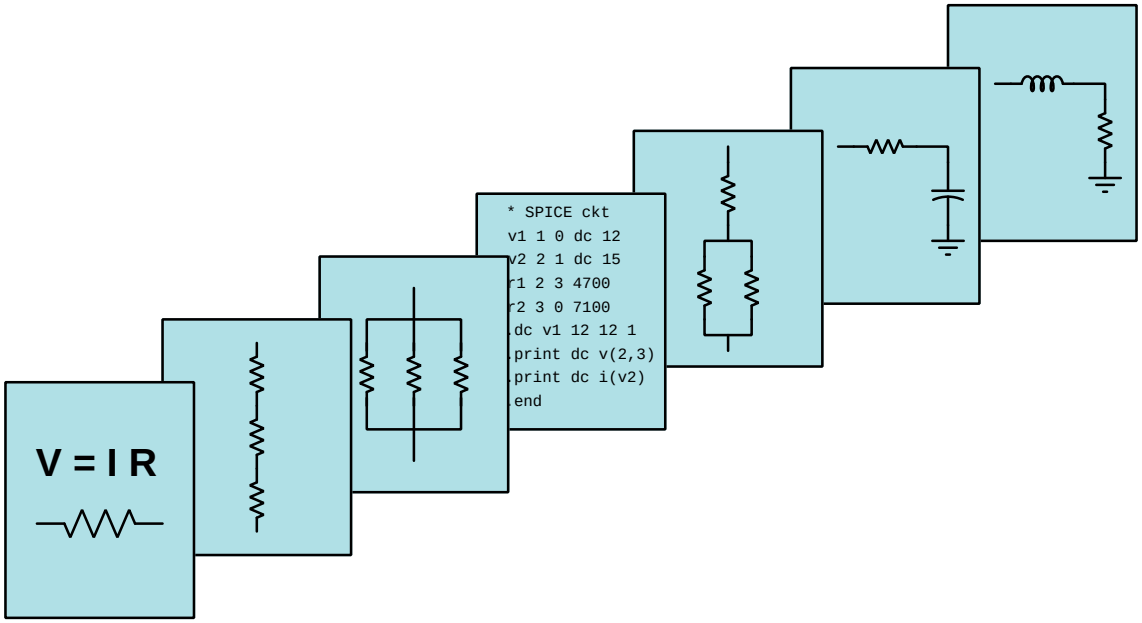
Modular Electronics Learning (ModEL) project



Trigonometry

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Chapter 1

Introduction

1.1 Recommendations for students

Trigonometry is a branch of mathematics related to triangles, and is extremely useful in a variety of technical disciplines including electric circuit analysis. Specifically, trigonometric functions describe oscillations in AC (alternating current) circuits. This module is explores basic trigonometric principles.

Important concepts related to trigonometry include **angles**, **degrees**, **radians**, **minutes** and **seconds** of degrees, the **Pythagorean Theorem**, mathematical **functions**, **ratios**, **inverse functions**, **waves**, the **Law of Sines**, and the **Law of Cosines**.

Here are some good questions to ask of yourself while studying this subject:

- What are some of the different ways we have to quantify angles?
- Why is the "radian" considered a natural unit of angular measurement?
- How are the various sides of a right triangle named, in relation to its angles?
- What are some applications for the Pythagorean Theorem?
- How are the basic trigonometric functions (sine, cosine, tangent) defined?
- What is the range of the sine function (i.e. minimum to maximum value)?
- What is the range of the cosine function (i.e. minimum to maximum value)?
- What is the range of the tangent function (i.e. minimum to maximum value)?
- How do circles relate to trigonometric functions?
- Why does the voltage output by an AC generator trace the shape of a sine wave over time?
- Does the Law of Sines apply to right triangles as well as non-right triangles?
- Does the Law of Cosines apply to right triangles as well as non-right triangles?

1.2 Challenging concepts related to trigonometry

The following list cites concepts related to this module's topic that are easily misunderstood, along with suggestions for properly understanding them:

- **Radians** – degrees are so commonplace for the measurement of angles that is strikes many students as unfathomable we may have alternative units of measurement. Interestingly, since the angular unit of the *radian* is actually based on the length of a perfect circle's radius, the unit of the radian is actually considered a *natural* expression for angles unlike the degree which is completely arbitrary in its division of one rotation into 360 equal parts. Confusion between degrees and radians also expresses itself in the use of scientific calculators, where one must carefully select the proper unit for angles before using any trigonometric functions.

- **Adjacent and Opposite definitions** – while the *hypotenuse* of a right triangle is always its longest side, how we define *adjacent* versus *opposite* depends entirely on the interior angle we happen to be referencing. In other words, the hypotenuse is defined absolutely while the opposite and adjacent sides are relative to a chosen angle. One way to help students clarify which is which is to imagine the right triangle being a top-down view of a three-sided room, with the student standing in one corner. The "opposite" side is that which lies directly opposite the room from that corner, while the "adjacent" side is the non-hypotenuse side joining at that corner.

- **Mathematical functions** – sine, cosine, tangent, and other trigonometric functions take in a single argument (an angle) and return a result based on that argument. This is different from the more familiar with arithmetic functions such as addition, subtraction, multiplication, and division which take in two or more arguments and return a single result. Functional notation is different too: for example, $\sin x$ often makes students think we're multiplying something called sin with x, but this is not the case. One helpful notational tool is to always enclose a function's argument(s) in parentheses (e.g. $\sin(x)$ rather than $\sin x$) just like computer programming languages usually do, to clearly show that the argument x is data going in to the function. Inverse functions, of course, work the same way except that for trigonometric inverse functions the argument is a ratio and the returned result is an angle. A good exploratory tool is a scientific calculator, especially when combined with a table of trigonometric functions where students can experiment with entering angles into trig functions to get ratios and vice-versa for inverse trig functions.

The *Derivations and Technical References* chapter contains a section with a table of sine, cosine, and tangent values for angles from 0 to 90^o in one-degree increments. Although electronic calculators make trig tables obsolete, the tables are still an excellent learning tool for those new to trigonometry.

1.3 Recommendations for instructors

This section lists realistic student learning outcomes supported by the content of the module as well as suggested means of assessing (measuring) student learning. The outcomes state what learners should be able to do, and the assessments are specific challenges to prove students have learned.

- **Outcome** – Demonstrate effective technical reading and writing

  Assessment – Students present their outlines of this module's instructional chapters (e.g. Case Tutorial, Tutorial, Historical References, etc.) ideally as an entry to a larger Journal document chronicling their learning. These outlines should exhibit good-faith effort at summarizing major concepts explained in the text.

  Assessment – Students present their "Conceptual Toolbox" Theory Journal entries listing important concepts applied in the Tutorial text(s). A "Conceptual Toolbox" serves as inspiration when solving problems, each concept suggesting a potential problem-solving action. Challenge students to collaboratively generate a Conceptual Toolbox for the day's topic during the class session and then use it when solving problems together, thus fostering the habit of reasoning from principles rather than recalling procedures!

- **Outcome** – Apply the concept of trigonometric functions to the calculation of right-triangle dimensions

  Assessment – complete the side length and angle determinations for a right triangle given some of the side lengths and angles; e.g. pose problems in the form of the "Unknown angles and side lengths" Quantitative Reasoning question.

- **Outcome** – Convert between radians and degrees

  Assessment – Accurately convert from one unit of angle measurement into another; e.g. pose problems in the form of the "Radian/degree angle conversions" Quantitative Reasoning question.

- **Outcome** – Prove the concept of sine and cosine waves by experiment

  Assessment – populate a computer spreadsheet with a sequential list of angle values, then use the built-in trigonometric functions to plot sine waves and cosine waves. Incorporate multiplier coefficients to alter the frequency and/or amplitude of these waves.

  Assessment – do the same using on paper, plotting points of sine and/or cosine waves manually while using either a trig table or a scientific calculator to produce the ratio values for various angle values.

Chapter 2

Tutorial

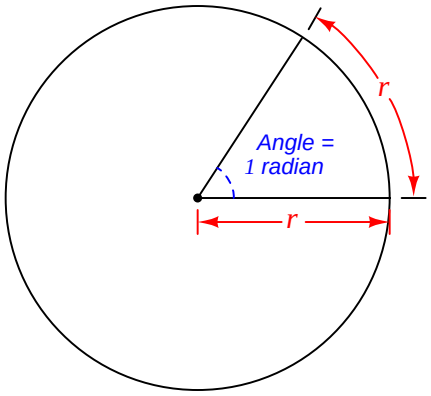
Trigonometry is the mathematical study of triangles. While this may seem at first to be overly specialized, trigonometric functions are extremely useful in the analysis of alternating-current (AC) electric circuits, and in fact for any field of study involving *waves* of any kind.

2.1 Measuring angles

An *angle* refers to a rotary displacement between two positions sharing a common center of rotation. The customary unit of measurement for angles is the *degree* there being 360 degrees in a full circle. It is also customary in mathematics that positive angles are associated with counter-clockwise displacement while negative angle values represent clockwise.

When the measurement of an angle demands precision less than a degree, we may use decimal values (e.g. 34.075 degrees) or use *minutes* and *seconds* to subdivide each degree. There are 60 minutes to every degree of rotation, and 60 seconds to every minute of rotation; not unlike there being 60 minutes of time within every hour and 60 seconds within every minute. Degrees, minutes, and seconds are represented by the superscripted symbols ^{o}, ', and " respectively. For example, 34.075^o could be alternatively expressed as $34^o4'30"$.

A *radian* is an alternative unit of measurement for angles. Whereas a "degree" is defined as an arbitrary division ($\frac{1}{360}$) of a full rotation, a "radian" is defined by the shape of a circle itself, namely the angle formed by marking one radius-length of a circle around its own circumference:



Since the circumference of a circle is equal to $2\pi r$, it means there are exactly 2π radians (and also 360 degrees) in one full circular rotation. This makes one radian approximately equal to 57.296^o, or $57^o17'44.806"$.

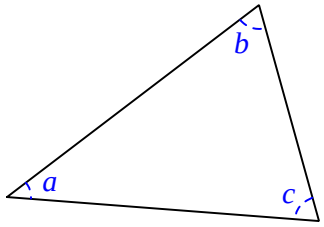
Radians are considered a "natural" unit of measurement[1] for angles, since it is based on the geometry of a perfect circle. Degrees, by comparison, are a completely arbitrary subdivision of a circle. Another unit of angular measurement – rarely used – is the *grad*, there being exactly 400 grads in a full circle.

"Scientific" electronic calculators all provide a *mode* function whereby the user gets to set the preferred unit of angular measurement. For most electrical/electronic circuit calculations the preferred unit is the *degree*, but in pure mathematical studies the preferred unit is the *radian*.

[1] In fact, radians are so natural for the measurement of angles that technically the "radian" is not even a true *unit*! Unlike other units of measurement which are either based on arbitrary artifact-standards or based on constants of nature to be applied to all manner of uses, the radian is fundamental to what a circle *is*.

