operating system on my personal computers almost exclusively since then. It is *free*, it is completely *configurable*, and it permits the continued use of highly efficient `Unix` applications and scripting languages (e.g. shell scripts, Makefiles, `sed`, `awk`) developed over many decades. `Linux` not only provided me with a powerful computing platform, but its open design served to inspire my life's work of creating open-source educational resources.

Bram Moolenaar's `Vim` text editor

> Writing code for any code-based computer application requires a *text editor*, which may be thought of as a word processor strictly limited to outputting plain-ASCII text files. Many good text editors exist, and one's choice of text editor seems to be a deeply personal matter within the programming world. I prefer `Vim` because it operates very similarly to `vi` which is ubiquitous on `Unix`/`Linux` operating systems, and because it may be entirely operated via keyboard (i.e. no mouse required) which makes it fast to use.

Donald Knuth's TEX typesetting system

Developed in the late 1970's and early 1980's by computer scientist extraordinaire Donald Knuth to typeset his multi-volume magnum opus *The Art of Computer Programming*, this software allows the production of formatted text for screen-viewing or paper printing, all by writing plain-text code to describe how the formatted text is supposed to appear. TEX is not just a markup language for documents, but it is also a Turing-complete programming language in and of itself, allowing useful algorithms to be created to control the production of documents. Simply put, TEX *is a programmer's approach to word processing*. Since TEX is controlled by code written in a plain-text file, this means anyone may read that plain-text file to see exactly how the document was created. This openness afforded by the code-based nature of TEX makes it relatively easy to learn how other people have created their own TEX documents. By contrast, examining a beautiful document created in a conventional WYSIWYG word processor such as Microsoft `Word` suggests nothing to the reader about *how* that document was created, or what the user might do to create something similar. As Mr. Knuth himself once quipped, conventional word processing applications should be called WYSIAYG (What You See Is *All* You Get).

Leslie Lamport's LATEX extensions to TEX

Like all true programming languages, TEX is inherently extensible. So, years after the release of TEX to the public, Leslie Lamport decided to create a massive extension allowing easier compilation of book-length documents. The result was LATEX, which is the markup language used to create all ModEL module documents. You could say that TEX is to LATEX as `C` is to `C++`. This means it is permissible to use any and all TEX commands within LATEX source code, and it all still works. Some of the features offered by LATEX that would be challenging to implement in TEX include automatic index and table-of-content creation.

Tim Edwards' `Xcircuit` drafting program

This wonderful program is what I use to create all the schematic diagrams and illustrations (but not photographic images or mathematical plots) throughout the ModEL project. It natively outputs PostScript format which is a true vector graphic format (this is why the images do not pixellate when you zoom in for a closer view), and it is so simple to use that I have never had to read the manual! Object libraries are easy to create for `Xcircuit`, being plain-text files using PostScript programming conventions. Over the years I have collected a large set of object libraries useful for drawing electrical and electronic schematics, pictorial diagrams, and other technical illustrations.

`Gimp` graphic image manipulation program

Essentially an open-source clone of Adobe's `PhotoShop`, I use `Gimp` to resize, crop, and convert file formats for all of the photographic images appearing in the ModEL modules. Although `Gimp` does offer its own scripting language (called `Script-Fu`), I have never had occasion to use it. Thus, my utilization of `Gimp` to merely crop, resize, and convert graphic images is akin to using a sword to slice bread.

`SPICE` circuit simulation program

`SPICE` is to circuit analysis as TEX is to document creation: it is a form of markup language designed to describe a certain object to be processed in plain-ASCII text. When the plain-text "source file" is compiled by the software, it outputs the final result. More modern circuit analysis tools certainly exist, but I prefer `SPICE` for the following reasons: it is *free*, it is *fast*, it is *reliable*, and it is a fantastic tool for *teaching* students of electricity and electronics how to write simple code. I happen to use rather old versions of `SPICE`, version 2g6 being my "go to" application when I only require text-based output. `NGSPICE` (version 26), which is based on Berkeley `SPICE` version 3f5, is used when I require graphical output for such things as time-domain waveforms and Bode plots. In all `SPICE` example netlists I strive to use coding conventions compatible with all `SPICE` versions.

Andrew D. Hwang's `ePiX` mathematical visualization programming library

This amazing project is a `C++` library you may link to any `C/C++` code for the purpose of generating PostScript graphic images of mathematical functions. As a completely free and open-source project, it does all the plotting I would otherwise use a Computer Algebra System (CAS) such as `Mathematica` or `Maple` to do. It should be said that `ePiX` is *not* a Computer Algebra System like `Mathematica` or `Maple`, but merely a mathematical *visualization* tool. In other words, it won't determine integrals for you (you'll have to implement that in your own `C/C++` code!), but it can graph the results, and it does so beautifully. What I really admire about `ePiX` is that it is a `C++` programming library, which means it builds on the existing power and toolset available with that programming language. Mr. Hwang could have probably developed his own stand-alone application for mathematical plotting, but by creating a `C++` library to do the same thing he accomplished something much greater.

`gnuplot` mathematical visualization software

Another open-source tool for mathematical visualization is `gnuplot`. Interestingly, this tool is *not* part of Richard Stallman's GNU project, its name being a coincidence. For this reason the authors prefer "gnu" *not* be capitalized at all to avoid confusion. This is a much "lighter-weight" alternative to a spreadsheet for plotting tabular data, and the fact that it easily outputs directly to an X11 console or a file in a number of different graphical formats (including PostScript) is very helpful. I typically set my `gnuplot` output format to default (X11 on my Linux PC) for quick viewing while I'm developing a visualization, then switch to PostScript file export once the visual is ready to include in the document(s) I'm writing. As with my use of `Gimp` to do rudimentary image editing, my use of `gnuplot` only scratches the surface of its capabilities, but the important points are that it's *free* and that it *works well*.