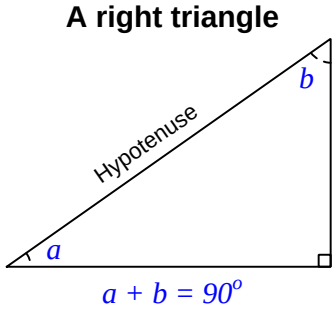
2.2 Right triangles

A *triangle* is a geometric object consisting of three line-segments joined together at their ends. The sum of the interior angles of any triangle, no matter its size or shape, always equals 180 degrees or π radians:



$a + b + c = 180^o$

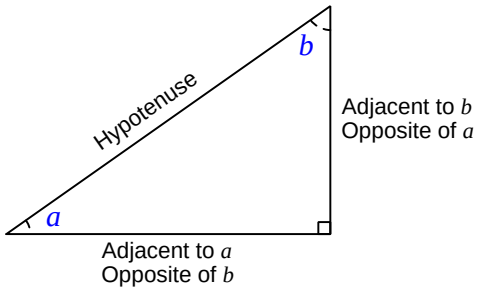
Any triangle containing one angle at 90^o ($\frac{\pi}{2}$ radians) is called a *right triangle*, with 90^o being called a *right angle*. Right triangles form the basis for most of trigonometry:



$a + b = 90^o$

Since the sum of all interior angles of a triangle must equal 180^o, and one of those angles in a right triangle is 90^o, then the sum of the other two angles (shown here as a and b) must comprise the other 90 degrees. The side of a right triangle opposite its right angle is called the *hypotenuse*.

The other two sides of a right triangle have special names as well, but their positions are relative to which of the non-right angles is referenced. The *opposite* side of a right triangle is always directly opposite the referenced angle, while the *adjacent* side of a right triangle meets at the vertex of the referenced angle.



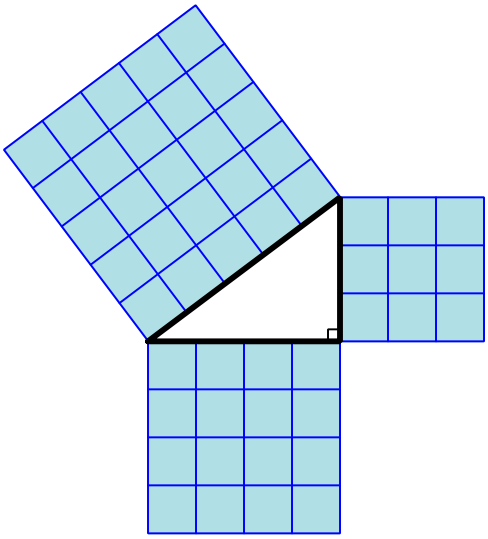
2.3 The Pythagorean Theorem

An interesting property unique to *right* triangles is the *Pythagorean Theorem*, which states that the square of the hypotenuse's length is equal to the sum of the squares of the other two sides' lengths. Expressed as a formula:

$$A^2 + B^2 = C^2 \qquad C = \sqrt{A^2 + B^2}$$

Where,
A = Length of side opposite angle a
B = Length of side opposite angle b
C = Length of hypotenuse

An interesting geometric illustration of the Pythagorean Theorem appears below, showing the square of each side length as the area of a square-shaped grid. Note how the number of internal squares held within each of the three large squares forms a sum, $9 + 16 = 25$, which is what we get for a right triangle consisting of side lengths[2] 3, 4, and 5:



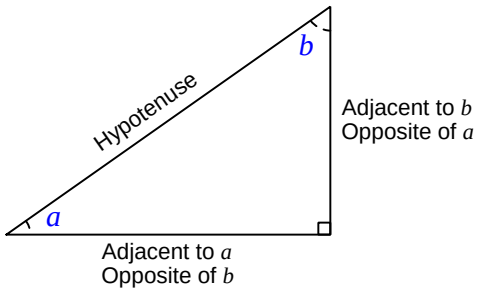
Using the Pythagorean Theorem, we may solve for the length of any side of a right triangle knowing the side-lengths of the other two.

$$C = \sqrt{A^2 + B^2} \qquad A = \sqrt{C^2 - B^2} \qquad B = \sqrt{C^2 - A^2}$$

[2]The values of 3, 4, and 5 happen to be the smallest whole numbers satisfying the Pythagorean Theorem.

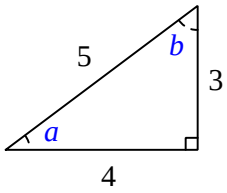
2.4 Angular functions

While the Pythagorean Theorem relates the lengths of a right triangle's sides to one another, another set of functions relate those side-lengths to their associated angles. The three basic trigonometric functions[3] are *sine*, *cosine*, and *tangent*, abbreviated sin, cos, and tan:



$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Let us return to our simple 3-4-5 right triangle to illustrate:



$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \sin a = \frac{3}{5} = 0.6 \qquad \sin b = \frac{4}{5} = 0.8$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \cos a = \frac{4}{5} = 0.8 \qquad \cos b = \frac{3}{5} = 0.6$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} \qquad \tan a = \frac{3}{4} = 0.75 \qquad \tan b = \frac{4}{3} = 1.\overline{33}$$

We would use the sine function, for example, to compute the ratio between the opposite and hypotenuse lengths given a known angle value. Cosine would be useful for computing the ratio of adjacent to hypotenuse side-lengths given a known angle. With tangent it's the ratio of opposite to adjacent we could compute from a given angle.

[3] A popular mnemonic device for remembering these relationships is the made-up word *SOHCAHTOA*, pronounced "Soe-Kah-Toe-Ah", which reminds us that Sine is the ratio of Opposite to Hypotenuse (SOH), Cosine is the ratio of Adjacent to Hypotenuse (CAH), and that Tangent is the ratio of Opposite to Adjacent (TOA).

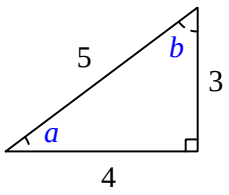
Each of these trigonometric functions has its corresponding *inverse*, or *arc-function*. These are used to compute the value of an angle given the side-lengths.

$$\sin^{-1}\left(\frac{\text{Opposite}}{\text{Hypotenuse}}\right) = \theta \qquad \cos^{-1}\left(\frac{\text{Adjacent}}{\text{Hypotenuse}}\right) = \theta \qquad \tan^{-1}\left(\frac{\text{Opposite}}{\text{Adjacent}}\right) = \theta$$

Please note that the $^{-1}$ superscript does *not* refer to reciprocation as it does when applied to a *variable*. When applied to a *function* the $^{-1}$ superscript represents mathematical inversion, i.e. that which un-does the original function. An alternative way to express these inverse functions is to call them *arcsine*, *arccosine*, and *arctangent*, respectively:

$$\arcsin\left(\frac{\text{Opposite}}{\text{Hypotenuse}}\right) = \theta \qquad \arccos\left(\frac{\text{Adjacent}}{\text{Hypotenuse}}\right) = \theta \qquad \arctan\left(\frac{\text{Opposite}}{\text{Adjacent}}\right) = \theta$$

Returning to our 3-4-5 right triangle once more, we may show multiple ways to calculate the values of the two non-right angles:



$$a = \arcsin\left(\frac{3}{5}\right) = \arccos\left(\frac{4}{5}\right) = \arctan\left(\frac{3}{4}\right) = 36.8699^o$$

$$b = \arcsin\left(\frac{4}{5}\right) = \arccos\left(\frac{3}{5}\right) = \arctan\left(\frac{4}{3}\right) = 53.1301^o$$

Of course, these functions work on *any* right triangle, not just our simple 3-4-5 example. So long as the triangle in question has one 90^o angle, we know we may freely apply the Pythagorean Theorem and/or these angular functions to the calculation of angles and/or side lengths.

Modern "scientific" calculators provide sine, cosine, and tangent functions as well as their respective arc-functions. Prior to the advent of electronic calculators, though, people relied on tables of pre-computed numbers for these functions. While practically obsolete, such "trig tables" are nevertheless still useful as educational tools because they allow one to survey the range of each function at a glance for the domain of angle values encompassing a full circle. This document contains a simple set of sine, cosine, and tangent tables in the Technical References chapter, section 3.1 beginning on page 18.

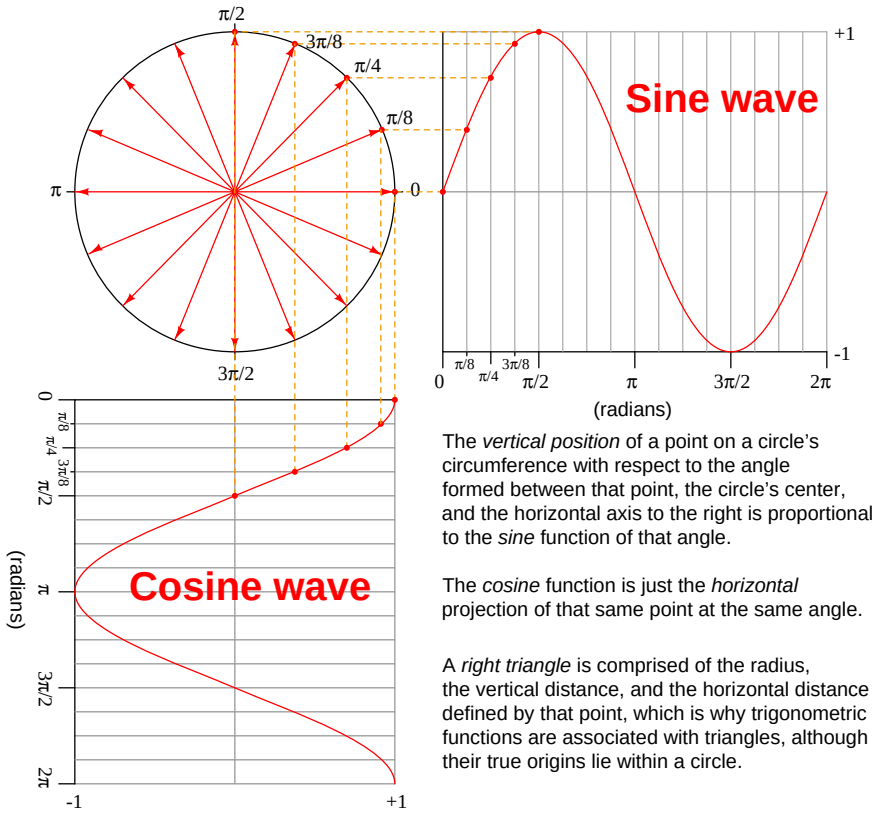
In the study of trigonometry we also encounter other angle-based functions which are nothing more than the reciprocals of sine, cosine, and tangent – these are *cosecant*, *secant*, and *cotangent*, respectively:

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{Hypotenuse}}{\text{Opposite}} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{\text{Hypotenuse}}{\text{Adjacent}} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\text{Adjacent}}{\text{Opposite}}$$

And, of course, each of these reciprocal functions has its inverse or "arc"-function as well. In most practical applications of trigonometry we rarely see these functions, but simply use sine, cosine, and tangent.

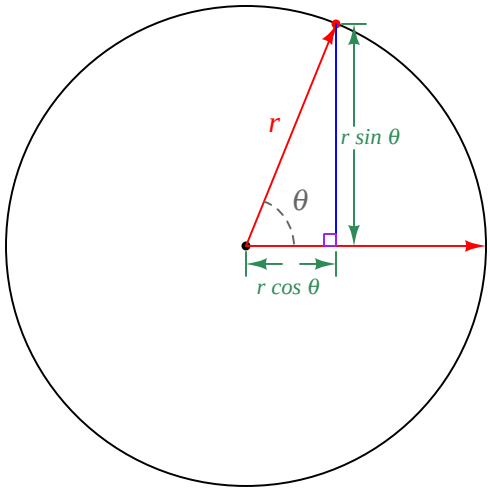
2.5 Sinusoidal waves

The angular functions of a right triangle are really related to *circles*, because the sine and cosine of any given angle really represents the projection of a point along a circle's circumference corresponding to that angle. If we plot the sine and cosine values for angles spanning the rotation of a full circle, we obtain a pair of *waves* having the same basic shape but shifted from one another by a quarter-rotation (90^o, or $\frac{\pi}{2}$ radians):



For a *unit circle* where the radius has a length of one (1), the sine of a given angle represents the vertical position of the point on the circle's circumference while the cosine of that same angle represents the horizontal position of that same point.

We may see the relationship between right triangles and circles very clearly if we inscribe such a triangle inside a circle:



The length of the inscribed triangle's vertical side with relation to the angle θ is the product of the circle's radius and the sine of that angle: $r \sin \theta$. Similarly, the length of the inscribed triangle's horizontal side to the angle θ involves the cosine of that angle: $r \cos \theta$.

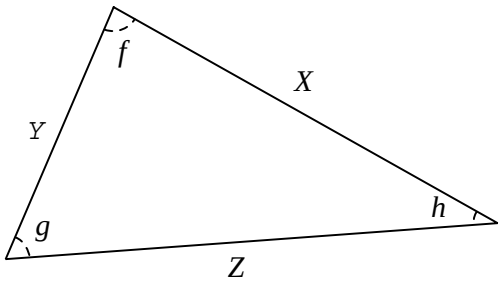
Visualizing the side-lengths of an inscribed triangle while angle θ progresses from a value of zero (0) to one full turn of the circle enables us to understand why the sine and cosine functions oscillate between -1 and $+1$ over this interval. When $\theta = 0$ the triangle is "flat" and consists merely of a horizontal line equal in length to the circle's radius; at this angle the value of the cosine function is one (1) while the sine function is zero (0). As θ increases to equal 90^o or $\frac{\pi}{2}$ radians, the triangle "flattens" in the vertical direction, making the sine function have a value of one (1) and the cosine function a value of zero (0) at this angle. As θ increases to equal 180^o or π radians, the triangle is once again "flat" but this time as a horizontal line to the *left* of the circle's center, making the cosine function have a value of negative one (-1) and the sine function zero (0). Moving on to 270^o or $\frac{3\pi}{2}$ radians, the "flattened" triangle now forms a vertical line below the circle's center which makes the sine function have a value of negative one (-1) and the cosine function a value of zero (0).

Sinusoidal waves are ubiquitous in nature, and they appear frequently in electric circuits as well. As it so happens, when any system *resonates* at its natural frequency of motion, the pattern it traces will be sinusoidal[4] in shape. In AC electric power systems the reason why voltage and current follow a sinusoidal trajectory over time is due to the way AC generators are built: by rotating a magnetic field past a set of fixed coils of wire (called "windings"). The rate-of-change of magnetic flux experienced by these wire coils naturally grows and decays according to the pattern of a sinusoid precisely because the magnetic field rotates in a *circular* manner past those coils, and therefore the voltage induced in these coils follows that same pattern.

[4] "Sinusoidal" simply means "like the shape of a sine wave".

2.6 Non-right triangle properties

Although angle-based functions such as sine and cosine are defined by the side-lengths of right triangles, we also find them useful for some non-right triangle applications as well. The first and simplest application is the *Law of Sines*, applying to *any* type of triangle. This law states that the ratio of any side-length of any triangle to the sine of its opposite angle is the same as for any other side and opposite angle in that triangle:



$$\frac{X}{\sin g} = \frac{Y}{\sin h} = \frac{Z}{\sin f}$$

Of course, the Law of Sines also works for right triangles, since this is a universal law for triangles and right triangles are merely special cases of triangles. In the case of a right triangle, the ratio of the longest side (the hypotenuse) to the sine of the largest angle ($\sin 90^o$) is simply the length of the hypotenuse, since $\sin 90^o = 1$.

The *Law of Cosines* similarly applies to any and all triangles, although its form is more complex than the Law of Sines. It states that the square of any given side-length is equal to the sum of other side-lengths' squares minus two times each of the other side lengths and the cosine of its opposite angle. Illustrating on the same triangle shown above:

$$X^2 = Y^2 + Z^2 - 2YZ\cos g$$

$$Y^2 = X^2 + Z^2 - 2XZ\cos h$$

$$Z^2 = X^2 + Y^2 - 2XY\cos f$$

Chapter 3

Derivations and Technical References

This chapter is where you will find mathematical derivations too detailed to include in the tutorial, and/or tables and other technical reference material.

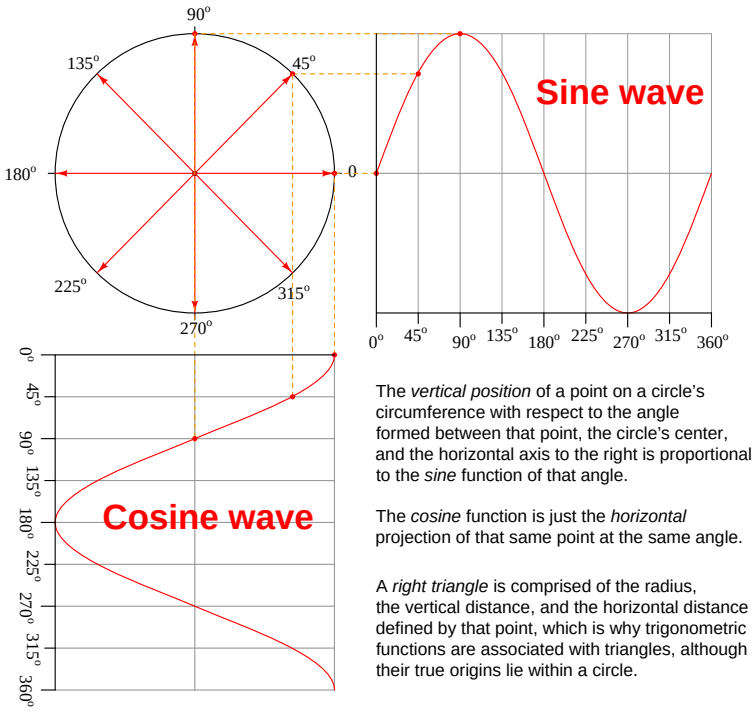
3.1 Table of sine, cosine, and tangent values

Angle	Sine	Cosine	Tangent	– – – –	Angle	Sine	Cosine	Tangent
0^o	0.000	1.000	0.000		45^o	0.707	0.707	1.000
1^o	0.017	1.000	0.017		46^o	0.719	0.695	1.036
2^o	0.035	0.999	0.035		47^o	0.731	0.682	1.072
3^o	0.052	0.999	0.052		48^o	0.743	0.669	1.111
4^o	0.070	0.998	0.070		49^o	0.755	0.656	1.150
5^o	0.087	0.996	0.087		50^o	0.766	0.643	1.192
6^o	0.105	0.995	0.105		51^o	0.777	0.629	1.235
7^o	0.122	0.993	0.123		52^o	0.788	0.616	1.280
8^o	0.139	0.990	0.141		53^o	0.799	0.602	1.327
9^o	0.156	0.988	0.158		54^o	0.809	0.588	1.376
10^o	0.174	0.985	0.176		55^o	0.819	0.574	1.428
11^o	0.191	0.982	0.194		56^o	0.829	0.559	1.483
12^o	0.208	0.978	0.213		57^o	0.839	0.545	1.540
13^o	0.225	0.974	0.231		58^o	0.848	0.530	1.600
14^o	0.242	0.970	0.249		59^o	0.857	0.515	1.664
15^o	0.259	0.966	0.268		60^o	0.866	0.500	1.732
16^o	0.276	0.961	0.287		61^o	0.875	0.485	1.804
17^o	0.292	0.956	0.306		62^o	0.883	0.469	1.881
18^o	0.309	0.951	0.325		63^o	0.891	0.454	1.963
19^o	0.326	0.946	0.344		64^o	0.899	0.438	2.050
20^o	0.342	0.940	0.364		65^o	0.906	0.423	2.145
21^o	0.358	0.934	0.384		66^o	0.914	0.407	2.246
22^o	0.375	0.927	0.404		67^o	0.921	0.391	2.356
23^o	0.391	0.921	0.424		68^o	0.927	0.375	2.475
24^o	0.407	0.914	0.445		69^o	0.934	0.358	2.605
25^o	0.423	0.906	0.466		70^o	0.940	0.342	2.747
26^o	0.438	0.899	0.488		71^o	0.946	0.326	2.904
27^o	0.454	0.891	0.510		72^o	0.951	0.309	3.078
28^o	0.469	0.883	0.532		73^o	0.956	0.292	3.271
29^o	0.485	0.875	0.554		74^o	0.961	0.276	3.487
30^o	0.500	0.866	0.577		75^o	0.966	0.259	3.732

Angle	**Sine**	**Cosine**	**Tangent**	– – – –	**Angle**	**Sine**	**Cosine**	**Tangent**
31^o	0.515	0.857	0.601		76^o	0.970	0.242	4.011
32^o	0.530	0.848	0.625		77^o	0.974	0.225	4.331
33^o	0.545	0.839	0.649		78^o	0.978	0.208	4.705
34^o	0.559	0.829	0.675		79^o	0.982	0.191	5.145
35^o	0.574	0.819	0.700		80^o	0.985	0.174	5.671
36^o	0.588	0.809	0.727		81^o	0.988	0.156	6.314
37^o	0.602	0.799	0.754		82^o	0.990	0.139	7.115
38^o	0.616	0.788	0.781		83^o	0.993	0.122	8.144
39^o	0.629	0.777	0.810		84^o	0.995	0.105	9.514
40^o	0.643	0.766	0.839		85^o	0.996	0.087	11.430
41^o	0.656	0.755	0.869		86^o	0.998	0.070	14.301
42^o	0.669	0.743	0.900		87^o	0.999	0.052	19.081
43^o	0.682	0.731	0.933		88^o	0.999	0.035	28.636
44^o	0.695	0.719	0.966		89^o	1.000	0.017	57.290
					90^o	1.000	0.000	∞

3.2 Trigonometric identities

3.2.1 The Unit Circle



The *vertical position* of a point on a circle's circumference with respect to the angle formed between that point, the circle's center, and the horizontal axis to the right is proportional to the *sine* function of that angle.

The *cosine* function is just the *horizontal* projection of that same point at the same angle.

A *right triangle* is comprised of the radius, the vertical distance, and the horizontal distance defined by that point, which is why trigonometric functions are associated with triangles, although their true origins lie within a circle.