`Python` programming language

Both Python and C++ find extensive use in these modules as instructional aids and exercises, but I'm listing Python here as a *tool* for myself because I use it almost daily as a *calculator*. If you open a Python interpreter console and type `from math import *` you can type mathematical expressions and have it return results just as you would on a hand calculator. Complex-number (i.e. *phasor*) arithmetic is similarly supported if you include the complex-math library (`from cmath import *`). Examples of this are shown in the Programming References chapter (if included) in each module. Of course, being a fully-featured programming language, Python also supports conditionals, loops, and other structures useful for calculation of quantities. Also, running in a console environment where all entries and returned values show as text in a chronologically-ordered list makes it easy to copy-and-paste those calculations to document exactly how they were performed.

Appendix D

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Appendix E

Version history

This is a list showing all significant additions, corrections, and other edits made to this learning module. Each entry is referenced by calendar date in reverse chronological order (newest version first), which appears on the front cover of every learning module for easy reference. Any contributors to this open-source document are listed here as well.

4 February 2026 – corrected a grammatical error courtesy of Wyatt Gleason.

3 September 2024 – minor edits to the Simplified Tutorial chapter regarding trigonometric addition of orthogonal phasor quantities.

28 August 2024 – divided the Introduction chapter into sections, one with recommendations for students, one with a listing of challenging concepts, and one with recommendations for instructors. Also edited image_3152 to show dashed projection lines.

31 January 2024 – emphasized in the Simplified Tutorial that the Pythagorean Theorem applied to phasor diagrams only works for representing quantities that *add* in the circuit.

21-22 December 2023 – expanded the angular velocity Full Tutorial section to include the s variable, and replaced "angular velocity" with "natural frequency" throughout that section.

18 December 2023 – added more references to $X_L = j\omega L$ and $X_C = \frac{1}{j\omega C}$ to the Full Tutorial. Also ensured that every instance of ω was referred to as "angular velocity" rather than "frequency".

13 September 2023 – added index references to scalar versus phasor quantities.

28 November 2022 – placed questions at the top of the itemized list in the Introduction chapter prompting students to devise experiments related to the tutorial content.

23 June 2022 – editing some instructor answers.

22 May 2022 – added one more comment in the Simplified Tutorial section regarding the use of calculators capable of complex-number arithmetic, and also added a "Programming References"

chapter to include instructions on how to use Python as a scientific calculator capable of complex-number arithmetic.

18 May 2022 – added a Case Tutorial section with complex-number arithmetic practice problems.

16 May 2022 – added a Case Tutorial section showing the effective use of a triangle-wave signal generator rather than a sine-wave signal generator to provide AC excitatin in simple RLC network experiments. It's not "perfect" like a sine-wave signal generator would (ideally) be, but the results are quite close to what they ought to be, usually within the tolerances of the components.

3 March 2022 – corrected a typographical error (X_C mistakenly shown as X_L). This error was pointed out to me courtesy of Ron Felix.

20 December 2021 – added another Tutorial example solution, showing a parallel network calculated "longhand" rather than using a complex-number calculator. Also made some minor formatting changes in Example #5 and corrected a few typographical errors in the rest of the text.

8 December 2021 – added a Case Tutorial section on testing custom-wound inductors.

8 May 2021 – commented out or deleted empty chapters.

21 February 2021 – added content to the Simplified Tutorial showing the R-X-Z triangle as a means to relate resistance, reactance, and impedance trigonometrically rather than by the use of complex numbers.

3 February 2021 – added content to the Introduction, and made minor edits to the Simplified Tutorial. Also added notes about the "phase reference" for an AC circuit's calculations into the example problems in the Full Tutorial.

2 February 2021 – added a Case Tutorial chapter.

30 January 2021 – minor edit to "Simple AC circuits" Quantitative question, specifying that the source in each case was the 0^o phase reference.

10 January 2021 – added new Quantitative question on simple AC circuits.

2 November 2020 – corrected typographical error in instructor notes for the "Protective relay burdens" question.

1 November 2020 – added photograph of a metal pipe being heated by induction.

5 October 2020 – significantly edited the Introduction chapter to make it more suitable as a pre-study guide and to provide cues useful to instructors leading "inverted" teaching sessions.

15 September 2020 – minor edits to Full Tutorial chapter.

5 April 2020 – included the term "insertion resistance" in Conceptual Reasoning question about phantom voltage measurement.

5 June 2019 – minor edits. Most significant was an edit to the final paragraph of the Simplified Tutorial, where I expanded the commonality between DC and AC circuit analysis from just series and parallel network principles, to *any* principle of voltage or current. This includes Ohm's Law, Kirchhoff's Laws, and network theorems.

26 May 2019 – added combinations of R and X to the Simplified Tutorial, as well as added some questions.

11 May 2019 – minor clarifying edits to Simplified Tutorial.

October 2018 – replaced "reactor" with "inductor" in one paragraph, plus numerous edits throughout the Full Tutorial made for the sake of clarity. Corrected a number of errors resulting from copy-and-pasting similar paragraphs, mostly confusing inductance and capacitance (thanks to Ron Felix for locating these errors).

August 2018 – added content to the Introduction.

July 2018 – added discussion of unit phasors to both the Simplified and Full Tutorial chapters. This lets readers know how to convert a scalar reactance value in their electronic calculator to a complex (phasor) impedance value without having to re-enter the reactance quantity.

May 2018 – added a Simplified Tutorial chapter, as well as added some phasor diagrams to the worked example circuits in the Full Tutorial.

October 2017 – document first created.

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