3.2.2 Pythagorean identity for sine and cosine

$$\sin^2 x + \cos^2 x = 1$$

3.2.3 Odd and even functions

Any mathematical function $f(x)$ is considered *odd* if $f(-x) = -f(x)$, as is the case with polynomial functions having only odd degree terms (e.g. $2x^7 + 8x^5 - x^3 + 9x$). Conversely, any mathematical function is considered *even* if $f(-x) = f(x)$, as with polynomial functions having only even (or zero) degree terms (e.g. $x^6 - 6x^4 + 3x^2 - 8$). The sine function is considered odd while the cosine function is even: for example, $\sin(-10) = -\sin(10)$ while $\cos(-10) = \cos(10)$.

$$\sin(-x) = -\sin(x) \qquad \cos(-x) = \cos(x)$$

3.2.4 Sums and differences of angles

$$\sin(x \pm y) = (\sin x \cos y) \pm (\cos x \sin y) \qquad \cos(x \pm y) = (\cos x \cos y) \mp (\sin x \sin y)$$

3.2.5 Products and sums of sine and cosine functions

$$\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2} \qquad \cos x \cos y = \frac{\cos(x-y) + \cos(x+y)}{2}$$

$$\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2} \qquad \cos x \sin y = \frac{\sin(x+y) - \sin(x-y)}{2}$$

$$\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

3.2.6 Squares of sine and cosine functions

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

3.2.7 Doubled angles

$$\sin(2x) = 2 \sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

3.2.8 Halved angles

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{2}} \qquad \cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + \cos x}{2}}$$

Chapter 4

Questions

This learning module, along with all others in the ModEL collection, is designed to be used in an inverted instructional environment where students independently read[1] the tutorials and attempt to answer questions on their own *prior* to the instructor's interaction with them. In place of lecture[2], the instructor engages with students in Socratic-style dialogue, probing and challenging their understanding of the subject matter through inquiry.

Answers are not provided for questions within this chapter, and this is by design. Solved problems may be found in the Tutorial and Derivation chapters, instead. The goal here is *independence*, and this requires students to be challenged in ways where others cannot think for them. Remember that you always have the tools of *experimentation* and *computer simulation* (e.g. SPICE) to explore concepts!

The following lists contain ideas for Socratic-style questions and challenges. Upon inspection, one will notice a strong theme of *metacognition* within these statements: they are designed to foster a regular habit of examining one's own thoughts as a means toward clearer thinking. As such these sample questions are useful both for instructor-led discussions as well as for self-study.

[1] Technical reading is an essential academic skill for any technical practitioner to possess for the simple reason that the most comprehensive, accurate, and useful information to be found for developing technical competence is in textual form. Technical careers in general are characterized by the need for continuous learning to remain current with standards and technology, and therefore any technical practitioner who cannot read well is handicapped in their professional development. An excellent resource for educators on improving students' reading prowess through intentional effort and strategy is the book textitReading For Understanding – How Reading Apprenticeship Improves Disciplinary Learning in Secondary and College Classrooms by Ruth Schoenbach, Cynthia Greenleaf, and Lynn Murphy.

[2] Lecture is popular as a teaching method because it is easy to implement: any reasonably articulate subject matter expert can talk to students, even with little preparation. However, it is also quite problematic. A good lecture always makes complicated concepts seem easier than they are, which is bad for students because it instills a false sense of confidence in their own understanding; reading and re-articulation requires more cognitive effort and serves to verify comprehension. A culture of teaching-by-lecture fosters a debilitating dependence upon direct personal instruction, whereas the challenges of modern life demand independent and critical thought made possible only by gathering information and perspectives from afar. Information presented in a lecture is ephemeral, easily lost to failures of memory and dictation; text is forever, and may be referenced at any time.

General challenges following tutorial reading

- Summarize as much of the text as you can in one paragraph of your own words. A helpful strategy is to explain ideas as you would for an intelligent child: as simple as you can without compromising too much accuracy.
- Simplify a particular section of the text, for example a paragraph or even a single sentence, so as to capture the same fundamental idea in fewer words.
- Where did the text make the most sense to you? What was it about the text's presentation that made it clear?
- Identify where it might be easy for someone to misunderstand the text, and explain why you think it could be confusing.
- Identify any new concept(s) presented in the text, and explain in your own words.
- Identify any familiar concept(s) such as physical laws or principles applied or referenced in the text.
- Devise a proof of concept experiment demonstrating an important principle, physical law, or technical innovation represented in the text.
- Devise an experiment to disprove a plausible misconception.
- Did the text reveal any misconceptions you might have harbored? If so, describe the misconception(s) and the reason(s) why you now know them to be incorrect.
- Describe any useful problem-solving strategies applied in the text.
- Devise a question of your own to challenge a reader's comprehension of the text.

General follow-up challenges for assigned problems

- Identify where any fundamental laws or principles apply to the solution of this problem, especially before applying any mathematical techniques.
- Devise a thought experiment to explore the characteristics of the problem scenario, applying known laws and principles to mentally model its behavior.
- Describe in detail your own strategy for solving this problem. How did you identify and organized the given information? Did you sketch any diagrams to help frame the problem?
- Is there more than one way to solve this problem? Which method seems best to you?
- Show the work you did in solving this problem, even if the solution is incomplete or incorrect.
- What would you say was the most challenging part of this problem, and why was it so?
- Was any important information missing from the problem which you had to research or recall?
- Was there any extraneous information presented within this problem? If so, what was it and why did it not matter?
- Examine someone else's solution to identify where they applied fundamental laws or principles.
- Simplify the problem from its given form and show how to solve this simpler version of it. Examples include eliminating certain variables or conditions, altering values to simpler (usually whole) numbers, applying a limiting case (i.e. altering a variable to some extreme or ultimate value).
- For quantitative problems, identify the real-world meaning of all intermediate calculations: their units of measurement, where they fit into the scenario at hand. Annotate any diagrams or illustrations with these calculated values.
- For quantitative problems, try approaching it qualitatively instead, thinking in terms of "increase" and "decrease" rather than definite values.
- For qualitative problems, try approaching it quantitatively instead, proposing simple numerical values for the variables.
- Were there any assumptions you made while solving this problem? Would your solution change if one of those assumptions were altered?
- Identify where it would be easy for someone to go astray in attempting to solve this problem.
- Formulate your own problem based on what you learned solving this one.

General follow-up challenges for experiments or projects

- In what way(s) was this experiment or project easy to complete?
- Identify some of the challenges you faced in completing this experiment or project.

- Show how thorough documentation assisted in the completion of this experiment or project.
- Which fundamental laws or principles are key to this system's function?
- Identify any way(s) in which one might obtain false or otherwise misleading measurements from test equipment in this system.
- What will happen if (component X) fails (open/shorted/etc.)?
- What would have to occur to make this system unsafe?

4.1 Conceptual reasoning

These questions are designed to stimulate your analytic and synthetic thinking[3]. In a Socratic discussion with your instructor, the goal is for these questions to prompt an extended dialogue where assumptions are revealed, conclusions are tested, and understanding is sharpened. Your instructor may also pose additional questions based on those assigned, in order to further probe and refine your conceptual understanding.

Questions that follow are presented to challenge and probe your understanding of various concepts presented in the tutorial. These questions are intended to serve as a guide for the Socratic dialogue between yourself and the instructor. Your instructor's task is to ensure you have a sound grasp of these concepts, and the questions contained in this document are merely a means to this end. Your instructor may, at his or her discretion, alter or substitute questions for the benefit of tailoring the discussion to each student's needs. The only absolute requirement is that each student is challenged and assessed at a level equal to or greater than that represented by the documented questions.

It is far more important that you convey your *reasoning* than it is to simply convey a correct answer. For this reason, you should refrain from researching other information sources to answer questions. What matters here is that *you* are doing the thinking. If the answer is incorrect, your instructor will work with you to correct it through proper reasoning. A correct answer without an adequate explanation of how you derived that answer is unacceptable, as it does not aid the learning or assessment process.

You will note a conspicuous lack of answers given for these conceptual questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. The best way by far is to debate the answers with fellow students and also with the instructor during the Socratic dialogue sessions intended to be used with these learning modules. Reasoning through challenging questions with other people is an excellent tool for developing strong reasoning skills.

Another means of checking your conceptual answers, where applicable, is to use circuit simulation software to explore the effects of changes made to circuits. For example, if one of these conceptual questions challenges you to predict the effects of altering some component parameter in a circuit, you may check the validity of your work by simulating that same parameter change within software and seeing if the results agree.

[3] *Analytical* thinking involves the "disassembly" of an idea into its constituent parts, analogous to dissection. *Synthetic* thinking involves the "assembly" of a new idea comprised of multiple concepts, analogous to construction. Both activities are high-level cognitive skills, extremely important for effective problem-solving, necessitating frequent challenge and regular practice to fully develop.

4.1.1 Reading outline and reflections

"Reading maketh a full man; conference a ready man; and writing an exact man" – Francis Bacon

Francis Bacon's advice is a blueprint for effective education: reading provides the learner with knowledge, writing focuses the learner's thoughts, and critical dialogue equips the learner to confidently communicate and apply their learning. Independent acquisition and application of knowledge is a powerful skill, well worth the effort to cultivate. To this end, students should read these educational resources closely, journal their own reflections on the reading, and discuss in detail their findings with classmates and instructor(s). You should be able to do all of the following after reading any instructional text:

☑ Briefly SUMMARIZE THE TEXT in the form of a journal entry documenting your learning as you progress through the course of study. Share this summary in dialogue with your classmates and instructor. Journaling is an excellent self-test of thorough reading because you cannot clearly express what you have not read or did not comprehend.

☑ Demonstrate ACTIVE READING STRATEGIES, including verbalizing your impressions as you read, simplifying long passages to convey the same ideas using fewer words, annotating text and illustrations with your own interpretations, working through mathematical examples shown in the text, cross-referencing passages with relevant illustrations and/or other passages, identifying problem-solving strategies applied by the author, etc. Technical reading is a special case of problem-solving, and so these strategies work precisely because they help solve any problem: paying attention to your own thoughts (metacognition), eliminating unnecessary complexities, identifying what makes sense, paying close attention to details, drawing connections between separated facts, and noting the successful strategies of others.

☑ Identify IMPORTANT THEMES, especially GENERAL LAWS and PRINCIPLES, expounded in the text and express them in the simplest of terms as though you were teaching an intelligent child. This emphasizes connections between related topics and develops your ability to communicate complex ideas to anyone.

☑ Form YOUR OWN QUESTIONS based on the reading, and then pose them to your instructor and classmates for their consideration. Anticipate both correct and incorrect answers, the incorrect answer(s) assuming one or more plausible misconceptions. This helps you view the subject from different perspectives to grasp it more fully.

☑ Devise EXPERIMENTS to test claims presented in the reading, or to disprove misconceptions. Predict possible outcomes of these experiments, and evaluate their meanings: what result(s) would confirm, and what would constitute disproof? Running mental simulations and evaluating results is essential to scientific and diagnostic reasoning.

☑ Specifically identify any points you found CONFUSING. The reason for doing this is to help diagnose misconceptions and overcome barriers to learning.

4.1.2 Foundational concepts

Correct analysis and diagnosis of electric circuits begins with a proper understanding of some basic concepts. The following is a list of some important concepts referenced in this module's full tutorial. Define each of them in your own words, and be prepared to illustrate each of these concepts with a description of a practical example and/or a live demonstration.

Variable

Superscript

Graphing as a problem-solving strategy

Inverse functions

Radius

Circumference

Angles

Degree

Radian

Right angle

Right triangle

Hypotenuse

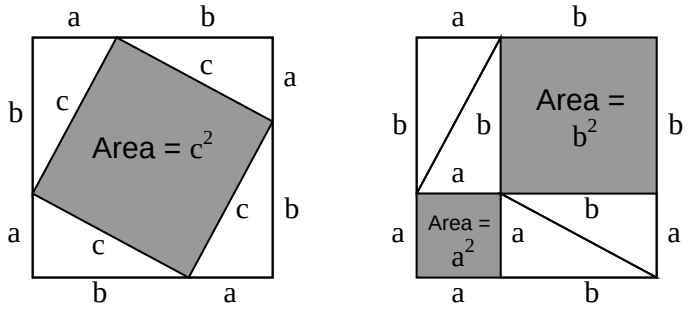
Adjacent

Opposite

Minutes and Seconds

4.1.3 Geometric proof of the Pythagorean Theorem

Geometry may be used to elegantly prove the Pythagorean Theorem. Shown here are two squares of equal area, each with four right triangles embedded within, occupying area of their own:



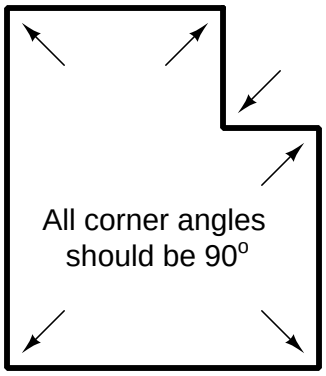
Explain how these two figures prove the Pythagorean Theorem, where $a^2 + b^2 = c^2$.

Challenges

- Write an equation for the area of each large square.

4.1.4 Squaring a house foundation

Suppose you were laying out the forms for a rectangular house foundation, and had to ensure the corners were "square" (i.e. exactly 90^o):



How could you ensure this, using trigonometry? How would you ensure this if you had no calculator with you to calculate trig functions, and no protractor or framing square with you to measure angles?

Challenges

- What important assumption(s) is/are implicit in your technique?

4.2 Quantitative reasoning

These questions are designed to stimulate your computational thinking. In a Socratic discussion with your instructor, the goal is for these questions to reveal your mathematical approach(es) to problem-solving so that good technique and sound reasoning may be reinforced. Your instructor may also pose additional questions based on those assigned, in order to observe your problem-solving firsthand.

Mental arithmetic and estimations are strongly encouraged for all calculations, because without these abilities you will be unable to readily detect errors caused by calculator misuse (e.g. keystroke errors).

You will note a conspicuous lack of answers given for these quantitative questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. My advice is to use circuit simulation software such as SPICE to check the correctness of quantitative answers. Refer to those learning modules within this collection focusing on SPICE to see worked examples which you may use directly as practice problems for your own study, and/or as templates you may modify to run your own analyses and generate your own practice problems.

Completely worked example problems found in the Tutorial may also serve as "test cases[4]" for gaining proficiency in the use of circuit simulation software, and then once that proficiency is gained you will never need to rely[5] on an answer key!

[4] In other words, set up the circuit simulation software to analyze the same circuit examples found in the Tutorial. If the simulated results match the answers shown in the Tutorial, it confirms the simulation has properly run. If the simulated results disagree with the Tutorial's answers, something has been set up incorrectly in the simulation software. Using every Tutorial as practice in this way will quickly develop proficiency in the use of circuit simulation software.

[5] This approach is perfectly in keeping with the instructional philosophy of these learning modules: *teaching students to be self-sufficient thinkers*. Answer keys can be useful, but it is even more useful to your long-term success to have a set of tools on hand for checking your own work, because once you have left school and are on your own, there will no longer be "answer keys" available for the problems you will have to solve.

4.2.1 Miscellaneous physical constants

Note: constants shown in **bold** type are *exact*, not approximations. Values inside of parentheses show one standard deviation (σ) of uncertainty in the final digits: for example, the magnetic permeability of free space value given as $1.25663706212(19) \times 10^{-6}$ H/m represents a center value (i.e. the location parameter) of $1.25663706212 \times 10^{-6}$ Henrys per meter with one standard deviation of uncertainty equal to $0.0000000000019 \times 10^{-6}$ Henrys per meter.

Avogadro's number (N_A) = **6.02214076** $\times 10^{23}$ **per mole** (mol^{-1})

Boltzmann's constant (k) = **1.380649** $\times 10^{-23}$ **Joules per Kelvin** (J/K)

Electronic charge (e) = **1.602176634** $\times 10^{-19}$ **Coulomb** (C)

Faraday constant (F) = **96,485.33212...** $\times 10^{4}$ **Coulombs per mole** (C/mol)

Magnetic permeability of free space (μ_0) = $1.25663706212(19) \times 10^{-6}$ Henrys per meter (H/m)

Electric permittivity of free space (ϵ_0) = $8.8541878128(13) \times 10^{-12}$ Farads per meter (F/m)

Characteristic impedance of free space (Z_0) = 376.730313668(57) Ohms (Ω)

Gravitational constant (G) = $6.67430(15) \times 10^{-11}$ cubic meters per kilogram-seconds squared ($m^3/kg\text{-}s^2$)

Molar gas constant (R) = **8.314462618... Joules per mole-Kelvin** (J/mol-K) = 0.08205746(14) liters-atmospheres per mole-Kelvin

Planck constant (h) = **6.62607015** $\times 10^{-34}$ **joule-seconds** (J-s)

Stefan-Boltzmann constant (σ) = **5.670374419...** $\times 10^{-8}$ **Watts per square meter-Kelvin**^{4} ($W/m^2 \cdot K^4$)

Speed of light in a vacuum (c) = **299,792,458 meters per second** (m/s) = 186282.4 miles per second (mi/s)

Note: All constants taken from NIST data "Fundamental Physical Constants – Complete Listing", from `http://physics.nist.gov/constants`, National Institute of Standards and Technology (NIST), 2018 CODATA Adjustment.

4.2.2 Introduction to spreadsheets

A powerful computational tool you are encouraged to use in your work is a *spreadsheet*. Available on most personal computers (e.g. Microsoft Excel), *spreadsheet* software performs numerical calculations based on number values and formulae entered into cells of a grid. This grid is typically arranged as lettered columns and numbered rows, with each cell of the grid identified by its column/row coordinates (e.g. cell `B3`, cell `A8`). Each cell may contain a string of text, a number value, or a mathematical formula. The spreadsheet automatically updates the results of all mathematical formulae whenever the entered number values are changed. This means it is possible to set up a spreadsheet to perform a series of calculations on entered data, and those calculations will be re-done by the computer any time the data points are edited in any way.

For example, the following spreadsheet calculates average speed based on entered values of distance traveled and time elapsed:

	A	B	C	D
1	Distance traveled	46.9	Kilometers	
2	Time elapsed	1.18	Hours	
3	Average speed	= B1 / B2	km/h	
4				
5				

Text labels contained in cells `A1` through `A3` and cells `C1` through `C3` exist solely for readability and are not involved in any calculations. Cell `B1` contains a sample distance value while cell `B2` contains a sample time value. The formula for computing speed is contained in cell `B3`. Note how this formula begins with an "equals" symbol (`=`), references the values for distance and speed by lettered column and numbered row coordinates (`B1` and `B2`), and uses a forward slash symbol for division (`/`). The coordinates `B1` and `B2` function as *variables*[6] would in an algebraic formula.

When this spreadsheet is executed, the numerical value `39.74576` will appear in cell `B3` rather than the formula `= B1 / B2`, because 39.74576 is the computed speed value given 46.9 kilometers traveled over a period of 1.18 hours. If a different numerical value for distance is entered into cell `B1` or a different value for time is entered into cell `B2`, cell `B3`'s value will automatically update. All you need to do is set up the given values and any formulae into the spreadsheet, and the computer will do all the calculations for you.

Cell `B3` may be referenced by other formulae in the spreadsheet if desired, since it is a variable just like the given values contained in `B1` and `B2`. This means it is possible to set up an entire chain of calculations, one dependent on the result of another, in order to arrive at a final value. The arrangement of the given data and formulae need not follow any pattern on the grid, which means you may place them anywhere.

[6] Spreadsheets may also provide means to attach text labels to cells for use as variable names (Microsoft Excel simply calls these labels "names"), but for simple spreadsheets such as those shown here it's usually easier just to use the standard coordinate naming for each cell.

Common[7] arithmetic operations available for your use in a spreadsheet include the following:

- Addition (`+`)
- Subtraction (`-`)
- Multiplication (`*`)
- Division (`/`)
- Powers (`^`)
- Square roots (`sqrt()`)
- Logarithms (`ln()` , `log10()`)

Parentheses may be used to ensure[8] proper order of operations within a complex formula. Consider this example of a spreadsheet implementing the *quadratic formula*, used to solve for roots of a polynomial expression in the form of $ax^2 + bx + c$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

	A	B
1	x_1	= (-B4 + sqrt((B4^2) - (4*B3*B5))) / (2*B3)
2	x_2	= (-B4 - sqrt((B4^2) - (4*B3*B5))) / (2*B3)
3	a =	9
4	b =	5
5	c =	-2

This example is configured to compute roots[9] of the polynomial $9x^2 + 5x - 2$ because the values of 9, 5, and -2 have been inserted into cells `B3`, `B4`, and `B5`, respectively. Once this spreadsheet has been built, though, it may be used to calculate the roots of *any* second-degree polynomial expression simply by entering the new a, b, and c coefficients into cells `B3` through `B5`. The numerical values appearing in cells `B1` and `B2` will be automatically updated by the computer immediately following any changes made to the coefficients.

[7] Modern spreadsheet software offers a bewildering array of mathematical functions you may use in your computations. I recommend you consult the documentation for your particular spreadsheet for information on operations other than those listed here.

[8] Spreadsheet programs, like text-based programming languages, are designed to follow standard order of operations by default. However, my personal preference is to use parentheses even where strictly unnecessary just to make it clear to any other person viewing the formula what the intended order of operations is.

[9] Reviewing some algebra here, a *root* is a value for x that yields an overall value of zero for the polynomial. For this polynomial ($9x^2 + 5x - 2$) the two roots happen to be $x = 0.269381$ and $x = -0.82494$, with these values displayed in cells `B1` and `B2`, respectively upon execution of the spreadsheet.

Alternatively, one could break up the long quadratic formula into smaller pieces like this:

$$y = \sqrt{b^2 - 4ac} \qquad z = 2a$$

$$x = \frac{-b \pm y}{z}$$

	A	B	C
1	x_1	= (-B4 + C1) / C2	= sqrt((B4^2) - (4*B3*B5))
2	x_2	= (-B4 - C1) / C2	= 2*B3
3	a =	9	
4	b =	5	
5	c =	-2	

Note how the square-root term (y) is calculated in cell `C1`, and the denominator term (z) in cell `C2`. This makes the two final formulae (in cells `B1` and `B2`) simpler to interpret. The positioning of all these cells on the grid is completely arbitrary[10] – all that matters is that they properly reference each other in the formulae.

Spreadsheets are particularly useful for situations where the same set of calculations representing a circuit or other system must be repeated for different initial conditions. The power of a spreadsheet is that it automates what would otherwise be a tedious set of calculations. One specific application of this is to simulate the effects of various components within a circuit failing with abnormal values (e.g. a shorted resistor simulated by making its value nearly zero; an open resistor simulated by making its value extremely large). Another application is analyzing the behavior of a circuit design given new components that are out of specification, and/or aging components experiencing drift over time.

[10] My personal preference is to locate all the "given" data in the upper-left cells of the spreadsheet grid (each data point flanked by a sensible name in the cell to the left and units of measurement in the cell to the right as illustrated in the first distance/time spreadsheet example), sometimes coloring them in order to clearly distinguish which cells contain entered data versus which cells contain computed results from formulae. I like to place all formulae in cells below the given data, and try to arrange them in logical order so that anyone examining my spreadsheet will be able to figure out *how* I constructed a solution. This is a general principle I believe all computer programmers should follow: *document and arrange your code to make it easy for other people to learn from it.*

4.2.3 Radian/degree angle conversions

Convert between the following angle units:

- 3.1 radians = ??? degrees
- 1.5 radians = ??? degrees
- 22 radians = ??? degrees
- 199^o = ??? radians
- 40^o = ??? radians
- 203^o = ??? radians

Challenges

- Which of the above angles represents more than one full rotation?

4.2.4 Plotting sine and cosine functions

Either plot by hand or use a computer program to plot a sine function and a cosine function given angles spanning the rotation of a full circle (i.e. from 0 degrees to 360 degrees, or from 0 radians to 2π radians).

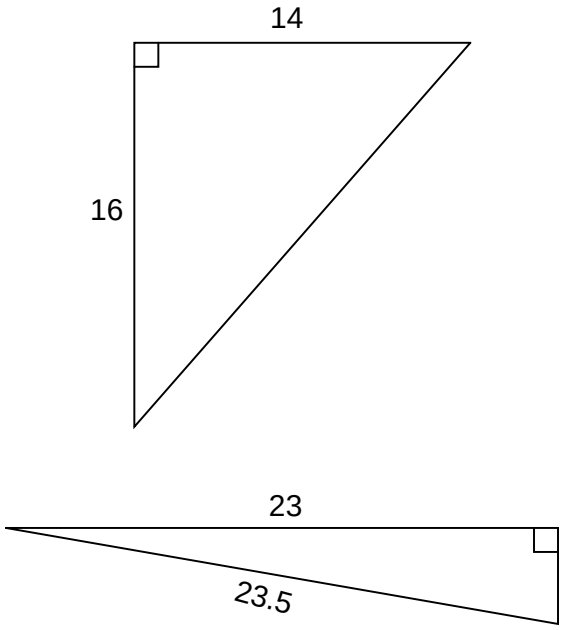
Reference a *trig table* and explain where the values for sine and cosine functions contained in that table exist on the plot you created.

Challenges

- At which angle value(s) does the sine function reach its positive peak value?
- At which angle value(s) does the sine function reach its negative peak value?
- At which angle value(s) does the cosine function reach its positive peak value?
- At which angle value(s) does the cosine function reach its negative peak value?

4.2.5 Unknown side lengths

Determine the lengths of the unmarked sides on these triangles:

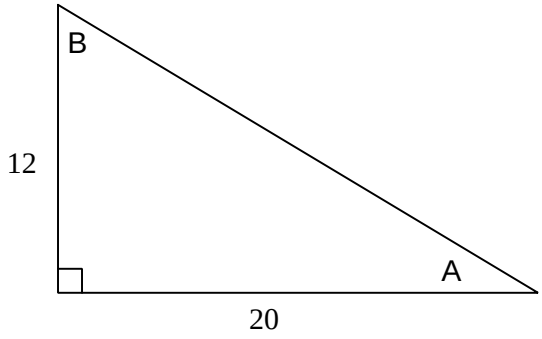


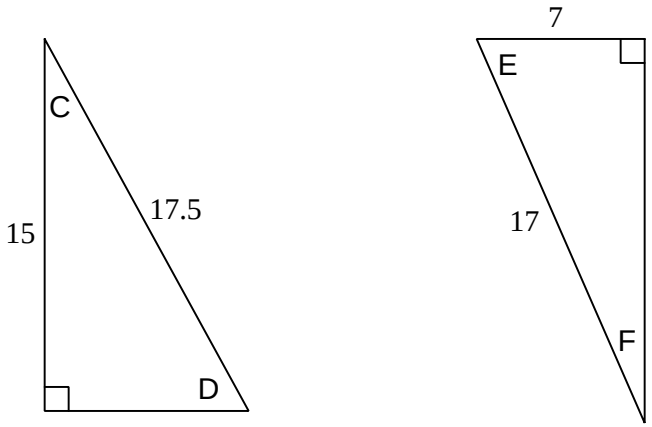
Challenges

- Why is it important for us to know that these are *right* triangles?

4.2.6 Unknown angles and side lengths

Determine the lengths of the unmarked sides and the values of the unmarked angles on these triangles:





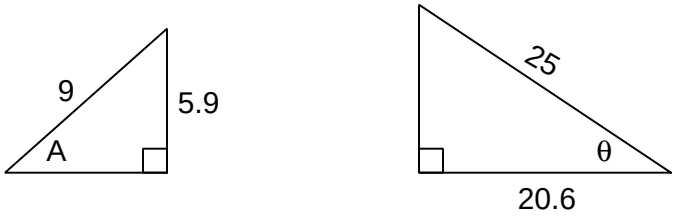
Next, show how it is possible to approximately calculate these same side lengths and angles, but without using any trigonometric function on your calculator!

Challenges

- Propose *multiple* methods for calculating each of the angles in these triangles.

4.2.7 Using trig tables

Suppose you were wishing to calculate angles A and θ for these triangles, using a trig function table rather than a *scientific* calculator:



If you are using a trig function table specifying angles to the nearest whole degree, you will find the values do not precisely match a whole-number angle, but is very close to one. What is the nearest whole-number value for angle A, and also for angle θ in these two triangles? Is the true value for angle A less than or greater than the nearest whole-number angle found in a trig table? How about the true value of angle θ?

Challenges

- Apply linear interpolation to get a closer estimate for both angles.

4.2.8 Sine and cosine values of common angles

Express the sine and cosine values for these angles as fractions of whole numbers and square roots:

- 0^o
- 30^o
- 45^o
- 60^o
- 90^o

Challenges

- What easy-to-remember pattern do see in these results?

4.3 Diagnostic reasoning

These questions are designed to stimulate your deductive and inductive thinking, where you must apply general principles to specific scenarios (deductive) and also derive conclusions about the failed circuit from specific details (inductive). In a Socratic discussion with your instructor, the goal is for these questions to reinforce your recall and use of general circuit principles and also challenge your ability to integrate multiple symptoms into a sensible explanation of what's wrong in a circuit. Your instructor may also pose additional questions based on those assigned, in order to further challenge and sharpen your diagnostic abilities.

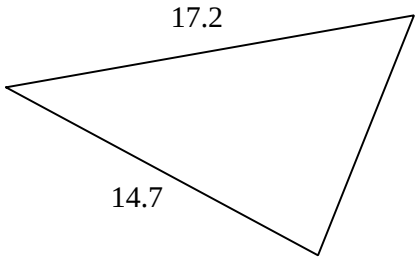
As always, your goal is to fully *explain* your analysis of each problem. Simply obtaining a correct answer is not good enough – you must also demonstrate sound reasoning in order to successfully complete the assignment. Your instructor's responsibility is to probe and challenge your understanding of the relevant principles and analytical processes in order to ensure you have a strong foundation upon which to build further understanding.

You will note a conspicuous lack of answers given for these diagnostic questions. Unlike standard textbooks where answers to every other question are given somewhere toward the back of the book, here in these learning modules students must rely on other means to check their work. The best way by far is to debate the answers with fellow students and also with the instructor during the Socratic dialogue sessions intended to be used with these learning modules. Reasoning through challenging questions with other people is an excellent tool for developing strong reasoning skills.

Another means of checking your diagnostic answers, where applicable, is to use circuit simulation software to explore the effects of faults placed in circuits. For example, if one of these diagnostic questions requires that you predict the effect of an open or a short in a circuit, you may check the validity of your work by simulating that same fault (substituting a very high resistance in place of that component for an open, and substituting a very low resistance for a short) within software and seeing if the results agree.

4.3.1 Find the mistake(s)

A student attempts to calculate the length of the unmarked side in this triangle, and obtains a result of 8.9303:



Explain why this side length is not necessarily correct for this triangle, and identify the unfounded assumption made by the student.

Challenges

- What other mistakes might a person make based on the same misconception/oversight?

Appendix A

Problem-Solving Strategies

The ability to solve complex problems is arguably one of the most valuable skills one can possess, and this skill is particularly important in any science-based discipline.

- Study principles, not procedures. Don't be satisfied with merely knowing how to compute solutions – learn *why* those solutions work.
- Identify what it is you need to solve, identify all relevant data, identify all units of measurement, identify any general principles or formulae linking the given information to the solution, and then identify any "missing pieces" to a solution. Annotate all diagrams with this data.
- Sketch a diagram to help visualize the problem. When building a real system, always devise a plan for that system and analyze its function *before* constructing it.
- Follow the units of measurement and meaning of every calculation. If you are ever performing mathematical calculations as part of a problem-solving procedure, and you find yourself unable to apply each and every intermediate result to some aspect of the problem, it means you don't understand what you are doing. Properly done, every mathematical result should have practical meaning for the problem, and not just be an abstract number. You should be able to identify the proper units of measurement for each and every calculated result, and show where that result fits into the problem.
- Perform "thought experiments" to explore the effects of different conditions for theoretical problems. When troubleshooting real systems, perform *diagnostic tests* rather than visually inspecting for faults, the best diagnostic test being the one giving you the most information about the nature and/or location of the fault with the fewest steps.
- Simplify the problem until the solution becomes obvious, and then use that obvious case as a model to follow in solving the more complex version of the problem.
- Check for exceptions to see if your solution is incorrect or incomplete. A good solution will work for *all* known conditions and criteria. A good example of this is the process of testing scientific hypotheses: the task of a scientist is not to find support for a new idea, but rather to *challenge* that new idea to see if it holds up under a battery of tests. The philosophical

principle of *reductio ad absurdum* (i.e. disproving a general idea by finding a specific case where it fails) is useful here.

- Work "backward" from a hypothetical solution to a new set of given conditions.
- Add quantities to problems that are qualitative in nature, because sometimes a little math helps illuminate the scenario.
- Sketch graphs illustrating how variables relate to each other. These may be quantitative (i.e. with realistic number values) or qualitative (i.e. simply showing increases and decreases).
- Treat quantitative problems as qualitative in order to discern the relative magnitudes and/or directions of change of the relevant variables. For example, try determining what happens if a certain variable were to increase or decrease before attempting to precisely calculate quantities: how will each of the dependent variables respond, by increasing, decreasing, or remaining the same as before?
- Consider limiting cases. This works especially well for qualitative problems where you need to determine which direction a variable will change. Take the given condition and magnify that condition to an extreme degree as a way of simplifying the direction of the system's response.
- Check your work. This means regularly testing your conclusions to see if they make sense. This does *not* mean repeating the same steps originally used to obtain the conclusion(s), but rather to use some other means to check validity. Simply repeating procedures often leads to *repeating the same errors* if any were made, which is why alternative paths are better.

Appendix B

Instructional philosophy

B.1 First principles of learning

- **Anyone can learn anything** given appropriate time, effort, resources, challenges, encouragement, and expectations. Dedicating time and investing effort are the student's responsibility; providing resources, challenges, and encouragement are the teacher's responsibility; high expectations are a responsibility shared by both student and teacher.

- **Transfer is not automatic.** The human mind has a natural tendency to compartmentalize information, which means the process of taking knowledge learned in one context and applying it to another usually does not come easy and therefore should never be taken for granted.

- **Learning is iterative.** The human mind rarely learns anything perfectly on the first attempt. Anticipate mistakes and plan for multiple tries to achieve full understanding, using the lessons of those mistakes as feedback to guide future attempts.

- **Information is absorbed, but understanding is created.** Facts and procedures may be memorized easily enough by repeated exposure, but the ability to reliably apply principles to novel scenarios only comes through intense personal effort. This effort is fundamentally creative in nature: casting new concepts in one's own words, running mental experiments to perceive patterns and trends, etc. These acts of making knowledge "one's own" need not be perfect in order to be effective, as the value lies in the activity and not necessarily the finished product. Depth of understanding and permanence of memory are both directly related to mental effort during the learning process – i.e. *easy learning is shallow and fleeting.*

- **Education trumps training.** There is no such thing as an entirely isolated subject, as all fields of knowledge are connected. Training is narrowly-focused and task-oriented. Education is broad-based and principle-oriented. When preparing for a life-long technical career, education beats training every time.

- **Character matters.** Poor habits are more destructive than deficits of knowledge or skill. This is especially true in collective endeavors, where a team's ability to function depends on trust between its members. Simply put, no one wants an untrustworthy person on their team. An essential component of education then, is character development.

- **People learn to be responsible by bearing responsibility.** An irresponsible person is someone who has never *had* to be responsible for anything that mattered enough to them. Just as anyone can learn anything, anyone can become responsible if the personal cost of irresponsibility becomes high enough.

- **What gets measured, gets done.** Accurate and relevant assessment of learning is key to ensuring all students learn. Therefore, it is imperative to measure what matters.

- **Failure is nothing to fear.** Every human being fails, and fails in multiple ways at multiple times. Eventual success only happens when we don't stop trying.

B.2 Proven strategies for instructors

- Assume every student is capable of learning anything they desire given the proper conditions. Treat them as capable adults by granting real responsibility and avoiding artificial incentives such as merit or demerit points.
- Create a consistent culture of high expectations across the entire program of study. Demonstrate and encourage patience, persistence, and a healthy sense of self-skepticism. Anticipate and de-stigmatize error. Teach respect for the capabilities of others as well as respect for one's own fallibility.
- Replace lecture with "inverted" instruction, where students first encounter new concepts through reading and then spend class time in Socratic dialogue with the instructor exploring those concepts and solving problems individually. There is a world of difference between observing someone solve a problem versus actually solving a problem yourself, and so the point of this form of instruction is to place students in a position where they *cannot* passively observe.
- Require students to read extensively, write about what they learn, and dialogue with you and their peers to sharpen their understanding. Apply Francis Bacon's advice that "reading maketh a full man; conference a ready man; and writing an exact man". These are complementary activities helping students expand their confidence and abilities.
- Use artificial intelligence (AI) to challenge student understanding rather than merely provide information. Find productive ways for AI to critique students' clarity of thought and of expression, for example by employing AI as a Socratic-style interlocutor or as a reviewer of students' journals. Properly applied, AI has the ability to expand student access to critical review well outside the bounds of their instructor's reach.
- Build frequent and rapid feedback into the learning process so that students know at all times how well they are learning, to identify problems early and fix them before they grow. Model the intellectual habit of self-assessing and self-correcting your own understanding (i.e. a cognitive *feedback loop*), encouraging students to do the same.
- Use "mastery" as the standard for every assessment, which means the exam or experiment or project must be done with 100% competence in order to pass. Provide students with multiple opportunity for re-tries (different versions of the assessment every time).
- Require students to devise their own hypotheses and procedures on all experiments, so that the process is truly a scientific one. Have students assess their proposed experimental procedures for risk and devise mitigations for those risks. Let nothing be pre-designed about students' experiments other than a stated task (i.e. what principle the experiment shall test) at the start and a set of demonstrable knowledge and skill objectives at the end.
- Have students build as much of their lab equipment as possible: building power sources, building test assemblies[1], and building complete working systems (no kits!). In order to provide

[1] In the program I teach, every student builds their own "Development Board" consisting of a metal chassis with DIN rail, terminal blocks, and an AC-DC power supply of their own making which functions as a portable lab environment they can use at school as well as take home.

this same "ground-up" experience for every new student, this means either previous students take their creations with them, or the systems get disassembled in preparation for the new students, or the systems grow and evolve with each new student group.

- Incorporate external accountability for you and for your students, continuously improving the curriculum and your instructional methods based on proven results. Have students regularly network with active professionals through participation in advisory committee meetings, service projects, tours, jobshadows, internships, etc. Practical suggestions include requiring students to design and build projects for external clients (e.g. community groups, businesses, different departments within the institution), and also requiring students attend all technical advisory committee meetings and to dialogue with the industry representatives at those meetings.
- Repeatedly explore difficult-to-learn concepts across multiple courses, so that students have multiple opportunities to build their understanding.
- Relate all new concepts, whenever possible, to previous concepts and to relevant physical laws. Challenge each and every student, every day, to *reason* from concept to concept and to explain the logical connections between. Challenge students to verify their conclusions by multiple approaches (e.g. double-checking their work using different methods). Ask *"Why?"* often.
- Maintain detailed records on each student's performance and share these records privately with them. These records should include academic performance as well as professionally relevant behavioral tendencies.
- Hold mandatory "check-in" meetings between all program faculty and each new student during their first term. Offer these to all other students as an option, except for any students continuing to manifest unprofessional behaviors, poor academic performance, or who have some other need for a face-to-face meeting with faculty.
- Address problems while they are small, before they grow larger. This is equally true for tutoring technical concepts as it is for helping students build professional habits.
- Build rigorous quality control into the curriculum to ensure every student masters every important concept, and that the mastery is retained over time. This includes (1) review questions added to every exam to re-assess knowledge taught in previous terms, (2) cumulative exams at the end of every term to re-assess all important concepts back to the very beginning of the program, and (3) review assessments in practical (hands-on) coursework to ensure critically-important skills were indeed taught and are still retained. What you will find by doing this is that it actually boosts retention of students by ensuring that important knowledge gets taught and is retained over long spans of time. In the absence of such quality control, student learning and retention tends to be spotty and this contributes to drop-out and failure rates later in their education.
- Finally, *never rush learning.* Education is not a race. Give your students ample time to digest complex ideas, as you continually remind yourself of just how long it took you to achieve mastery! Long-term retention and the consistently correct application of concepts are always the result of *focused effort over long periods of time* which means there are no shortcuts to learning.

B.3 Proven strategies for students

The single most important piece of advice I have for any student of any subject is to take responsibility for your own development in all areas of life including mental development. Expecting others in your life to entirely guide your own development is a recipe for disappointment. This is just as true for students enrolled in formal learning institutions as it is for auto-didacts pursuing learning entirely on their own. Learning to think in new ways is key to being able to gainfully use information, to make informed decisions about your life, and to best serve those you care about. With this in mind, I offer the following advice to students:

- **Approach all learning as valuable**. No matter what course you take, no matter who you learn from, no matter the subject, there is something useful in every learning experience. If you don't see the value of every new experience, you are not looking closely enough!
- **Continually challenge yourself**. Let other people take shortcuts and find easy answers to easy problems. The purpose of education is to stretch your mind, in order to shape it into a more powerful tool. This doesn't come by taking the path of least resistance. An excellent analogy for an empowering education is productive physical exercise: becoming stronger, more flexible, and more persistent only comes through intense personal effort.
- **Master the use of language**. This includes reading extensively, writing every day, listening closely, and speaking articulately. To a great extent language channels and empowers thought, so the better you are at wielding language the better you will be at grasping abstract concepts and articulating them not only for your benefit but for others as well.
- **Do not limit yourself to the resources given to you**. Read books that are not on the reading list. Run experiments that aren't assigned to you. Form study groups outside of class. Take an entrepreneurial approach to your own education, as though it were a business you were building for your future benefit.
- **Express and share what you learn**. Take every opportunity to teach what you have learned to others, as this will not only help them but will also strengthen your own understanding[2].
- Realize that **no one can give you understanding**, just as no one can give you physical fitness. These both must be *built*.
- **Above all, recognize that learning is hard work, and that a certain level of frustration is unavoidable**. There are times when you will struggle to grasp some of these concepts, and that struggle is a natural thing. Take heart that it will yield with persistent and varied[3] effort, and never give up! That concepts don't immediately come to you is not a sign of something wrong, but rather of something right: that you have found a worthy challenge!

[2] On a personal note, I was surprised to learn just how much my own understanding of electronics and related subjects was strengthened by becoming a teacher. When you are tasked every day with helping other people grasp complex topics, it catalyzes your own learning by giving you powerful incentives to study, to articulate your thoughts, and to reflect deeply on the process of learning.

[3] As the old saying goes, "Insanity is trying the same thing over and over again, expecting different results." If you find yourself stumped by something in the text, you should attempt a different approach. Alter the thought experiment, change the mathematical parameters, do whatever you can to see the problem in a slightly different light, and then the solution will often present itself more readily.

B.4 Design of these learning modules

"The unexamined circuit is not worth energizing" – Socrates (if he had taught electricity)

These learning modules, although useful for self-study, were designed to be used in a formal learning environment where a subject-matter expert challenges students to digest the content and exercise their critical thinking abilities in the answering of questions and in the construction and testing of working circuits. Every effort has been made to embed the following instructional and assessment philosophies within:

- The first goal of education is to enhance clear and independent thought, in order that every student reach their fullest potential in a highly complex and inter-dependent world. Robust reasoning is *always* more important than particulars of any subject matter, because its application is universal.
- Literacy is fundamental to independent learning and thought because text continues to be the most efficient way to communicate complex ideas over space and time. Those who cannot read with ease are limited in their ability to acquire knowledge and perspective.
- Articulate communication is fundamental to work that is complex and interdisciplinary.
- Faulty assumptions and poor reasoning are best corrected through challenge, not presentation. The rhetorical technique of *reductio ad absurdum* (disproving an assertion by exposing an absurdity) works well to discipline student's minds, not only to correct the problem at hand but also to learn how to detect and correct future errors.
- Important principles should be repeatedly explored and widely applied throughout a course of study, not only to reinforce their importance and help ensure their mastery, but also to showcase the interconnectedness and utility of knowledge.

These learning modules were expressly designed to be used in an "inverted" teaching environment[4] where students first read the introductory and tutorial chapters on their own, then individually attempt to answer the questions and construct working circuits according to the experiment and project guidelines. The instructor never lectures, but instead meets regularly with each individual student to review their progress, answer questions, identify misconceptions, and challenge the student to new depths of understanding through further questioning. Regular meetings between instructor and student should resemble a Socratic[5] dialogue, where questions serve as scalpels to dissect topics and expose assumptions. The student passes each module only after consistently demonstrating their ability to logically analyze and correctly apply all major concepts in each question or project/experiment. The instructor must be vigilant in probing each student's understanding to ensure they are truly *reasoning* and not just *memorizing*. This is why "Challenge" points appear throughout, as prompts for students to think deeper about topics and as starting points for instructor queries. Sometimes these challenge points require additional knowledge that hasn't been covered in the series to answer in full. This is okay, as the major purpose of the Challenges is to stimulate analysis and synthesis on the part of each student.

The instructor must possess enough mastery of the subject matter and awareness of students' reasoning to generate their own follow-up questions to practically any student response. Even completely correct answers given by the student should be challenged by the instructor for the purpose of having students practice articulating their thoughts and defending their reasoning. Conceptual errors committed by the student should be exposed and corrected not by direct instruction, but rather by reducing the errors to an absurdity[6] through well-chosen questions and thought experiments posed by the instructor. Becoming proficient at this style of instruction requires time and dedication, but the positive effects on critical thinking for both student and instructor are spectacular.

An inspection of these learning modules reveals certain unique characteristics. One of these is a bias toward thorough explanations in the tutorial chapters. Without a live instructor to explain concepts and applications to students, the text itself must fulfill this role. This philosophy results in lengthier explanations than what you might typically find in a textbook, each step of the reasoning process fully explained, including footnotes addressing common questions and concerns students raise while learning these concepts. Each tutorial seeks to not only explain each major concept in sufficient detail, but also to explain the logic of each concept and how each may be developed

[4] In a traditional teaching environment, students first encounter new information via *lecture* from an expert, and then independently apply that information via *homework*. In an "inverted" course of study, students first encounter new information via *homework*, and then independently apply that information under the scrutiny of an expert. The expert's role in lecture is to simply *explain*, but the expert's role in an inverted session is to *challenge*, *critique*, and if necessary *explain* where gaps in understanding still exist.

[5] Socrates is a figure in ancient Greek philosophy famous for his unflinching style of questioning. Although he authored no texts, he appears as a character in Plato's many writings. The essence of Socratic philosophy is to leave no question unexamined and no point of view unchallenged. While purists may argue a topic such as electric circuits is too narrow for a true Socratic-style dialogue, I would argue that the essential thought processes involved with scientific reasoning on *any* topic are not far removed from the Socratic ideal, and that students of electricity and electronics would do very well to challenge assumptions, pose thought experiments, identify fallacies, and otherwise employ the arsenal of critical thinking skills modeled by Socrates.

[6] This rhetorical technique is known by the Latin phrase *reductio ad absurdum*. The concept is to expose errors by counter-example, since only one solid counter-example is necessary to disprove a universal claim. As an example of this, consider the common misconception among beginning students of electricity that voltage cannot exist without current. One way to apply *reductio ad absurdum* to this statement is to ask how much current passes through a fully-charged battery connected to nothing (i.e. a clear example of voltage existing without current).

from "first principles". Again, this reflects the goal of developing clear and independent thought in students' minds, by showing how clear and logical thought was used to forge each concept. Students benefit from witnessing a model of clear thinking in action, and these tutorials strive to be just that.

Another feature of these learning modules is that they do not treat topics in isolation. Rather, important concepts are introduced early in the series, and appear repeatedly as stepping-stones toward other concepts in subsequent modules. This helps to avoid the "compartmentalization" of knowledge, demonstrating the inter-connectedness of concepts and simultaneously reinforcing them. Each module is fairly complete in itself, reserving the beginning of its tutorial to a review of foundational concepts.

To high standards of education,

Tony R. Kuphaldt

Appendix C

Tools used

I am indebted to the developers of many open-source software applications in the creation of these learning modules. The following is a list of these applications with some commentary on each.

You will notice a theme common to many of these applications: a bias toward *code*. Although I am by no means an expert programmer in any computer language, I understand and appreciate the flexibility offered by code-based applications where the user (you) enters commands into a plain ASCII text file, which the software then reads and processes to create the final output. Code-based computer applications are by their very nature *extensible*, while WYSIWYG (What You See Is What You Get) applications are generally limited to whatever user interface the developer makes for you.

The `GNU/Linux` computer operating system

> There is so much to be said about Linus Torvalds' `Linux` and Richard Stallman's `GNU` project. First, to credit just these two individuals is to fail to do justice to the *mob* of passionate volunteers who contributed to make this amazing software a reality. I first learned of `Linux` back in 1996, and have been using this operating system on my personal computers almost exclusively since then. It is *free*, it is completely *configurable*, and it permits the continued use of highly efficient `Unix` applications and scripting languages (e.g. shell scripts, Makefiles, `sed`, `awk`) developed over many decades. `Linux` not only provided me with a powerful computing platform, but its open design served to inspire my life's work of creating open-source educational resources.

Bram Moolenaar's `Vim` text editor

> Writing code for any code-based computer application requires a *text editor*, which may be thought of as a word processor strictly limited to outputting plain-ASCII text files. Many good text editors exist, and one's choice of text editor seems to be a deeply personal matter within the programming world. I prefer `Vim` because it operates very similarly to `vi` which is ubiquitous on `Unix`/`Linux` operating systems, and because it may be entirely operated via keyboard (i.e. no mouse required) which makes it fast to use.

Donald Knuth's TEX typesetting system

Developed in the late 1970's and early 1980's by computer scientist extraordinaire Donald Knuth to typeset his multi-volume magnum opus *The Art of Computer Programming*, this software allows the production of formatted text for screen-viewing or paper printing, all by writing plain-text code to describe how the formatted text is supposed to appear. TEX is not just a markup language for documents, but it is also a Turing-complete programming language in and of itself, allowing useful algorithms to be created to control the production of documents. Simply put, TEX *is a programmer's approach to word processing*. Since TEX is controlled by code written in a plain-text file, this means anyone may read that plain-text file to see exactly how the document was created. This openness afforded by the code-based nature of TEX makes it relatively easy to learn how other people have created their own TEX documents. By contrast, examining a beautiful document created in a conventional WYSIWYG word processor such as Microsoft `Word` suggests nothing to the reader about *how* that document was created, or what the user might do to create something similar. As Mr. Knuth himself once quipped, conventional word processing applications should be called WYSIAYG (What You See Is *All* You Get).

Leslie Lamport's LATEX extensions to TEX

Like all true programming languages, TEX is inherently extensible. So, years after the release of TEX to the public, Leslie Lamport decided to create a massive extension allowing easier compilation of book-length documents. The result was LATEX, which is the markup language used to create all ModEL module documents. You could say that TEX is to LATEX as `C` is to `C++`. This means it is permissible to use any and all TEX commands within LATEX source code, and it all still works. Some of the features offered by LATEX that would be challenging to implement in TEX include automatic index and table-of-content creation.

Tim Edwards' `Xcircuit` drafting program

This wonderful program is what I use to create all the schematic diagrams and illustrations (but not photographic images or mathematical plots) throughout the ModEL project. It natively outputs PostScript format which is a true vector graphic format (this is why the images do not pixellate when you zoom in for a closer view), and it is so simple to use that I have never had to read the manual! Object libraries are easy to create for `Xcircuit`, being plain-text files using PostScript programming conventions. Over the years I have collected a large set of object libraries useful for drawing electrical and electronic schematics, pictorial diagrams, and other technical illustrations.

`Gimp` graphic image manipulation program

Essentially an open-source clone of Adobe's `PhotoShop`, I use `Gimp` to resize, crop, and convert file formats for all of the photographic images appearing in the ModEL modules. Although `Gimp` does offer its own scripting language (called `Script-Fu`), I have never had occasion to use it. Thus, my utilization of `Gimp` to merely crop, resize, and convert graphic images is akin to using a sword to slice bread.

`SPICE` circuit simulation program

`SPICE` is to circuit analysis as TEX is to document creation: it is a form of markup language designed to describe a certain object to be processed in plain-ASCII text. When the plain-text "source file" is compiled by the software, it outputs the final result. More modern circuit analysis tools certainly exist, but I prefer `SPICE` for the following reasons: it is *free*, it is *fast*, it is *reliable*, and it is a fantastic tool for *teaching* students of electricity and electronics how to write simple code. I happen to use rather old versions of `SPICE`, version 2g6 being my "go to" application when I only require text-based output. `NGSPICE` (version 26), which is based on Berkeley `SPICE` version 3f5, is used when I require graphical output for such things as time-domain waveforms and Bode plots. In all `SPICE` example netlists I strive to use coding conventions compatible with all `SPICE` versions.

Andrew D. Hwang's `ePiX` mathematical visualization programming library

This amazing project is a `C++` library you may link to any `C/C++` code for the purpose of generating PostScript graphic images of mathematical functions. As a completely free and open-source project, it does all the plotting I would otherwise use a Computer Algebra System (CAS) such as `Mathematica` or `Maple` to do. It should be said that `ePiX` is *not* a Computer Algebra System like `Mathematica` or `Maple`, but merely a mathematical *visualization* tool. In other words, it won't determine integrals for you (you'll have to implement that in your own `C/C++` code!), but it can graph the results, and it does so beautifully. What I really admire about `ePiX` is that it is a `C++` programming library, which means it builds on the existing power and toolset available with that programming language. Mr. Hwang could have probably developed his own stand-alone application for mathematical plotting, but by creating a `C++` library to do the same thing he accomplished something much greater.

`gnuplot` mathematical visualization software

Another open-source tool for mathematical visualization is `gnuplot`. Interestingly, this tool is *not* part of Richard Stallman's GNU project, its name being a coincidence. For this reason the authors prefer "gnu" *not* be capitalized at all to avoid confusion. This is a much "lighter-weight" alternative to a spreadsheet for plotting tabular data, and the fact that it easily outputs directly to an X11 console or a file in a number of different graphical formats (including PostScript) is very helpful. I typically set my `gnuplot` output format to default (X11 on my Linux PC) for quick viewing while I'm developing a visualization, then switch to PostScript file export once the visual is ready to include in the document(s) I'm writing. As with my use of `Gimp` to do rudimentary image editing, my use of `gnuplot` only scratches the surface of its capabilities, but the important points are that it's *free* and that it *works well*.

`Python` programming language

Both Python and C++ find extensive use in these modules as instructional aids and exercises, but I'm listing Python here as a *tool* for myself because I use it almost daily as a *calculator*. If you open a Python interpreter console and type `from math import *` you can type mathematical expressions and have it return results just as you would on a hand calculator. Complex-number (i.e. *phasor*) arithmetic is similarly supported if you include the complex-math library (`from cmath import *`). Examples of this are shown in the Programming References chapter (if included) in each module. Of course, being a fully-featured programming language, Python also supports conditionals, loops, and other structures useful for calculation of quantities. Also, running in a console environment where all entries and returned values show as text in a chronologically-ordered list makes it easy to copy-and-paste those calculations to document exactly how they were performed.

Appendix D

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Appendix E

References

Appendix F

Version history

This is a list showing all significant additions, corrections, and other edits made to this learning module. Each entry is referenced by calendar date in reverse chronological order (newest version first), which appears on the front cover of every learning module for easy reference. Any contributors to this open-source document are listed here as well.

30 August 2026 – added +1 and −1 labels to image_0405.

27 January 2026 – corrected some typographical errors and clarified some unclear prose, courtesy of Wyatt Gleason.

25 August 2024 – divided the Introduction chapter into sections, one with recommendations for students, one with a listing of challenging concepts, and one with recommendations for instructors.

12 March 2023 – added a new Quantitative Reasoning question on plotting sine and cosine functions. Also corrected a mis-spelled word.

13 June 2022 – corrected an error in the equation for cotangent, courtesy of Caleb Colby.

18 May 2022 – minor edits made to the "Find the mistake(s)" Diagnostic Reasoning question.

8 May 2021 – commented out or deleted empty chapters.

20 January 2021 – significantly edited the Introduction chapter to make it more suitable as a pre-study guide and to provide cues useful to instructors leading "inverted" teaching sessions.

2-3 January 2021 – added content to Tutorial as well as Questions chapters.

1 January 2021 – document first created.

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